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Fracpairs and fractions over a reduced commutative ring

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Abstract

In the well-known construction of the field of fractions of an integral domain, division by zero is excluded. We introduce "fracpairs" as pairs subject to laws consistent with the use of the pair as a fraction, but do not exclude denominators to be zero. We investigate fracpairs over a reduced commutative ring (that is, a commutative ring that has no nonzero nilpotent elements) and provide these with natural definitions for addition, multiplication, and additive and multiplicative inverse. We find that modulo a simple congruence these fracpairs constitute a "common meadow", which is a commutative monoid both for addition and multiplication, extended with a weak additive inverse, a multiplicative inverse except for zero, and an additional element $\mathbf{a}$ that is the image of the multiplicative inverse on zero and that propagates through all operations. Considering $\mathbf{a}$ as an error-value supports the intuition.

The equivalence classes of fracpairs thus obtained are called common cancellation fractions (cc-fractions), and cc-fractions over the integers constitute a homomorphic pre-image of the common meadow $\mathbb{Q}_\mathbf{a}$, the field $\mathbb{Q}$ of rational numbers expanded with an $\mathbf{a}$-totalized inverse. Moreover, the initial common meadow is isomorphic to the initial algebra of cc-fractions over the integer numbers. Next, we define canonical term algebras (and therewith normal forms) for cc-fractions over the integers and some meadows that model the rational numbers expanded with a totalized inverse, and we provide some negative results concerning their associated term rewriting properties. Then we consider reduced commutative rings in which the sum of two squares plus one cannot be a zero divisor: by extending the equivalence relation on fracpairs we obtain an initial algebra that is isomorphic to $\mathbb{Q}_\mathbf{a}$. Finally, we express some negative conjectures concerning alternative specifications for these (concrete) datatypes.

Keywords: Fraction as a pair; Common meadow; Division by zero; Abstract datatype; Rational numbers; Term rewriting

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1. Introduction

In this paper we introduce *fracpairs*, where the idea that "a fraction is a pair" is formalized, though without the constraint that the second element of the pair must not be zero. We provide fracpairs with natural definitions for addition, multiplication, and additive and multiplicative inverse. In order to further model this approach to a "fraction", one can consider fracpairs modulo any equivalence that is a congruence with respect to addition, multiplication, and additive and multiplicative inverse, and we will consider two such equivalence relations.

This set-up is comparable to the construction of the field of fractions of an integral domain,[1] which we recall here. Given an integral domain R, the elements of the field of fractions $Q(R)$ are equivalence classes in $R \times R \setminus \{0\}$ that are often represented as

$$\frac{p}{q}$$

(in-line written as p/q), where the equivalence $\sim$ is defined by

$$\frac{p}{q} \sim \frac{r}{s} \quad \text{if, and only if } p \cdot s = q \cdot r \text{ holds in } R.$$

In $Q(R)$, addition, multiplication, and additive inverse are defined by

$$\frac{p}{q} + \frac{r}{s} = \frac{p \cdot s + r \cdot q}{q \cdot s} \quad \text{and} \quad \frac{p}{q} \cdot \frac{r}{s} = \frac{p \cdot r}{q \cdot s} \quad \text{and} \quad -\frac{p}{q} = \frac{-p}{q} \tag{1}$$

and these definitions are independent from the particular choice of a representative p/q. These fractions satisfy the axioms CR given in Table 1 of commutative rings with $0/p = 0/1$ for the zero and $1/1$ for the multiplicative unit 1. Because each $p/q \in Q(R)$ different from the zero has an inverse q/p, $Q(R)$ is a field, and it is the smallest field in which R can be embedded. Identifying $p \in R$ with (the equivalence class of) $p/1$ makes R a subring of $Q(R)$.

In this paper we will consider fracpairs defined over a commutative ring R that is *reduced* (see [10]), i.e., R has no nonzero nilpotent elements, or equivalently, R satisfies the property

$$x \cdot x = 0 \;\Rightarrow\; x = 0. \tag{2}$$

The integral domain $\mathbb{Z}$ of integers is a prime example of a reduced commutative ring,[2] and other examples that are not an integral domain are the ring $\mathbb{Z}/6\mathbb{Z}$ and the ring $\mathbb{Z} \times \mathbb{Z}$.

We recall the following familiar consequences of the axioms CR for commutative rings:

$$-0 = 0, \qquad 0 \cdot x = 0, \qquad -(-x) = x, \quad \text{and} \quad -(x \cdot y) = x \cdot (-y).$$

As is common, we assume that $\cdot$ binds stronger than $+$ and we will often omit brackets (as in $x \cdot y + x \cdot z$).

Fracpairs over a reduced commutative ring are provided with definitions for addition, multiplication, and additive inverse as described in (1), and – more interesting – also with a multiplicative inverse. Our first main result (Theorem 2.3.2) is that fracpairs modulo a natural congruence relation constitute a so-called *common meadow*. The equivalence classes of fracpairs thus obtained will be called "common cancellation fractions", or *cc-fractions* for short. It follows

[1] Integral domain: a nonzero commutative ring in which the product of any two nonzero elements is nonzero.

[2] Terminology: Lam [13, p. 194] uses "commutative reduced ring" and "noncommutative reduced ring".

Table 1
CR, axioms for commutative rings with 0 as the zero and 1 as the multiplicative unit.

$(x+y)+z = x+(y+z)$	$(x \cdot y) \cdot z = x \cdot (y \cdot z)$
$x+y = y+x$	$x \cdot y = y \cdot x$
$x+0 = x$	$1 \cdot x = x$
$x+(-x) = 0$	$x \cdot (y+z) = (x \cdot y) + (x \cdot z)$

that cc-fractions over $\mathbb{Z}$ constitute a homomorphic pre-image of the common meadow $\mathbb{Q}_{\mathbf{a}}$, that is, the field $\mathbb{Q}$ of rational numbers expanded with an **a**-totalized inverse (that is, $0^{-1} = \mathbf{a}$). A further result is the characterization of the initial common meadow as the initial algebra of cc-fractions over $\mathbb{Z}$ (Theorem 2.3.4). Finally, for fracpairs over a reduced commutative ring that satisfies a particular property we consider a more identifying equivalence relation in order to define "rational fractions", and prove that the rational fractions over $\mathbb{Z}$ represent $\mathbb{Q}_{\mathbf{a}}$ (Theorem 3.3.3). These results reinforce our idea that common meadows can be used in the development of alternative foundations of elementary (educational) mathematics from a perspective of abstract datatypes, term rewriting, and mathematical logic. We will return to this point in Section 4.

The paper is structured as follows. In Section 2 we introduce fracpairs and cc-fractions over a reduced commutative ring and prove our main results. In Section 3 we discuss some term rewriting issues for meadows in the context of fracpairs, and define canonical term algebras that represent these meadows, including a representation of $\mathbb{Q}_{\mathbf{a}}$ as an initial algebra of rational fractions. In Section 4, we end the paper with some conclusions and a brief digression. In Appendix A we analyze the cc-fractions over $\mathbb{Z}/6\mathbb{Z}$, and in Appendix B we prove some elementary identities for common meadows.

2. Fracpairs and fractions over a reduced commutative ring

In Section 2.1 we define fracpairs and an equivalence on these, and establish some elementary properties. In Section 2.2 we define common cancellation fractions and relate these to the setting of common meadows, and in Section 2.3 we present our main results.

2.1. Fracpairs and common cancellation equivalence: some elementary properties

Given a reduced commutative ring R, a *fracpair over R* is an element of $R \times R$ with special notation

$$\frac{p}{q},$$

which will be in-line written as p/q. Note that for any $p \in R$, $p/0$ is a fracpair over R. When considering a fracpair p/q over R as an expression, we will use some common terminology:

$$\frac{p}{q} \text{ has } \textit{numerator } p \text{ and } \textit{denominator } q.$$

We will consider fracpairs modulo some 'cancellation equivalence', that is, an equivalence generated by a set of 'cancellation identities', where a cancellation identity has the form $(x \cdot y)/(x \cdot z) = y/z$.

Definition 2.1.1. Let R be a reduced commutative ring. The cancellation equivalence generated by the common cancellation axiom CC defined in Table 2 for fracpairs is called

cc-equivalence, notation $=_{\mathrm{cc}}$.

Table 2
CC, the Common Cancellation axiom for fracpairs.

$\frac{x \cdot z}{y \cdot (z \cdot z)} = \frac{x}{y \cdot z}$	(CC)

In the proposition below we state a few simple properties of cc-equivalence.

Proposition 2.1.2. *For fracpairs over a reduced commutative ring R, the following identities hold:*

$$\frac{x}{x \cdot x} =_{\mathrm{cc}} \frac{1}{x}, \qquad \frac{x \cdot z}{z \cdot z} =_{\mathrm{cc}} \frac{x}{z},$$
$$\frac{x}{0} =_{\mathrm{cc}} \frac{0}{0}, \qquad \frac{x}{-y} =_{\mathrm{cc}} \frac{-x}{y}.$$

Proof. The two topmost identities are trivial: take $x = y = 1$ respectively $y = 1$ in CC and apply the axioms CR for commutative rings. Furthermore, by the top rightmost identity we immediately find

$$\frac{x}{0} =_{\mathrm{cc}} \frac{x \cdot 0}{0 \cdot 0} = \frac{0}{0},$$

and

$$\frac{x}{-y} =_{\mathrm{cc}} \frac{x \cdot (-y)}{(-y) \cdot (-y)} = \frac{(-x) \cdot y}{y \cdot y} =_{\mathrm{cc}} \frac{-x}{y}. \quad \Box$$

Consistency of the construction of fracpairs over R amounts to the absence of unexpected identifications (thus, separations) in the case that R is nontrivial ($0 \neq 1$).

Proposition 2.1.3. *Let R be a nontrivial reduced commutative ring. For fracpairs $p/0$ and q/r over R with nonzero r it holds that*

$$\frac{p}{0} \neq_{\mathrm{cc}} \frac{q}{r}.$$

Proof. Each instance of CC leaves the denominator 0 of $p/0$ invariant: if $s \cdot t = 0$, then $s \cdot (t \cdot t) = 0$ by CR, and if $s \cdot (t \cdot t) = 0$, then $(s \cdot t) \cdot (s \cdot t) = 0$ by CR and thus $s \cdot t = 0$ by property (2) that defines reduced rings. Hence, during a sequence of proof steps this denominator cannot transform from zero to nonzero or from nonzero to zero. $\Box$

In the remainder of this section we establish some more elementary properties of fracpairs, and discuss a related approach to "fractions as pairs". Let

$$n(R) = \{x \in R \mid \forall y \in R : \ x \cdot y = 0 \ \Rightarrow \ y = 0\}$$

be the set of nonzero divisors of R. It easily follows that

$$p \cdot q \in n(R) \iff p \in n(R) \quad \text{and} \quad q \in n(R).[3] \tag{3}$$

[3] Observe that for this property to hold, it suffices that R is a commutative ring.

Let R be a reduced commutative ring, then the relation $(p, q) \sim (r, s)$ defined by $p \cdot s = q \cdot r$ is an equivalence relation on $R \times n(R)$. We show transitivity: assume $(p, q) \sim (r, s) \sim (u, v)$, then $p \cdot s \cdot v = q \cdot r \cdot v = q \cdot s \cdot u$, hence $s \cdot (p \cdot v - q \cdot u) = 0$. Since $s \in n(R)$, $p \cdot v - q \cdot u = 0$, that is, $p \cdot v = q \cdot u$ and thus $(p, q) \sim (u, v)$.

Proposition 2.1.4. *Let R be a reduced commutative ring and $p, q, r, s \in R$. If*

$$\frac{p}{q} =_{\mathrm{cc}} \frac{r}{s} \quad \textit{and} \quad q \in n(R),$$

then $s \in n(R)$ and $R \models p \cdot s = q \cdot r$.

Proof. This follows by induction on the length of a proof of $p/q =_{\mathrm{cc}} r/s$. It suffices to show that each CC-instance

$$\frac{u \cdot w}{v \cdot (w \cdot w)} =_{\mathrm{cc}} \frac{u}{v \cdot w}$$

implies

$$v \cdot (w \cdot w) \in n(R) \iff v \cdot w \in n(R),$$

which follows from (3), and that each such instance satisfies $\sim$, which follows from CR because $(u \cdot w) \cdot (v \cdot w) = (v \cdot (w \cdot w)) \cdot u$. Hence, $s \in n(R)$ and thus $(p, q) \sim (r, s)$, and thus $R \models p \cdot s = q \cdot r$. $\square$

Propositions 2.1.3 and 2.1.4 imply the following corollary.

Corollary 2.1.5. *Let R be a nontrivial reduced commutative ring, then*

1. *the fracpairs $0/1$, $1/1$, and $1/0$ over R are pairwise distinct,*
2. *for $p, q \in R$, if $p/1 =_{\mathrm{cc}} q/1$, then $R \models p = q$,*
3. *for $p, q \in n(R)$, if $1/p =_{\mathrm{cc}} 1/q$, then $R \models p = q$.*

We conclude this section by discussing a construction that generalizes the notion of the field of fractions in a related way. Let R be an arbitrary commutative ring, and let S be a multiplicative subset of R (that is, $1 \in S$ and if $u, v \in S$, then $u \cdot v \in S$). Then $S^{-1}R$, the *localization of R with respect to S* (see, e.g., [14]), is defined as the set of equivalence classes of pairs $(p, q) \in R \times S$ under the equivalence relation

$$(p, q) \sim (r, s) \iff \exists u \in S : u \cdot (p \cdot s - q \cdot r) = 0.$$

Addition and multiplication in $S^{-1}R$ are defined as usual (cf. the definitions in (1)):

$$(p, q) + (r, s) = (p \cdot s + q \cdot r, q \cdot s) \quad \text{and} \quad (p, q) \cdot (r, s) = (p \cdot r, q \cdot s).$$

For $S = n(R)$ this yields the *total quotient ring of R*, also called the *total ring of fractions of R* (see [14]). If R is a domain, then $S = R \setminus \{0\}$ and the total quotient ring is the same as the field of fractions $Q(R)$. Since S in the construction contains no zero divisors, the natural ring homomorphism $R \to Q(R)$ is injective, so the total quotient ring is an extension of R. In the general case, the ring homomorphism from R to $S^{-1}R$ might fail to be injective. For example, if $0 \in S$, then $S^{-1}R$ is the trivial ring. In the case that S is also *saturated*, that is, $x \cdot y \in S \Rightarrow x \in S$, we have the following connection with fracpairs over R.

Table 3
Defining equations for the operations and constants of Σ_{cm} on fracpairs.

$\left(\frac{x}{y}\right)+\left(\frac{u}{v}\right)=\frac{(x\cdot v)+(u\cdot y)}{y\cdot v}$	(F1)	$0=\frac{0}{1}$	(F5)
$\left(\frac{x}{y}\right)\cdot\left(\frac{u}{v}\right)=\frac{x\cdot u}{y\cdot v}$	(F2)	$1=\frac{1}{1}$	(F6)
$-\left(\frac{x}{y}\right)=\frac{-x}{y}$	(F3)	$\mathbf{a}=\frac{1}{0}$	(F7)
$\left(\frac{x}{y}\right)^{-1}=\frac{y\cdot y}{x\cdot y}$	(F4)		

Proposition 2.1.6. *Let R be a reduced commutative ring, and let S be a multiplicative subset of R that is saturated.*

If $p/q =_{\rm cc} r/s$ for $p, r, s \in R$ and $q \in S$, then $s \in S$ and $S^{-1}R \models (p,q) \sim (r,s)$.

Proof. Because S is multiplicatively closed and saturated, $s \in S$ follows from $q \in S$ by induction on the length of a proof for $p/q =_{\rm cc} r/s$. By Proposition 2.1.4 it follows that $R \models p\cdot s = q\cdot r$, and because $1 \in S$, $S^{-1}R \models (p,q) \sim (r,s)$. □

In Appendix A we show how Proposition 2.1.6 can be used to prove separation of fracpairs with respect to cc-equivalence: we show that for certain fracpairs p/q and r/s over $\mathbb{Z}/6\mathbb{Z}$, $p/q \neq_{\rm cc} r/s$.

2.2. *Fracpairs and common cancellation fractions: constants and operations*

In Table 3 we define constants and operations for fracpairs that are tailored to the setting of common meadows [5], that is, structures over the signature

$$\Sigma_{cm} = \{0, 1, \mathbf{a}, -(\_), (\_)^{-1}, +, \cdot\}.$$

In the next section we explain the concept of a common meadow and discuss the role of the constants 0, 1, and $\mathbf{a}$. Note that the defining equations for addition (F1), multiplication (F2), and additive inverse (F3) all have a familiar form. The defining equation for the multiplicative inverse (F4) ensures that if a denominator of a fracpair has a factor 0, then that of its inverse also has a factor 0. We shall sometimes omit brackets in sums and products of fracpairs and write

$$\frac{p}{q}+\frac{r}{s} \quad\text{and}\quad \frac{p}{q}\cdot\frac{r}{s}.$$

Given a reduced commutative ring R we define

$$F(R)$$

as the set of fracpairs over R.

Proposition 2.2.1. *Let R be a reduced commutative ring, and let the meadow operations from Σ_{cm} be defined on $F(R)$ by Eqs. (F2)–(F4) in Table 3. Then the relation $=_{\rm cc}$ is a congruence on $F(R)$.*

Proof. It suffices to show that if p/q can be proven equal to r/s with finitely many instances of the axiom CC, then the same holds for their image under the meadow operations as defined in Table 3.

Let $A = (p \cdot r)/(q \cdot (r \cdot r))$ and $B = p/(q \cdot r)$, so $A =_{\mathrm{cc}} B$. Then

- $A + (s/t) =_{\mathrm{cc}} B + (s/t)$ because

$$\begin{aligned}
\frac{p \cdot r}{q \cdot (r \cdot r)} + \frac{s}{t} &= \frac{(p \cdot r) \cdot t + s \cdot (q \cdot (r \cdot r))}{(q \cdot (r \cdot r)) \cdot t} && \text{by (F1)}\\
&= \frac{(p \cdot t + s \cdot (q \cdot r)) \cdot r}{(q \cdot t) \cdot (r \cdot r)} =_{\mathrm{cc}} \frac{p \cdot t + s \cdot (q \cdot r)}{(q \cdot t) \cdot r}\\
&= \frac{p \cdot t + s \cdot (q \cdot r)}{(q \cdot r) \cdot t}\\
&= \frac{p}{q \cdot r} + \frac{s}{t}, && \text{by (F1)}
\end{aligned}$$

  and $(s/t) + A =_{\mathrm{cc}} (s/t) + B$ follows in a similar way,
- $A \cdot (s/t) =_{\mathrm{cc}} B \cdot (s/t)$ follows immediately from (F2), and so does

$$(s/t) \cdot A =_{\mathrm{cc}} (s/t) \cdot B,$$

- $-A =_{\mathrm{cc}} -B$: trivial (by (F3)),
- $A^{-1} =_{\mathrm{cc}} B^{-1}$ because

$$\begin{aligned}
\left(\frac{p \cdot r}{q \cdot (r \cdot r)}\right)^{-1} &= \frac{(q \cdot (r \cdot r)) \cdot (q \cdot (r \cdot r))}{(p \cdot r) \cdot (q \cdot (r \cdot r))} && \text{by (F4)}\\
&= \frac{((q \cdot (r \cdot r)) \cdot (q \cdot r)) \cdot r}{((p \cdot r) \cdot q) \cdot (r \cdot r)} =_{\mathrm{cc}} \frac{(q \cdot (r \cdot r)) \cdot (q \cdot r)}{((p \cdot r) \cdot q) \cdot r}\\
&= \frac{((q \cdot (r \cdot r)) \cdot q) \cdot r}{(p \cdot q) \cdot (r \cdot r)} =_{\mathrm{cc}} \frac{(q \cdot (r \cdot r)) \cdot q}{(p \cdot q) \cdot r}\\
&= \frac{(q \cdot r) \cdot (q \cdot r)}{p \cdot (q \cdot r)}\\
&= \left(\frac{p}{q \cdot r}\right)^{-1}. && \text{by (F4)} \quad \square
\end{aligned}$$

Eqs. (F5), (F6), and (F7) in Table 3 define the constants 0, 1, and **a** from the common meadow signature Σ_{cm} as fracpairs. So, in the setting with fracpairs, these constants can be seen as abbreviations for $0/1$, $1/1$, and $1/0$, respectively.

Definition 2.2.2. Let R be a reduced commutative ring.

1. A **common cancellation fraction over** R, **cc-fraction** for short, is a fracpair over R modulo cc-equivalence. For a fracpair p/q over R, $[p/q]_{\mathrm{cc}}$ is the cc-fraction represented by p/q.
2. The **initial algebra of cc-fractions over** R equipped with the constants and operations of Table 3, notation

$$\mathbb{F}_{cm}(R)$$

   is defined by dividing out cc-congruence on $F(R)$. Thus, for fracpairs p/q and r/s over R,

$$\mathbb{F}_{cm}(R) \models \frac{p}{q} = \frac{r}{s} \iff (\mathrm{CC} + \text{Table 3}) \vdash \frac{p}{q} =_{\mathrm{cc}} \frac{r}{s} \iff [p/q]_{\mathrm{cc}} = [r/s]_{\mathrm{cc}}.$$

We will write p/q for a cc-fraction $[p/q]_{\mathrm{cc}}$ if it is clear from the context that a cc-fraction is meant.

2.3. Common cancellation fractions constitute a common meadow

With the aim of regarding the multiplicative inverse as a total operation, *meadows* were introduced by Bergstra and Tucker in [7] as alternatives for fields with a purely equational axiomatization.[4] Meadows are commutative von Neumann regular rings (vNrr's) equipped with a weak multiplicative inverse x^{-1} (thus $0^{-1} = 0$) that is an involution (thus $(x^{-1})^{-1} = x$). In particular, the class of meadows is a variety, so each substructure of a meadow is a meadow, which is not the case for commutative vNrr's (cf. [2]). In this paper we will mainly consider a variation of the concept of a meadow, and therefore meadows will be further referred to as *involutive meadows*.

A *common meadow* [5] is a structure with addition, multiplication, and a multiplicative inverse, and differs from an involutive meadow in that the inverse of zero is not zero, but equal to an additional constant $\mathbf{a}$ that propagates through all operations. Considering $\mathbf{a}$ as an error-value supports the intuition. Common meadows are formally defined as structures over the signature $\Sigma_{cm} = \{0, 1, \mathbf{a}, -(\_), (\_)^{-1}, +, \cdot\}$ that satisfy the axioms in Table 4, and we write $\mathrm{Md}_{\mathbf{a}}$ for this set of axioms. We further assume that the inverse operation $(\_)^{-1}$ binds stronger than $\cdot$ and omit brackets whenever possible, e.g., $x \cdot (y^{-1})$ is written as $x \cdot y^{-1}$.

The use of the constant $\mathbf{a}$ is a matter of convenience only, it constitutes a derived constant with defining equation $\mathbf{a} = 0^{-1}$, so all uses of $\mathbf{a}$ can be avoided.

Before relating cc-fractions to common meadows, we provide some more introduction to the latter. The axioms of $\mathrm{Md}_{\mathbf{a}}$ that feature a (sub)term of the form $0 \cdot t$ cover the case that t equals $\mathbf{a}$, for example, $\mathbf{a} + (-\mathbf{a}) = 0 \cdot \mathbf{a} = \mathbf{a}$. Some typical $\mathrm{Md}_{\mathbf{a}}$-consequences are these:

$$x = x + 0 \cdot x, \qquad 0 \cdot 0 = 0, \qquad -0 = 0, \quad \text{and} \quad -(x \cdot y) = x \cdot (-y)$$

(we prove the last identity in Appendix B). Another $\mathrm{Md}_{\mathbf{a}}$-consequence can be called the *weak additive inverse property*:

$$x + (-x) + x = x$$

(which follows with the axiom $x + (-x) = 0 \cdot x$), and thus by the axiom $-(-x) = x$ also $(-x) + x + (-x) = -x$. We show that given x, any y satisfying $x + y + x = x$ and $y + x + y = y$ is unique (implicitly using commutativity and associativity):

$$\begin{aligned}
y &= y + x + y \\
&= y + x + (-x) + x + y \\
&= y + x + (-x) && \text{by } y + x + y = y \\
&= y + x + (-x) + x + (-x) \\
&= (-x) + x + (-x) && \text{by } x + y + x = x \\
&= -x.
\end{aligned}$$

Furthermore, by $-(x \cdot y) = x \cdot (-y)$ we find $-\mathbf{a} = \mathbf{a}$, and with the axiom $(x^{-1})^{-1} = x + 0 \cdot x^{-1}$ we find $\mathbf{a}^{-1} = \mathbf{a}$. In summary, a common meadow is a commutative monoid both for addition and multiplication, extended with a weak additive inverse, a multiplicative inverse except for zero, and the additional element $\mathbf{a}$ that is the image of the multiplicative inverse on zero and

4 An overview of meadows as a new theme in the theory of rings and fields is available at https://meadowsite.wordpress.com/.

Table 4
$\mathrm{Md}_{\mathbf{a}}$, a set of axioms for common meadows.

$(x+y)+z = x+(y+z)$	$-(-x) = x$
$x+y = y+x$	$0\cdot(x\cdot x) = 0\cdot x$
$x+0 = x$	$(x^{-1})^{-1} = x+(0\cdot(x^{-1}))$
$x+(-x) = 0\cdot x$	$x\cdot(x^{-1}) = 1+(0\cdot(x^{-1}))$
$(x\cdot y)\cdot z = x\cdot(y\cdot z)$	$(x\cdot y)^{-1} = (x^{-1})\cdot(y^{-1})$
$x\cdot y = y\cdot x$	$1^{-1} = 1$
$1\cdot x = x$	$0^{-1} = \mathbf{a}$
$x\cdot(y+z) = (x\cdot y)+(x\cdot z)$	$x+\mathbf{a} = \mathbf{a}$
	$x\cdot\mathbf{a} = \mathbf{a}$

propagates through all operations. Let $\mathbb{M}_1$ and $\mathbb{M}_2$ be common meadows, then

$$f : \mathbb{M}_1 \to \mathbb{M}_2$$

is a homomorphism if f preserves $1, 0$, and $\mathbf{a}$, and the operations have the morphism property (that is, $f(x+y) = f(x)+f(y)$, $f(x\cdot y) = f(x)\cdot f(y)$, $f(-x) = -f(x)$, and $f(x^{-1}) = (f(x))^{-1}$). In the case that $\mathbb{M}_1$ is a minimal algebra (that is, each of its elements is represented by a closed term over Σ_{cm}), f is unique.

Proposition 2.3.1. *Let $\mathbb{M}_1$ and $\mathbb{M}_2$ be common meadows and $f : \mathbb{M}_1 \to \mathbb{M}_2$ be a function that satisfies*

$$\begin{aligned} f(x+y) &= f(x)+f(y)\\ f(x\cdot y) &= f(x)\cdot f(y)\\ f(x^{-1}) &= (f(x))^{-1}\\ f(1) &= 1, \end{aligned}$$

then f is a homomorphism.

Proof. Write 1_i for the unit in $\mathbb{M}_i$ and 0_i for its zero. We first show $f(0_1) = 0_2$: observe that $0_i = 1_i + (-1_i)$, and in $\mathbb{M}_2$, $1_2 = f(1_1) = f(1_1+0_1) = f(1_1)+f(0_1) = 1_2 + f(0_1)$. Hence $0_2 = 1_2+(-1_2) = 1_2+(-1_2)+f(0_1) = 0_2+f(0_1) = f(0_1)$. It follows that $f(\mathbf{a}) = f(0_1^{-1}) = 0_2^{-1} = \mathbf{a}$. Finally, we have to prove that $f(-x) = -f(x)$: observe $f(x)+(-f(x)) = 0_2\cdot f(x) = f(0_1)\cdot f(x) = f(0_1\cdot x) = f(x+(-x)) = f(x)+f(-x)$, and hence

$$\begin{aligned} f(-x) &= f((-x)+x+(-x)) && \text{by the weak additive inverse property}\\ &= f(-x)+f(x)+f(-x)\\ &= (-f(x))+f(x)+(-f(x))\\ &= -f(x) && \text{by the weak additive inverse property.} \quad \square \end{aligned}$$

Given a common meadow $\mathbb{M}$, we finally notice that after forgetting $(\_)^{-1}$, the substructure $\{x \in \mathbb{M} \mid 0\cdot x = 0\}$ is a commutative ring.

In the previous section we already suggested a strong connection between cc-fractions and common meadows if one forgets about the underlying ring R and the fracpairing operation. This yields the following elementary result, which together with the next corollary we see as our first main result.

Theorem 2.3.2. *Let R be a reduced commutative ring, then $\mathbb{F}_{cm}(R)$ is a common meadow.*

Proof. By Proposition 2.2.1, $=_{\mathrm{cc}}$ is a congruence with respect to Σ_{cm}. Therefore, showing that $\mathbb{F}_{cm}(R)$ is a common meadow only requires proof checking of all $\mathrm{Md}_{\mathbf{a}}$-axioms (see Table 4). We consider four cases, all other cases being equally straightforward:

$$\begin{aligned}
\frac{p}{q}\cdot\left(\frac{p}{q}\right)^{-1} &= \frac{p}{q}\cdot\frac{q\cdot q}{p\cdot q} = \frac{p\cdot(q\cdot q)}{q\cdot(p\cdot q)} = \frac{(p\cdot q)\cdot q}{p\cdot(q\cdot q)}\\
&= \frac{p\cdot q}{p\cdot q} = \frac{1\cdot(p\cdot q)+0\cdot 1}{1\cdot(p\cdot q)} = \frac{1}{1}+\frac{0}{p\cdot q}\\
&= 1+\frac{0}{1}\cdot\frac{q\cdot q}{p\cdot q} = 1+0\cdot\frac{q\cdot q}{p\cdot q} = 1+0\cdot\left(\frac{p}{q}\right)^{-1},\\
\left(\left(\frac{p}{q}\right)^{-1}\right)^{-1} &= \left(\frac{q\cdot q}{p\cdot q}\right)^{-1} = \frac{(p\cdot q)\cdot(p\cdot q)}{(p\cdot q)\cdot(q\cdot q)} = \frac{((p\cdot q)\cdot p)\cdot q}{(p\cdot q)\cdot(q\cdot q)}\\
&= \frac{(p\cdot q)\cdot p}{(p\cdot q)\cdot q} = \frac{p\cdot(p\cdot q)}{q\cdot(p\cdot q)} = \frac{p}{q}+\frac{0\cdot q}{p\cdot q}\\
&= \frac{p}{q}+\frac{0}{1}\cdot\frac{q\cdot q}{p\cdot q} = \frac{p}{q}+0\cdot\left(\frac{p}{q}\right)^{-1},
\end{aligned}$$

$$0^{-1} = \left(\frac{0}{1}\right)^{-1} = \frac{1\cdot 1}{0\cdot 1} = \frac{1}{0} = \mathbf{a}, \quad\text{and}\quad \frac{p}{q}+\mathbf{a} = \frac{p}{q}+\frac{1}{0} = \frac{q}{0} \stackrel{2.1.2}{=} \frac{1}{0} = \mathbf{a}. \qquad \square$$

The construction of $\mathbb{F}_{cm}(R)$ is arguably the most straightforward construction of a common meadow.

With $\mathbb{Q}_{\mathbf{a}}$ we denote the common meadow that is defined as the field $\mathbb{Q}$ of rational numbers expanded with an $\mathbf{a}$-totalized inverse (that is, $0^{-1} = \mathbf{a}$).[5] We have the following corollary of Theorem 2.3.2.

Corollary 2.3.3. *The unique homomorphism $f : \mathbb{F}_{cm}(\mathbb{Z}) \to \mathbb{Q}_{\mathbf{a}}$ is surjective, but not injective. Thus, the common meadow $\mathbb{F}_{cm}(\mathbb{Z})$ is a proper homomorphic pre-image of $\mathbb{Q}_{\mathbf{a}}$.*

Proof. Observe that by Corollary 2.1.5, $\mathbb{F}_{cm}(\mathbb{Z})$ is nontrivial ($0 = 0/1$, $1 = 1/1$, and $\mathbf{a} = 0/1$ are pairwise distinct). Define $f : \mathbb{F}_{cm}(\mathbb{Z}) \to \mathbb{Q}_{\mathbf{a}}$ by $f(n/m) = n\cdot m^{-1}$. Then f is well-defined and according to Proposition 2.3.1 a homomorphism:

- $f(x+y) = f(x)+f(y)$ because $\mathbb{Q}_{\mathbf{a}} \models x\cdot y^{-1}+u\cdot v^{-1} = (x\cdot v+u\cdot y)\cdot(y\cdot v)^{-1}$ (see Appendix B or [5, Prop. 2.2.2]),
- $f(x\cdot y) = f(x)\cdot f(y)$ follows trivially,
- for the case $f((x)^{-1})$ first observe that $\mathbb{Q}_{\mathbf{a}} \models 0\cdot x\cdot x^{-1} = 0\cdot(1+0\cdot x^{-1}) = 0\cdot x^{-1}$, hence $f((n/m)^{-1}) = f((m\cdot m)/(n\cdot m)) = m\cdot m\cdot(n\cdot m)^{-1} = m\cdot(1+0\cdot m^{-1})\cdot n^{-1} = (m+0\cdot m^{-1})\cdot n^{-1} = (m^{-1})^{-1}\cdot n^{-1} = (n\cdot m^{-1})^{-1} = (f(n/m))^{-1}$, and
- $f(1/1) = 1\cdot 1^{-1} = 1$.

Each element in $\mathbb{Q}_{\mathbf{a}}$ can be represented by $n\cdot m^{-1}$ with $n, m \in \mathbb{Z}$, hence f is surjective. However, f is not injective: $f(1/1) = f(2/2) = 1$, while $\mathbb{F}_{cm}(\mathbb{Z}) \not\models 1/1 = 2/2$ because otherwise the homomorphism from $\mathbb{F}_{cm}(\mathbb{Z})$ onto $\mathbb{F}_{cm}(\mathbb{Z}/6\mathbb{Z})$ implies $\mathbb{F}_{cm}(\mathbb{Z}/6\mathbb{Z}) \models 1/1 = 2/2$, and the latter is a contradiction by Proposition 2.1.6, as is spelled out in Appendix A.2. $\square$

[5] $\mathbb{Q}_{\mathbf{a}}$ is introduced in [5].

Our second main result is a characterization of the initial common meadow.

Theorem 2.3.4. *The initial common meadow* $\mathbb{I}(\Sigma_{cm}, \mathrm{Md}_{\boldsymbol{a}})$ *is isomorphic to* $\mathbb{F}_{cm}(\mathbb{Z})$.

Proof. We use the following two properties of common meadows. Firstly, for each closed term t over the meadow signature Σ_{cm}, there exist closed terms p and q over the signature $\Sigma_r = \{0, 1, -(\_), +, \cdot\}$ of rings such that $\mathrm{Md}_{\mathbf{a}} \vdash t = p \cdot q^{-1}$ (this follows by induction on the structure of t, applying the identity $\mathrm{Md}_{\mathbf{a}} \vdash x \cdot y^{-1} + u \cdot v^{-1} = (x \cdot v + u \cdot y) \cdot (y \cdot v)^{-1}$). Secondly, $\mathrm{Md}_{\mathbf{a}} \vdash x \cdot (x^{-1} \cdot x^{-1}) = x^{-1}$ (see Appendix B or [5, Prop. 2.2.1]), and hence

$$\mathrm{Md}_{\mathbf{a}} \vdash (x \cdot z) \cdot (y \cdot (z \cdot z))^{-1} = x \cdot (y \cdot z)^{-1},$$

which can be seen as a characterization of CC (see Table 2).

Because $\mathbb{F}_{cm}(\mathbb{Z})$ is a model of $\mathrm{Md}_{\mathbf{a}}$, there exists a homomorphism

$$f : \mathbb{I}(\Sigma_{cm}, \mathrm{Md}_{\mathbf{a}}) \to \mathbb{F}_{cm}(\mathbb{Z}).$$

For p a closed term over Σ_r, we find $f(p) = p/1$ (this follows by structural induction on p), and thus

$$f((p)^{-1}) = \left(\frac{p}{1}\right)^{-1} \overset{\text{(F4)}}{=} \frac{1 \cdot 1}{1 \cdot p} = \frac{1}{p}.$$

Hence, for p, q closed terms over Σ_r, $f(p \cdot q^{-1}) = p/q$.

It follows immediately that f is surjective. Also, f is injective: if for closed terms p, q over Σ_r, $f(p \cdot q^{-1}) = f(r \cdot s^{-1})$, thus

$$\mathbb{F}_{cm}(\mathbb{Z}) \models \frac{p}{q} = \frac{r}{s},$$

then we can find a proof using CC and the CR-axioms. For closed terms over Σ_r, $\mathrm{Md}_{\mathbf{a}}$ implies the CR-identities[6] and each CC-instance in this proof can be mimicked in $\mathbb{I}(\Sigma_{cm}, \mathrm{Md}_{\mathbf{a}})$ with an instance of the equation $(x \cdot z) \cdot (y \cdot (z \cdot z))^{-1} = x \cdot (y \cdot z)^{-1}$. Hence, $\mathrm{Md}_{\mathbf{a}} \vdash p \cdot q^{-1} = r \cdot s^{-1}$, so $\mathbb{I}(\Sigma_{cm}, \mathrm{Md}_{\mathbf{a}}) \models p \cdot q^{-1} = r \cdot s^{-1}$. □

3. Term rewriting for meadows

In Section 3.1 we provide details about canonical terms for involutive meadows, for common meadows, and for cc-fractions. Until now we have not been successful in resolving questions about the existence of specifications for meadows with nice term rewriting properties, and we provide in Section 3.2 a survey of relevant negative results. In Section 3.3 we define "rational fractions" by defining an initial algebra that is isomorphic to $\mathbb{Q}_{\mathbf{a}}$.

3.1. DDRSes and canonical terms

A so-called DDRS (datatype defining rewrite system, see [6]) is an equational specification over some given signature that, interpreted as a rewrite system by orienting the equations from left-to-right, is ground complete and thus defines (unique) normal forms for closed terms. Given some DDRS, its *canonical term algebra* (CTA) is determined as the algebra over that signature with the set of normal forms as its domain, and in the context of CTAs we prefer to speak of

[6] In particular, $p + (-p) = 0$ (or equivalently, $0 \cdot p = 0$); this follows by structural induction on p.

Table 5
A DDRS for $\mathbb{Z}$.

$-0=0$	(r1)	$0+x=x$	(r8)
$-(-x)=x$	(r2)	$(-1)+1=0$	(r9)
$x+(y+z)=(x+y)+z$	(r3)	$(-(x+1))+1=-x$	(r10)
$x+0=x$	(r4)	$(-x)+(-y)=-(x+y)$	(r11)
$1+(-1)=0$	(r5)	$x\cdot 0=0$	(r12)
$(x+1)+(-1)=x$	(r6)	$x\cdot 1=x$	(r13)
$x+(-(y+1))=(x+(-y))+(-1)$	(r7)	$x\cdot(-y)=(-x)\cdot y$	(r14)
		$x\cdot(y+z)=(x\cdot y)+(x\cdot z)$	(r15)

canonical terms rather than normal forms. An abstract datatype (ADT) may be understood as the isomorphism class of its instantiations which are in our case CTAs.

In Table 5 we provide a DDRS for the ADT $\mathbb{Z}$ over the signature $\Sigma_r = \{0, 1, -(\_), +, \cdot\}$ of rings. For a proof that the DDRS of Table 5 is terminating and ground-confluent we refer to [6]. Observe that the symmetric variant of Eq. (r7), that is, $(-x) + (y + 1) = ((-x) + y) + 1$, is an instance of Eq. (r3).

Definition 3.1.1. Positive numerals for $\mathbb{Z}$ are defined inductively: 1 is a positive numeral, and $n + 1$ is a positive numeral if n is one. **Negative numerals for** $\mathbb{Z}$ have the form $-(n)$ with n a positive numeral. A **numeral for** $\mathbb{Z}$ is either a positive or a negative numeral, or 0.

Canonical terms for $\mathbb{Z}$ are the numerals for $\mathbb{Z}$, and we write

$$\widehat{\mathbb{Z}}$$

for the canonical term algebra for integers with these canonical terms.

Thus, $\widehat{\mathbb{Z}}$ constitutes a datatype that implements (realizes) the ADT $\mathbb{Z}$ by the DDRS specified in Table 5. Some other specifications of $\mathbb{Z}$ in the "language of rings" are discussed in [1], but these have negative normal forms

$$-1,\ (-1)+(-1),\ ((-1)+(-1))+(-1), \ldots .$$

Below we define three more types of canonical terms and their associated canonical term algebras. The (involutive) meadow $\mathbb{Q}_0$ is defined as the field $\mathbb{Q}$ of rational numbers with a zero-totalized inverse (so $0^{-1} = 0$ and $(_)^{-1}$is an involution; see, e.g., [7,4,3]).

Definition 3.1.2. Canonical terms for $\mathbb{Q}_0$ are the canonical terms for $\mathbb{Z}$ (see Definition 3.1.1) and closed expressions of the form $n \cdot m^{-1}$ and $(-n) \cdot m^{-1}$ such that

* n is a positive numeral, and
* m is a positive numeral larger than 1, and
* n and m (viewed as natural numbers) are relatively prime.

With $\widehat{\mathbb{Q}_0}$ we denote the canonical term algebra for the abstract datatype $\mathbb{Q}_0$ with these canonical terms.

Thus $\widehat{\mathbb{Q}_0}$ is a datatype that implements the ADT $\mathbb{Q}_0$.

Definition 3.1.3. Canonical terms for $\mathbb{Q}_\mathbf{a}$ are the canonical terms for $\mathbb{Q}_0$ and the additional constant **a**.

With $\widehat{\mathbb{Q}_\mathbf{a}}$ we denote the canonical term algebra for the abstract datatype $\mathbb{Q}_\mathbf{a}$ with these canonical terms.

Thus $\widehat{\mathbb{Q}_{\mathbf{a}}}$ is a datatype that implements the ADT $\mathbb{Q}_{\mathbf{a}}$.

Definition 3.1.4. Canonical terms for $\mathbb{F}_{cm}(\mathbb{Z})$ are all fracpairs n/m with n and m canonical terms for $\mathbb{Z}$ (see Definition 3.1.1) and m not a negative numeral, such that one of the following conditions is met, where we write $m_{\mathbb{Z}}$ for the integer denoted by m:

* $n = 0$, and $m_{\mathbb{Z}}$ is squarefree, or
* $m = 0$ and $n = 1$, or
* $m = 1$, or
* $m \neq 0$ and $n \neq 0$ and $m \neq 1$ and for every prime p, if $m_{\mathbb{Z}}$ is a multiple of $p \cdot p$ then $n_{\mathbb{Z}}$ is not a multiple of p.

$\widehat{\mathbb{F}_{cm}}(\widehat{\mathbb{Z}})$ is the canonical term algebra with these canonical terms.

So, $\widehat{\mathbb{F}_{cm}}(\widehat{\mathbb{Z}})$ constitutes a datatype that implements the ADT $\mathbb{F}_{cm}(\mathbb{Z})$.

3.2. Nonexistence of DDRSes for $\widehat{\mathbb{F}_{cm}}(\widehat{\mathbb{Z}})$, for $\widehat{\mathbb{Q}_0}$, and for $\widehat{\mathbb{Q}_{\mathbf{a}}}$

In this section we prove some negative results concerning the existence of certain DDRSes.

Theorem 3.2.1. *There is no DDRS for $\widehat{\mathbb{F}_{cm}}(\widehat{\mathbb{Z}})$.*

Proof. Suppose E is a finite set of rewrite rules for the signature of $\widehat{\mathbb{F}_{cm}}(\widehat{\mathbb{Z}})$ that constitutes a DDRS. Notice that if m is a positive numeral with $m_{\mathbb{Z}}$ not squarefree, then the fracpair

$$\frac{0}{m}$$

is not a normal form. Assume that m exceeds the length of all left-hand sides of equations in E (for some suitable measure), thus $0/m$ must match with a left-hand side of say equation $e \in E$ that has the form

$$\frac{0}{x\underline{+k}} \quad \text{or} \quad \frac{y}{x\underline{+k}}$$

where we assume the following notational convention, writing $\equiv$ for syntactic equivalence:

$$x\underline{+0} \equiv x, \text{ and for all natural numbers } n,\ x\underline{+(n+1)} \equiv (x\underline{+n}) + 1.$$

Now choose a canonical term ℓ with $\ell_{\mathbb{Z}}$ squarefree and larger than $m_{\mathbb{Z}}$. It follows that e rewrites $0/\ell$ so that this term cannot be a normal form which contradicts the definition of canonical terms (Definition 3.1.4). $\square$

This proof works just as well if a DDRS is allowed to make use of auxiliary operations. Moreover, very similar proofs work for $\widehat{\mathbb{Q}_0}$ and $\widehat{\mathbb{Q}_{\mathbf{a}}}$, as we state in the next theorem.

Theorem 3.2.2. *There is no DDRS for $\widehat{\mathbb{Q}_0}$ and for $\widehat{\mathbb{Q}_{\mathbf{a}}}$.*

Proof. Suppose E is a finite set of rewrite rules for $\widehat{\mathbb{Q}_0}$ that constitutes a DDRS and consider a term $(1+1)/m$ with $m_{\mathbb{Z}}$ a multiple of 2 that exceeds the largest equation in E (for some suitable measure). Because $(1+1)/m$ is not a canonical term it is rewritten by say equation $e \in E$. The left-hand side of e must have the form $t/(x\underline{+k})$ for some t and k so that t matches with $1+1$. From this condition it follows that x is not a variable in t and without loss of generality we may assume that

$$t \in \{1+1, y, 1+y, y+1, y+y\}.$$

Table 6
RF, the rational fracpair axiom.

$\frac{x \cdot (((z \cdot z) + (u \cdot u)) + 1)}{y \cdot (((z \cdot z) + (u \cdot u)) + 1)} = \frac{x}{y}$	(RF)

Now let ℓ be a $\mathbb{Q}_0$ ($\mathbb{Q}_{\mathbf{a}}$) canonical term so that $\ell_{\mathbb{Z}}$ is odd and exceeds $m_{\mathbb{Z}}$. We find that $(1+1)/\ell$ is a canonical term according to the definition thereof but at the same time it is not a normal form because it can be rewritten by means of e. Thus, such E does not exist.

Finally, observe that the above reasoning also applies for the case of $\widehat{\mathbb{Q}_{\mathbf{a}}}$. $\square$

The above proof also demonstrates that auxiliary functions will not help, not even auxiliary sorts will enable the construction of a DDRS for $\widehat{\mathbb{Q}_0}$ or for $\widehat{\mathbb{Q}_{\mathbf{a}}}$. We notice that without the constraint that the normal forms are given in advance (by way of a choice of canonical terms) the matter is different because according to [8], a DDRS can be found with auxiliary functions for each computable datatype.

We return to the question of DDRSes for rational numbers in Section 4, where we express some (negative) conjectures about their existence.

3.3. *An initial algebra of fractions for rational numbers*

In this section we introduce "rational fractions", that is, fractions tailored to an initial specification of the rational numbers in the style of $\mathbb{F}_{cm}(\mathbb{Z})$. We start off with the definition of a certain class of reduced commutative rings.

Given a commutative ring R, consider the following conditional property:

$$\forall x, y, z \in R : x \cdot (y^2 + z^2 + 1) = 0 \;\Rightarrow\; x = 0. \tag{4}$$

We first show that not each commutative ring that satisfies condition (4) is reduced.[7] The commutative ring $\mathbb{Z}[X]/(X^2)$, i.e., the polynomial ring in one indeterminate X modulo the ideal generated by X^2, has as its elements polynomials of the form $nX + m$ with $n, m \in \mathbb{Z}$ (see e.g. [14]). This ring is not reduced ($X \cdot X = 0$ and $X \neq 0$), but satisfies property (4): suppose

$$(nX + m) \cdot ((pX + q)^2 + (rX + s)^2 + 1) = 0,$$

thus

$$(n(q^2 + s^2 + 1) + 2m(pq + rs))X + m(q^2 + s^2 + 1) = 0. \tag{5}$$

Hence, $m(q^2 + s^2 + 1) = 0$, thus $m = 0$, and hence we find for X's coefficient in (5) that also $n(q^2 + s^2 + 1) = 0$, thus $n = 0$, and therefore $nX + m = 0$.

Definition 3.3.1. Let R be a reduced commutative ring that satisfies property (4). The cancellation equivalence generated by the rational fracpair axiom RF defined in Table 6 and the common cancellation axiom CC for fracpairs (defined in Table 2) is called

rf-equivalence, notation $=_{\text{rf}}$.

[7] Of course, not every reduced commutative ring satisfies property (4), for example $\mathbb{Z}/6\mathbb{Z}$ does not.

Let R be a nontrivial reduced commutative ring that satisfies property (4). Concerning consistency (and thus separation) it follows for fracpairs p/q and $r/0$ over R that if q is nonzero, then

$$p/q \neq_{\mathrm{rf}} r/0$$

because (4) ensures that application of RF cannot turn a nonzero denominator into zero or vice versa (cf. Proposition 2.1.3). Furthermore, as in Corollary 2.1.5, it follows that

- for $p, q \in R$, if $p/1 =_{\mathrm{rf}} q/1$, then $R \models p = q$, and
- for $p, q \in n(R)$, if $1/p =_{\mathrm{rf}} 1/q$, then $R \models p = q$.

With respect to the operations on fracpairs in Table 3 (thus, with respect to the signature of common meadows) it follows that $=_{\mathrm{rf}}$ is a congruence on $F(R)$, the set of fracpairs over R (cf. Proposition 2.2.1).

Definition 3.3.2. Let R be a reduced commutative ring that satisfies property (4).

1. A **rational fraction over** R is a fracpair over R modulo rf-equivalence.
2. The **initial algebra of rational fractions over** R equipped with the constants and operations of Table 3, notation

   $$\mathbb{F}^r_{cm}(R)$$

   is defined by dividing out rf-congruence on $F(R)$.

We end this section with an elementary result for the particular case of $\mathbb{F}^r_{cm}(\mathbb{Z})$.

Theorem 3.3.3. *The structure $\mathbb{F}^r_{cm}(\mathbb{Z})$ is a common meadow that is isomorphic to $\mathbb{Q}_{\boldsymbol{a}}$.*

Proof. In [4] the following folk theorem in field theory is recalled (and proven, see Lemma 7): For each prime number p and $u \in \mathbb{Z}_p$, there exist $v, w \in \mathbb{Z}_p$ such that $u = v^2 + w^2$. This implies the following property (see [4, Corollary 1][8]):

$$\text{For each prime number } p \text{ there exist } a, b, m \in \mathbb{N} \text{ such that } m \cdot p = a^2 + b^2 + 1.\text{[9]} \tag{6}$$

Now, given some prime number p, let m, a, b be such that (6) is satisfied. For arbitrary $c, d \in \mathbb{N}$ we derive

$$\begin{aligned}
\frac{c \cdot p}{d \cdot p} &=_{\mathrm{rf}} \frac{c \cdot p \cdot (a^2 + b^2 + 1)}{d \cdot p \cdot (a^2 + b^2 + 1)} && \text{by RF} \\
&= \frac{c \cdot p \cdot m \cdot p}{d \cdot p \cdot m \cdot p} && \text{by (6)} \\
&=_{\mathrm{rf}} \frac{c \cdot m \cdot p}{d \cdot m \cdot p} && \text{by CC} \\
&= \frac{c \cdot (a^2 + b^2 + 1)}{d \cdot (a^2 + b^2 + 1)} && \text{by (6)} \\
&=_{\mathrm{rf}} \frac{c}{d}. && \text{by RF}
\end{aligned}$$

[8] The report version of this paper (arXiv:0907.0540v3) uses a different numbering: Lemma 6 and Corollary 1, respectively.

[9] A proof of (6) is the following: let $a, b \in \mathbb{Z}_p$ be such that $-1 = a^2 + b^2$. Then $a^2 + b^2 + 1$ is a multiple of $p \in \mathbb{N}$.

So, for $n, m \in \mathbb{N}$ it follows that:

* $n/m =_{\text{rf}} p/q$ with p, q relative prime if $n \neq 0 \neq m$,
* $n/m =_{\text{rf}} 0/1$ if $n = 0$ and $m \neq 0$,
* $n/m =_{\text{rf}} 1/0$ if $m = 0$ (cf. Proposition 2.1.2).

Hence, we can represent each rational fraction by a fracpair that matches the definition of canonical terms for $\mathbb{Q}_{\mathbf{a}}$, identifying n/m with $n \cdot m^{-1}$ if $n \neq 0$ and $m \notin \{0, 1\}$, with n if $m = 1$ or [$n = 0$ and $m \neq 0$], and with $\mathbf{a}$ if $m = 0$ (cf. Definition 3.1.2).

The observation that the defining equations for the constants and operations of common meadows given in Table 3 match those for $\mathbb{Q}_{\mathbf{a}}$ finishes the proof. □

4. Conclusions and digression

We lifted the notion of a quotient field construction by dropping the requirement that in a "fraction p/q" (over some integral domain) the q must not be equal to zero and came up with the notion of *fracpairs* defined over a reduced commutative ring R, and *common cancellation fractions* (cc-fractions) that are defined by a simple equivalence on fracpairs over R. Natural definitions of the constants and operations of a meadow on fracpairs yield a common meadow (Theorem 2.3.2), and this is arguably the most straightforward construction of a common meadow. Furthermore, we showed that the common meadow $\mathbb{F}_{cm}(\mathbb{Z})$ of common cancellation fractions over $\mathbb{Z}$ is a proper homomorphic pre-image of $\mathbb{Q}_{\mathbf{a}}$ (Corollary 2.3.3), and is isomorphic to the initial common meadow (Theorem 2.3.4; confer the characterization of the involutive meadow in [9]).

Then, in Section 3, we considered canonical terms and term rewriting for integers and for some meadows that model expanded versions of the rational numbers, and proved the nonexistence of DDRSes (datatype defining rewrite systems) for the associated canonical term algebras of $\mathbb{F}_{cm}(\mathbb{Z})$, $\mathbb{Q}_0$ and $\mathbb{Q}_{\mathbf{a}}$ (Theorem 3.2.1 and Theorem 3.2.2), each of which is based on a DDRS in which the integers are represented as 0, the positive numerals

$$1,\ 1+1,\ (1+1)+1, \ldots,$$

and the negations thereof. Moreover, we defined "rational fracpairs" that constitute an initial algebra that is isomorphic to $\mathbb{Q}_{\mathbf{a}}$ (Theorem 3.3.3).

We have the following four conjectures about the nonexistence of DDRS specifications for rational numbers:

1. The meadow of rationals $\mathbb{Q}_0$ admits an equational initial algebra specification (see [8] and a subsequent simplification in [4]). Now the conjecture is that no finite equational initial algebra specification of $\mathbb{Q}_0$ is both confluent and strongly terminating (interpreting the equations as left-to-right rewrite rules). This is irrespective of the choice of normal forms.
   Another formulation of this conjecture: $\mathbb{Q}_0$ cannot be specified by means of a DDRS.
2. We conjecture that for $\mathbb{Q}_{\mathbf{a}}$ the same situation applies as for $\mathbb{Q}_0$: No DDRS for it can be found irrespective of the normal forms one intends the DDRS to have.
3. The following conjecture (if true) seems to be simpler to prove: $\mathbb{F}_{cm}(\mathbb{Z})$ cannot be specified by means of a DDRS.
4. The above negative conjectures remain if one allows the DDRS to be modulo associativity of $+$ and $\cdot$, commutativity of $+$ and $\cdot$, or both associativity and commutativity $+$ and $\cdot$.

Concerning these matters, we should mention the work [11] of Contejean et al. in which normal forms for rational numbers are specified by a complete term rewriting system modulo commutativity and associativity of $+$ and $\cdot$. The associated datatype Rat comprises two functions

$$\mathtt{rat}, / : \mathbb{Z} \times \mathbb{Z} \to \mathtt{Rat},$$

where the symbol rat denotes any fraction, while / denotes irreducible fractions. Also in this work, division by zero is allowed, "but such alien terms can be avoided by introducing a sort for non-null integers" and is not considered any further. The main purpose of this work is to use the resulting datatype for computing Gröbner bases of polynomial ideals over $\mathbb{Q}$.

We conclude with some comments on the use of the word "fraction", a term that is sometimes used in the semantic sense, as in the field of fractions, and sometimes in the syntactic sense, as a fraction having a numerator and a denominator. For the latter interpretation we introduced the notion of a "fracpair" to be used if numerator and denominator are viewed as values, and in the case that we want to refer to the particular syntax of numerator and denominator, one can introduce the notion of a *fracterm*, that is, an "expression of type fracpair" (thus, not making any identifications that hold in the underlying ring). Rollnik [16] prefers to view fractions as values, over viewing fractions as pairs or viewing fractions as terms. He develops a detailed teaching method for fractions based on that viewpoint. Fracpairs provide an abstraction level in between of both views of fractions.

Finally we comment on a classic requirement on addition of fractions:

$$\frac{x}{y} + \frac{z}{y} = \frac{x+z}{y}. \tag{7}$$

With the axiom CC and the defining equation for + (see Table 3) a proof of this law is immediate:

$$\frac{x}{y} + \frac{z}{y} = \frac{(x \cdot y) + (z \cdot y)}{y \cdot y} = \frac{(x+z) \cdot y}{y \cdot y} =_{\mathrm{cc}} \frac{x+z}{y}.$$

Taking $\mathbb{Z}$ as the underlying reduced commutative ring, this relates to the notion of *quasi-cardinality* that emerged from educational mathematics and is due to Griesel [12] (see also Padberg [15, p. 30]). The aspect of quasi-cardinality for addition of fracpairs, which can also be called the *quasi-cardinality law*, is expressed by Eq. (7). So we find that the quasi-cardinality law, which features as a central fact in many textbooks on elementary arithmetic, follows from the equations for fracpairs, the definition of addition on fracpairs, and the CC-axiom.

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Appendix A. Fracpairs over $\mathbb{Z}/6\mathbb{Z}$ and the structure of $\mathbb{F}_{cm}(\mathbb{Z}/6\mathbb{Z})$

In Appendix A.1 we discuss cc-equivalence of fracpairs over $\mathbb{Z}/6\mathbb{Z}$, and in Appendix A.2 we analyze the structure of $\mathbb{F}_{cm}(\mathbb{Z}/6\mathbb{Z})$.

A.1. Cc-equivalence of fracpairs over $\mathbb{Z}/6\mathbb{Z}$

In this section we investigate which constants can be used to represent all fracpairs over the reduced commutative ring $\mathbb{Z}/6\mathbb{Z}$ modulo cc-equivalence. Recall that in $\mathbb{Z}/6\mathbb{Z}$,

$$-0 = 0, \qquad -1 = 5, \qquad -2 = 4, \quad \text{and} \quad -3 = 3,$$

and that addition and multiplication are defined by

+	1	2	3	4	5
1	2	3	4	5	0
2	3	4	5	0	1
3	4	5	0	1	2
4	5	0	1	2	3
5	0	1	2	3	4

·	1	2	3	4	5
1	1	2	3	4	5
2	2	4	0	2	4
3	3	0	3	0	3
4	4	2	0	4	2
5	5	4	3	2	1

From the constants in $\mathbb{Z}/6\mathbb{Z}$ one obtains 36 fracpairs, from which the following twelve can be used to represent all fracpairs modulo cc-equivalence:

$$\frac{1}{0}, \qquad \frac{0}{1}, \frac{1}{1}, \frac{2}{1}, \frac{3}{1}, \frac{4}{1}, \frac{5}{1}, \qquad \frac{0}{2}, \frac{1}{2}, \frac{2}{2}, \qquad \frac{0}{3}, \frac{1}{3}. \tag{A.1}$$

In Table A.7 we show that the fracpairs listed in (A.1) represent all fracpairs over $\mathbb{Z}/6\mathbb{Z}$ modulo cc-equivalence. For all $p, q \in \mathbb{Z}/6\mathbb{Z}$ and $r \in \mathbb{Z}/6\mathbb{Z} \setminus \{0\}$, Proposition 2.1.3 implies $p/r \neq_{cc} q/0$, and Proposition 2.1.2 implies $p/0 =_{cc} q/0$. Of course, we choose $1/0$ as the representing fracpair for the latter equivalence.

In the following we prove separation modulo cc-equivalence of various fracpairs over $\mathbb{Z}/6\mathbb{Z}$ using Proposition 2.1.6. There are three choices for a saturated subset S of $\mathbb{Z}/6\mathbb{Z}$ that yield a nontrivial localized ring $S^{-1}(\mathbb{Z}/6\mathbb{Z})$. We list the equivalences generated by each of these subsets:

$S = \{1, 5\}$: $(\boldsymbol{k}, \mathbf{1}) \sim (5k, 5)$ for $k \in \mathbb{Z}/6\mathbb{Z}$

$S = \{1, 3, 5\}$: $(0, 1) \sim (2, 1) \sim (4, 1) \sim (\mathbf{0}, \mathbf{3}) \sim (2, 3) \sim (4, 3) \sim (0, 5) \sim (2, 5) \sim (4, 5)$
$(1, 1) \sim (3, 1) \sim (5, 1) \sim (\mathbf{1}, \mathbf{3}) \sim (3, 3) \sim (5, 3) \sim (1, 5) \sim (3, 5) \sim (5, 5)$

$S = \{1, 2, 4, 5\}$: $(\mathbf{0}, \mathbf{1}) \sim (\mathbf{3}, \mathbf{1}) \sim (\mathbf{0}, \mathbf{2}) \sim (3, 2) \sim (0, 4) \sim (3, 4) \sim (0, 5) \sim (3, 5)$
$(\mathbf{1}, \mathbf{1}) \sim (\mathbf{4}, \mathbf{1}) \sim (\mathbf{2}, \mathbf{2}) \sim (5, 2) \sim (1, 4) \sim (4, 4) \sim (2, 5) \sim (5, 5)$
$(\mathbf{2}, \mathbf{1}) \sim (\mathbf{5}, \mathbf{1}) \sim (\mathbf{1}, \mathbf{2}) \sim (4, 2) \sim (2, 4) \sim (5, 4) \sim (1, 5) \sim (4, 5)$.

The proofs of these equivalences are trivial but cumbersome. With respect to fracpairs over $\mathbb{Z}/6\mathbb{Z}$, the following can be concluded:

1. The case $S = \{1, 5\}$ implies that $k/1 \neq_{cc} \ell/1$ if $k \neq \ell$, and that fracpairs of the form $x/5$ need not be considered. So this case yields six fracpairs that are distinct modulo cc-equivalence.
2. The case $S = \{1, 3, 5\}$ introduces six fracpairs of the form $x/3$, and implies that $0/3$ and $1/3$ are distinct, that $0/3$ is distinct from $1/1$, $3/1$ and $5/1$, and $1/3$ from $0/1$, $2/1$ and $4/1$. Furthermore, the identities in Table A.7 imply that $0/3$ and $1/3$ represent modulo cc-equivalence all fracpairs of the form $x/3$.
3. The case $S = \{1, 2, 4, 5\}$ introduces twelve fracpairs of the form $x/2$ or $x/4$, and the identities in Table A.7 imply that $0/2$ and $1/2$ and $2/2$ represent all fracpairs of this form. Furthermore, this case implies that $0/2$ and $1/2$ and $2/2$ are mutually distinct modulo cc-equivalence.

We note that e.g. $1/1$ and $2/2$ cannot be distinguished in this way. Separation of $1/1$ and $2/2$ in $\mathbb{F}_{cm}(\mathbb{Z}/6\mathbb{Z})$ can however be proven easily, as we show in Appendix A.2, and separations of the

Table A.7
Equivalences between fracpairs over $\mathbb{Z}/6\mathbb{Z}$ modulo cc-equivalence, where the right-hand sides occur in (A.1) and where Proposition 2.1.2 is repeatedly used.

$\frac{3}{2} =_{cc} \frac{3\cdot 2}{2\cdot 2} =_{cc} \frac{0\cdot 2}{2\cdot 2\cdot 2} = \frac{0}{2}$ $\frac{4}{2} = \frac{2\cdot 2}{2\cdot 2\cdot 2} =_{cc} \frac{2}{2\cdot 2} =_{cc} \frac{1}{2}$ $\frac{5}{2} =_{cc} \frac{5\cdot 2}{2\cdot 2} =_{cc} \frac{5\cdot 2\cdot 2}{2\cdot 2\cdot 2} = \frac{2}{2}$	$\frac{2}{3} = \frac{2}{3\cdot 3} =_{cc} \frac{2\cdot 3}{3\cdot 3\cdot 3} = \frac{0}{3}$ $\frac{3}{3} = \frac{3}{3\cdot 3} =_{cc} \frac{1}{3}$ $\frac{4}{3} = \frac{4}{3\cdot 3} =_{cc} \frac{4\cdot 3}{3\cdot 3\cdot 3} = \frac{0}{3}$ $\frac{5}{3} = \frac{5}{3\cdot 5\cdot 5} =_{cc} \frac{1}{3\cdot 5} = \frac{1}{3}$
$\frac{0}{4} = \frac{0\cdot 2}{2\cdot 2} =_{cc} \frac{0}{2}$ $\frac{1}{4} = \frac{1}{2\cdot 2} =_{cc} \frac{1\cdot 2}{2\cdot 2\cdot 2} = \frac{2}{2}$ $\frac{2}{4} = \frac{2}{2\cdot 2} =_{cc} \frac{1}{2}$ $\frac{3}{4} = \frac{3}{2\cdot 2} =_{cc} \frac{3\cdot 2}{2\cdot 2\cdot 2} = \frac{0}{2}$ $\frac{4}{4} = \frac{2\cdot 2}{2\cdot 2} =_{cc} \frac{2}{2}$ $\frac{5}{4} = \frac{5}{4\cdot 5\cdot 5} =_{cc} \frac{1}{4\cdot 5} = \frac{1}{2}$	$\frac{0}{5} = \frac{0\cdot 5}{5\cdot 5\cdot 5} =_{cc} \frac{0}{5\cdot 5} = \frac{0}{1}$ $\frac{1}{5} = \frac{5\cdot 5}{5\cdot 5\cdot 5} =_{cc} \frac{5}{5\cdot 5} = \frac{5}{1}$ $\frac{2}{5} = \frac{4\cdot 5}{5\cdot 5\cdot 5} =_{cc} \frac{4}{5\cdot 5} = \frac{4}{1}$ $\frac{3}{5} = \frac{3\cdot 5}{5\cdot 5\cdot 5} =_{cc} \frac{3}{5\cdot 5} = \frac{3}{1}$ $\frac{4}{5} = \frac{2\cdot 5}{5\cdot 5\cdot 5} =_{cc} \frac{2}{5\cdot 5} = \frac{2}{1}$ $\frac{5}{5} = \frac{5}{5\cdot 5\cdot 5} =_{cc} \frac{1}{5\cdot 5} = \frac{1}{1}$
$\frac{0}{0} =_{cc} \frac{2}{0} =_{cc} \frac{3}{0} =_{cc} \frac{4}{0} =_{cc} \frac{5}{0} =_{cc} \frac{1}{0}$	

remaining fracpairs from (A.1) that do not follow from the conclusions above can be proven in a similar fashion.

A.2. *The structure of* $\mathbb{F}_{cm}(\mathbb{Z}/6\mathbb{Z})$

By Theorem 2.3.2, $\mathbb{F}_{cm}(\mathbb{Z}/6\mathbb{Z})$ is a common meadow. Multiplication in $\mathbb{F}_{cm}(\mathbb{Z}/6\mathbb{Z})$ is defined in Table A.8, where we leave out $1/0 = \mathbf{a}$ (recall $\mathrm{Md}_{\mathbf{a}} \vdash x \cdot \mathbf{a} = \mathbf{a}$).

Separation of $1/1$ and $2/2$ in $\mathbb{F}_{cm}(\mathbb{Z}/6\mathbb{Z})$ now follows easily: if $1/1 =_{cc} 2/2$ then $\mathbb{F}_{cm}(\mathbb{Z}/6\mathbb{Z}) \models 1/2 = 2/2 \cdot 2/1 = 1/1 \cdot 2/1$ and $\mathbb{F}_{cm}(\mathbb{Z}/6\mathbb{Z}) \models 1/2 = 2/2 \cdot 5/1 = 1/1 \cdot 5/1$, hence $2/1 =_{cc} 5/1$, which contradicts their separation mentioned in Appendix A.1.

Addition in $\mathbb{F}_{cm}(\mathbb{Z}/6\mathbb{Z})$ is defined in the following table, where we leave out $0/1$ (the zero for $+$) and $1/0 = \mathbf{a}$ (recall $\mathrm{Md}_{\mathbf{a}} \vdash x + \mathbf{a} = \mathbf{a}$):

$x+y$	$\frac{1}{1}$	$\frac{2}{1}$	$\frac{3}{1}$	$\frac{4}{1}$	$\frac{5}{1}$	$\frac{0}{2}$	$\frac{1}{2}$	$\frac{2}{2}$	$\frac{0}{3}$	$\frac{1}{3}$
$\frac{1}{1}$	$\frac{2}{1}$	$\frac{3}{1}$	$\frac{4}{1}$	$\frac{5}{1}$	$\frac{0}{1}$	$\frac{2}{2}$	$\frac{0}{2}$	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{0}{3}$
$\frac{2}{1}$	$\frac{3}{1}$	$\frac{4}{1}$	$\frac{5}{1}$	$\frac{0}{1}$	$\frac{1}{1}$	$\frac{1}{2}$	$\frac{2}{2}$	$\frac{0}{2}$	$\frac{0}{3}$	$\frac{1}{3}$
$\frac{3}{1}$	$\frac{4}{1}$	$\frac{5}{1}$	$\frac{0}{1}$	$\frac{1}{1}$	$\frac{2}{1}$	$\frac{0}{2}$	$\frac{1}{2}$	$\frac{2}{2}$	$\frac{1}{3}$	$\frac{0}{3}$
$\frac{4}{1}$	$\frac{5}{1}$	$\frac{0}{1}$	$\frac{1}{1}$	$\frac{2}{1}$	$\frac{3}{1}$	$\frac{2}{2}$	$\frac{0}{2}$	$\frac{1}{2}$	$\frac{0}{3}$	$\frac{1}{3}$
$\frac{5}{1}$	$\frac{0}{1}$	$\frac{1}{1}$	$\frac{2}{1}$	$\frac{3}{1}$	$\frac{4}{1}$	$\frac{1}{2}$	$\frac{2}{2}$	$\frac{0}{2}$	$\frac{1}{3}$	$\frac{0}{3}$
$\frac{0}{2}$	$\frac{2}{2}$	$\frac{1}{2}$	$\frac{0}{2}$	$\frac{2}{2}$	$\frac{1}{2}$	$\frac{0}{2}$	$\frac{1}{2}$	$\frac{2}{2}$	**a**	**a**
$\frac{1}{2}$	$\frac{0}{2}$	$\frac{2}{2}$	$\frac{1}{2}$	$\frac{0}{2}$	$\frac{2}{2}$	$\frac{1}{2}$	$\frac{2}{2}$	$\frac{0}{2}$	**a**	**a**
$\frac{2}{2}$	$\frac{1}{2}$	$\frac{0}{2}$	$\frac{2}{2}$	$\frac{1}{2}$	$\frac{0}{2}$	$\frac{2}{2}$	$\frac{0}{2}$	$\frac{1}{2}$	**a**	**a**
$\frac{0}{3}$	$\frac{1}{3}$	$\frac{0}{3}$	$\frac{1}{3}$	$\frac{0}{3}$	$\frac{1}{3}$	**a**	**a**	**a**	$\frac{0}{3}$	$\frac{1}{3}$
$\frac{1}{3}$	$\frac{0}{3}$	$\frac{1}{3}$	$\frac{0}{3}$	$\frac{1}{3}$	$\frac{0}{3}$	**a**	**a**	**a**	$\frac{1}{3}$	$\frac{0}{3}$

Separation of $0/1$ and $0/2$ in $\mathbb{F}_{cm}(\mathbb{Z}/6\mathbb{Z})$ can be shown using the table above: if $0/1 =_{\text{cc}} 0/2$ then $\mathbb{F}_{cm}(\mathbb{Z}/6\mathbb{Z}) \models 1/1 = 0/1 + 1/1 = 0/2 + 1/1 = 2/2$, which contradicts their separation shown above.

Finally, we provide a table for both additive and multiplicative inverse:

x	**a**	$\frac{0}{1}$	$\frac{1}{1}$	$\frac{2}{1}$	$\frac{3}{1}$	$\frac{4}{1}$	$\frac{5}{1}$	$\frac{0}{2}$	$\frac{1}{2}$	$\frac{2}{2}$	$\frac{0}{3}$	$\frac{1}{3}$
$-x$	**a**	$\frac{0}{1}$	$\frac{5}{1}$	$\frac{4}{1}$	$\frac{3}{1}$	$\frac{2}{1}$	$\frac{1}{1}$	$\frac{0}{2}$	$\frac{2}{2}$	$\frac{1}{2}$	$\frac{0}{3}$	$\frac{1}{3}$
x^{-1}	**a**	**a**	$\frac{1}{1}$	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{2}{2}$	$\frac{5}{1}$	**a**	$\frac{1}{2}$	$\frac{2}{2}$	**a**	$\frac{1}{3}$

Note that in the particular case of $\mathbb{F}_{cm}(\mathbb{Z}/6\mathbb{Z})$, the equation $(x^{-1})^{-1} = x^{-1}$ is valid.

Appendix B. Some identities for common meadows

1. $\text{Md}_{\mathbf{a}} \vdash -(x \cdot y) = x \cdot (-y)$.

   Proof. First, $\text{Md}_{\mathbf{a}} \vdash 0 \cdot x = 0 \cdot (-x)$ because $0 \cdot x = x + -x = -x + -(-x) = 0 \cdot (-x)$. Hence

$$\begin{aligned}-(x \cdot y) &= -(x \cdot y) + 0 \cdot -(x \cdot y) \\ &= -(x \cdot y) + 0 \cdot x \cdot y\end{aligned}$$

Table A.8
Multiplication of fracpairs in $\mathbb{F}_{cm}(\mathbb{Z}/6\mathbb{Z})$.

$x \cdot y$	$\frac{0}{1}$	$\frac{1}{1}$	$\frac{2}{1}$	$\frac{3}{1}$	$\frac{4}{1}$	$\frac{5}{1}$	$\frac{0}{2}$	$\frac{1}{2}$	$\frac{2}{2}$	$\frac{0}{3}$	$\frac{1}{3}$
$\frac{0}{1}$	$\frac{0}{1}$	$\frac{0}{1}$	$\frac{0}{1}$	$\frac{0}{1}$	$\frac{0}{1}$	$\frac{0}{1}$	$\frac{0}{2}$	$\frac{0}{2}$	$\frac{0}{2}$	$\frac{0}{3}$	$\frac{0}{3}$
$\frac{1}{1}$	$\frac{0}{1}$	$\frac{1}{1}$	$\frac{2}{1}$	$\frac{3}{1}$	$\frac{4}{1}$	$\frac{5}{1}$	$\frac{0}{2}$	$\frac{1}{2}$	$\frac{2}{2}$	$\frac{0}{3}$	$\frac{1}{3}$
$\frac{2}{1}$	$\frac{0}{1}$	$\frac{2}{1}$	$\frac{4}{1}$	$\frac{0}{1}$	$\frac{2}{1}$	$\frac{4}{1}$	$\frac{0}{2}$	$\frac{2}{2}$	$\frac{1}{2}$	$\frac{0}{3}$	$\frac{0}{3}$
$\frac{3}{1}$	$\frac{0}{1}$	$\frac{3}{1}$	$\frac{0}{1}$	$\frac{3}{1}$	$\frac{0}{1}$	$\frac{3}{1}$	$\frac{0}{2}$	$\frac{0}{2}$	$\frac{0}{2}$	$\frac{0}{3}$	$\frac{1}{3}$
$\frac{4}{1}$	$\frac{0}{1}$	$\frac{4}{1}$	$\frac{2}{1}$	$\frac{0}{1}$	$\frac{4}{1}$	$\frac{2}{1}$	$\frac{0}{2}$	$\frac{1}{2}$	$\frac{2}{2}$	$\frac{0}{3}$	$\frac{0}{3}$
$\frac{5}{1}$	$\frac{0}{1}$	$\frac{5}{1}$	$\frac{4}{1}$	$\frac{3}{1}$	$\frac{2}{1}$	$\frac{1}{1}$	$\frac{0}{2}$	$\frac{2}{2}$	$\frac{1}{2}$	$\frac{0}{3}$	$\frac{1}{3}$
$\frac{0}{2}$	$\frac{0}{2}$	$\frac{0}{2}$	$\frac{0}{2}$	$\frac{0}{2}$	$\frac{0}{2}$	$\frac{0}{2}$	$\frac{0}{2}$	$\frac{0}{2}$	$\frac{0}{2}$	$\mathbf{a}$	$\mathbf{a}$
$\frac{1}{2}$	$\frac{0}{2}$	$\frac{1}{2}$	$\frac{2}{2}$	$\frac{0}{2}$	$\frac{1}{2}$	$\frac{2}{2}$	$\frac{0}{2}$	$\frac{2}{2}$	$\frac{1}{2}$	$\mathbf{a}$	$\mathbf{a}$
$\frac{2}{2}$	$\frac{0}{2}$	$\frac{2}{2}$	$\frac{1}{2}$	$\frac{0}{2}$	$\frac{2}{2}$	$\frac{1}{2}$	$\frac{0}{2}$	$\frac{1}{2}$	$\frac{2}{2}$	$\mathbf{a}$	$\mathbf{a}$
$\frac{0}{3}$	$\frac{0}{3}$	$\frac{0}{3}$	$\frac{0}{3}$	$\frac{0}{3}$	$\frac{0}{3}$	$\frac{0}{3}$	$\mathbf{a}$	$\mathbf{a}$	$\mathbf{a}$	$\frac{0}{3}$	$\frac{0}{3}$
$\frac{1}{3}$	$\frac{0}{3}$	$\frac{1}{3}$	$\frac{0}{3}$	$\frac{1}{3}$	$\frac{0}{3}$	$\frac{1}{3}$	$\mathbf{a}$	$\mathbf{a}$	$\mathbf{a}$	$\frac{0}{3}$	$\frac{1}{3}$

$$\begin{aligned}
&= -(x \cdot y) + x \cdot (y + (-y)) \\
&= -(x \cdot y) + x \cdot y + x \cdot (-y) \\
&= 0 \cdot (x \cdot y) + x \cdot (-y) \\
&= (x + 0 \cdot x) \cdot (-y) \\
&= x \cdot (-y). \quad \square
\end{aligned}$$

2. $\mathrm{Md}_{\mathbf{a}} \vdash x \cdot y^{-1} + u \cdot v^{-1} = (x \cdot v + u \cdot y) \cdot (y \cdot v)^{-1}$.

Proof. First, $\mathrm{Md}_{\mathbf{a}} \vdash 0 \cdot (x \cdot y) = 0 \cdot (x + y)$ because

$$\begin{aligned}
0 \cdot (x + y) &= 0 \cdot (x + y) \cdot (x + y) \\
&= 0 \cdot x + 0 \cdot x \cdot y + 0 \cdot y \cdot x + 0 \cdot y \\
&= (0 + 0 \cdot y) \cdot x + (0 + 0 \cdot x) \cdot y \\
&= 0 \cdot x \cdot y + 0 \cdot x \cdot y \\
&= 0 \cdot (x \cdot y),
\end{aligned}$$

and thus

$$\begin{aligned} x \cdot y \cdot y^{-1} &= x \cdot (1 + 0 \cdot y^{-1}) \\ &= x + 0 \cdot x \cdot y^{-1} \\ &= x + 0 \cdot x + 0 \cdot y^{-1} \\ &= x + 0 \cdot y^{-1}. \end{aligned} \tag{B.1}$$

Hence,

$$\begin{aligned} (x \cdot v + u \cdot y) \cdot (y \cdot v)^{-1} &= x \cdot y^{-1} \cdot v \cdot v^{-1} + u \cdot v^{-1} \cdot y \cdot y^{-1} \\ &= (x \cdot y^{-1} + 0 \cdot v^{-1}) + (u \cdot v^{-1} + 0 \cdot y^{-1}) \qquad \text{by (B.1)} \\ &= (x \cdot y^{-1} + 0 \cdot y^{-1}) + (u \cdot v^{-1} + 0 \cdot v^{-1}) \\ &= x \cdot y^{-1} + u \cdot v^{-1}. \quad \square \end{aligned}$$

3. $\mathrm{Md}_{\mathbf{a}} \vdash x \cdot (x^{-1} \cdot x^{-1}) = x^{-1}$.

   Proof. $x \cdot x^{-1} \cdot x^{-1} = (1 + 0 \cdot x^{-1}) \cdot x^{-1} = x^{-1} + 0 \cdot x^{-1} = x^{-1}$. $\square$

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