Exam Topology in Physics 2018

This is the **exam** for the course **topology in physics** academic year 2017/2018.
Some remarks beforehand:

- **the problem marked with a $\star$ should not be made by those students opting for the 6 EC version of the course;**
- those students doing the 8 EC version of the course have 3 hours to complete the exam, those students doing the 6 EC version have 2,5 hours to complete the exam;
- the homework grade based on either your best 11 (for 8 EC) or your best 9 (for 6 EC) will count towards 30% of the final grade;
- note that the amount of points that can be obtained is indicated next to each (sub)problem;
- the exam will be graded on a scale from 1-10 so that your grade equals $1 + 9 \times \frac{points}{100}$ for the 8 EC version and $1 + 9 \times \frac{points}{85}$ for the 6 EC version.

Good Luck!

Quickfire Questions 15 ***pts***

We will start the exam off with a lightning round. This means you do **not** need to motivate your answers for these quickfire questions.

5 pt i) Ind $\left(-\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} - \frac{\partial^2}{\partial z^2}\right) = ?$

5 pt ii) Suppose $E \to M$ is a vector bundle and ∇^i for $i = 1, 2, 3$ are connections on E. Which of the following defines a connection (only one answer is correct):

a) $-\nabla^1$,
b) $\nabla^1 + \nabla^2$,
c) $\nabla^1 - \nabla^2$,
d) $\nabla^1 - \nabla^2 + \nabla^3$.

5 pt iii) Which of the following differential operators is elliptic (only one answer is correct):

a) The Dirac operator on $\mathbb{R}^{3,1}$,
b) The Dirac operator on $\mathbb{R}^4$,
c) The operator $\frac{\partial}{\partial x} - \frac{\partial}{\partial y}$ on $\mathbb{R}^2$.

Problem 1: The Aharonov-Bohm Effect 25 *pts*

We parameterize $\mathbb{R}^3$ with three coordinates x, y and z. Instead of x and y, we will also use cylindrical coordinates r and θ in this exercise, where

$$x = r\cos\theta, \qquad y = r\sin\theta. \tag{1}$$

We study an infinitely long cylindrical solenoid along the z-axis, with radius R in the (x, y)-plane. For $r \leq R$, the solenoid contains a one-form gauge field

$$A = \frac{1}{2}r^2 d\theta \qquad \text{for } r \leq R \tag{2}$$

4 *pts* a. Write A in terms of x- and y-coordinates.

3 *pts* b. Show that the field strength two-form equals $F = dx \wedge dy$.

3 *pts* c. Argue that the previous result proves that A is not an exact form.

For $r \geq R$, we assume that the size of the gauge field becomes constant:

$$A = \frac{1}{2}R^2 d\theta \qquad \text{for } r \geq R \tag{3}$$

3 *pts* d. For both the region inside and outside the solenoid, explain whether there are electric and/or magnetic fields present (and if so, which of the two), and whether those fields point in the x-, y- or z-direction. You don't have to worry about signs, so you don't need to mention whether a field points in the positive or negative direction.

Quantum mechanics tells us that when an electron travels around a closed loop γ in space, its wave function picks up a phase

$$\phi = \frac{e}{\hbar}\int_\gamma A \tag{4}$$

Here, e is the electron charge and $\hbar$ is Planck's constant.

4 *pts* e. Use Stokes' theorem to compute the phase shift of an electron that moves around the solenoid once and returns to its original location. Again, you do not need to worry about the sign of the answer.

We now want to remove the solenoid ("shrink it to zero size") while keeping a nonzero gauge field

$$A = \frac{1}{2}C d\theta, \tag{5}$$

everywhere, with C some fixed constant.

4 *pts* f. Explain why we cannot do this in $\mathbb{R}^3$, but *can* do this if we take our space to be $M = \mathbb{R}^3 \setminus \{z\text{-axis}\}$.

In the lectures, we have seen that if the second cohomology group $H^2_{dR}(M)$ vanishes (as is the case here), one has the identity

$$\Omega^2_{cl}(M) \cong \frac{\Omega^1(M)/d\Omega^0(M)}{H^1_{dR}(M)} \tag{6}$$

4 *pts* g. Using this identity, explain why the field strength F, in the setup of exercise (f), can *not* be used to describe all physically inequivalent field configurations.
Note: you do not need to compute anything; an explanation in words (and/or symbols) suffices.

Problem 2: BF Theory 20 *pts*

In this exercise, we study a quantum field theory in 4 Euclidean dimensions known as BF theory. Its action is

$$S_{BF} = \int_M \text{Tr } (B \wedge F + \frac{\Lambda}{12} B \wedge B), \tag{7}$$

Here, M is a 4-dimensional manifold, which for now we assume to have no boundary. F is the field strength of a connection that in this exercise you can assume to be represented by a Lie algebra valued 1-form A, so $F = dA + A \wedge A$. The field B is a Lie algebra valued *2-form* field. Λ is a (real) numerical constant.

4 *pts* a. Find the equation of motion that results from the variation of the field B. Show that plugging the solution to this equation of motion into the action leads to

$$S_{EOM} = -\frac{3}{\Lambda} \int_M \text{Tr } (F \wedge F) \tag{8}$$

The path integral for BF theory can be written as

$$Z = \int DA \int DB \; e^{iS_{BF}} \tag{9}$$

4 *pts* b. Argue that after doing the B-integral, the resulting path integral equals

$$Z = \int DA \; e^{iS_{EOM}}. \tag{10}$$

That is: in this example the path integral over B can be carried out by simply inserting the solution to its equation of motion in the action. *Hint: since a path integral is not well-defined mathematically, we do* ***not*** *expect a rigorous proof here. Therefore, you are allowed to use any argument that would hold for an ordinary integral and assume without further proof that it holds for path integrals as well.*

3 *pts* c. Describe over which space the resulting A-integral should be performed.

4 *pts* d. Show that the expression Tr $(F \wedge F)$ appearing in the action is a total derivative, and explain why this implies that on a manifold M without boundary, BF theory is not a very interesting theory.

5 *pts* e. Using characteristic classes, explain how you could also have arrived at the conclusion of part (d) without a lengthy computation.

Problem 3: The Euler Characteristic 25 *pts*

Recall the definition of the Hodge star operator $\star\colon \Omega^p(M) \to \Omega^{n-p}(M)$ on a Riemannian manifold (M, g) of dimension $n = 2m$. It is given by

$$\star \left(dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_p}\right) = \frac{\sqrt{|g|}}{(n-p)!} \epsilon^{\mu_1 \cdots \mu_p}{}_{\nu_1 \cdots \nu_{n-p}} dx^{\nu_1} \wedge \cdots \wedge dx^{\nu_{n-p}}$$

on the basis p-forms. Recall in particular that $\star^2 \alpha = (-1)^{np+p} \alpha$ for a p-form α. Then set

$$d^* \alpha = (-1)^{np+n+1} \star d \star \alpha$$

and recall that this is the **adjoint** of the exterior derivative d for the usual non-degenerate (positive definite) bilinear pairing induced on $\Omega^\bullet(M)$ by g. In this problem we are considering $\Omega^\bullet(M)$ as sections of the exterior algebra of the complexified cotangent bundle, i.e. $\Omega^\bullet(M) = \Gamma\left(M; \bigwedge^\bullet T^* M_{\mathbb{C}}\right)$. In other words we are considering complex-valued differential forms.

3 *pts* a. Show that $\text{Ker } d + d^* = \text{Ker } d \cap \text{Ker } d^*$ if we view all these operators as acting on $\Omega^\bullet(M)$.
HINT: you do not need the explicit formula for the Hodge star.

It can be shown that the map $f\colon \text{Ker } d + d^* \to \bigoplus_{p=0}^n H^p_{\text{dR}}(M)$ that maps a form $\alpha \in \text{Ker } d + d^*$ to its cohomology class is an **isomorphism**.

Recall the definition of the Euler characteristic

$$\chi(M) := \sum_{i=0}^n (-1)^i \beta_i(M)$$

where $\beta_i(M) = \text{Dim } H^i_{\text{dR}}(M)$ are called the *Betti numbers* of M. From now on we set $X\colon E \to F$ to be the operator $d + d^*$ acting between the vector bundles

$$E = \textstyle\bigwedge^{even} T^* M_{\mathbb{C}} = \displaystyle\bigoplus_{i=0}^m \textstyle\bigwedge^{2i} T^* M_{\mathbb{C}} \quad \text{and} \quad F = \textstyle\bigwedge^{odd} T^* M_{\mathbb{C}} = \displaystyle\bigoplus_{i=0}^{m-1} \textstyle\bigwedge^{2i+1} T^* M_{\mathbb{C}}.$$

3 *pts* b. Show that $\text{Ind } X = \chi(M)$.

Since the operator X is elliptic we may use the result from (b) and the Atiyah–Singer index theorem to express $\chi(M)$ as the integral of a characteristic class. Recall the Atiyah–Singer index theorem for X

$$\text{Ind } X = (-1)^{\frac{n(n+1)}{2}} \int_M \text{Ch}\left(E - F\right) \frac{\text{Td}\left(TM_{\mathbb{C}}\right)}{\text{e}\left(TM\right)}.$$

Recall that the classes in the integrand can be given in terms of their corresponding invariant polynomials of the curvature F of a connection on the vector bundle V:

$$\mathrm{Ch}\,(V) = \mathrm{Tr}\, e^{\frac{iF}{2\pi}} \quad \text{and} \quad \mathrm{Td}\,(V) = \mathrm{Det}\, \frac{iF}{2\pi(1 - e^{-\frac{iF}{2\pi}})}.$$

To determine the Euler class we first recall the splitting principle which essentially says that we may consider $V = L_1 \oplus L_2 \oplus \ldots \oplus L_k$ a sum of line bundles L_i. For the complexified tangent bundle we find in particular $TM_{\mathbb{C}} = L_1 \oplus \overline{L_1} \oplus \ldots \oplus L_m \oplus \overline{L_m}$ with first chern classes $x_i = c_1(L_i) = -c_1(\overline{L_i})$; we have

$$\mathrm{e}\,(TM) = \prod_{i=1}^{m} x_i.$$

For the following subproblem it will also be useful to consider the facts that for vector bundles L and L' we have

- $c_1(L) = \mathrm{Tr}\, \frac{iF}{2\pi}$ for F the curvature of a connection on L. Note in particular that this means that $c_1(L) = \frac{iF}{2\pi}$ if L is a line bundle.
- $c_1(L) = -c_1(L^*)$,
- $\mathrm{Ch}\,(L \otimes L') = \mathrm{Ch}\,(L)\mathrm{Ch}\,(L')$,
- $\mathrm{Ch}\,(L \oplus L') = \mathrm{Ch}\,(L) + \mathrm{Ch}\,(L')$ and
- $\bigwedge^{\bullet} (L \oplus L') = (\bigwedge^{\bullet} L) \otimes (\bigwedge^{\bullet} L')$.

5 *pts* c. Show that

$$\mathrm{Ind}\, X = \int_M \mathrm{e}\,(TM).$$

Consider the torus $\mathbb{T}^2$ given by $S^1 \times S^1$ with coordinates (θ, ϕ) ranging from 0 to 2π. We equip it with the metric g obtained by restricting the Euclidean metric to the embedding $\iota \colon \mathbb{T}^2 \hookrightarrow \mathbb{R}^3$ given by

$$\iota(\theta, \phi) = ((2 + \cos\theta)\cos\phi, (2 + \cos\theta)\sin\phi, \sin\theta).$$

Note that this differs greatly from the metric induced by viewing $\mathbb{T}^2$ as the unit square with opposite sides identified, even though the topology remains unchanged.

Recall that the Christoffel symbols Γ^k_{ij} corresponding to a local frame $\{e_1, e_2\}$ are defined by

$$\nabla_{e_i} e_j = \sum_{k=1}^{2} \Gamma^k_{ij} e_k$$

where ∇ is the Levi-Civita connection on $T\mathbb{T}^2$. In the following you may use that the non-zero Christoffel symbols with respect to the frame $\{\frac{\partial}{\partial\theta}, \frac{\partial}{\partial\phi}\}$ are given by

$$\Gamma^{\phi}_{\theta\phi} = \Gamma^{\phi}_{\phi\theta} = \frac{-\sin\theta}{2+\cos\theta} \quad \text{and} \quad \Gamma^{\theta}_{\phi\phi} = \sin\theta(2+\cos\theta).$$

4 *pts* d. Determine the function $F(\theta,\phi)$ such that

$$\mathrm{e}\,(TM) = F(\theta,\phi)d\theta \wedge d\phi,$$

when we use the curvature of the Levi-Civita connection to find $\mathrm{e}\,(TM)$. *HINT: to obtain the correct normal form of the curvature you will have to change to the orthonormal frame* $\{\frac{\partial}{\partial\theta}, \frac{1}{2+\cos\theta}\frac{\partial}{\partial\phi}\}$.

6 *pts* e. How could you have determined the value of

$$\int_0^{2\pi}\int_0^{2\pi} F(\theta,\phi)d\theta d\phi$$

without the computation at (d)?

4 *pts* f. Use arguments and the previous results rather than complicated computations to determine the Betti numbers of the 2-torus.

⋆ Problem 4: The Clifford algebra and the rotation Lie algebra 15 *pts*

We consider the Clifford algebra Cliff_3 generated by $\psi_i,\ i = 1, 2, 3$ satisfying

$$\psi_i\psi_j = -\psi_j\psi_i, \quad i \neq j,$$
$$\psi_i^2 = -1.$$

4 *pts* a. Show that the elements $J_i \in \mathrm{Cliff}_3(\mathbb{R}),\ i = 1, 2, 3$, defined as

$$J_1 := \frac{1}{2}\psi_2\psi_3, \quad \text{and cyclic permutations,}$$

satisfy the commutation relations of the Lie algebra $\mathfrak{so}(3)$:

$$[J_1, J_2] = J_3,, \quad \text{and cyclic permutations.}$$

5 *pts* b. For any element $c \in \mathrm{Cliff}_3(\mathbb{R})$ define its exponential by the formal power series

$$\exp(c) := \sum_{k=0}^{\infty} \frac{c^k}{k!} \in \mathrm{Cliff}_3(\mathbb{R}).$$

Show that

$$\exp(\theta\psi_i\psi_j) = \cos(\theta) + \sin(\theta)\psi_i\psi_j,$$

for all $i, j = 1, 2, 3,\ i \neq j$ and $\theta \in \mathbb{R}$.

As usual we use $\psi(v)$ to denote $x\psi_1 + y\psi_2 + z\psi_3$ for $(x, y, z) = v \in \mathbb{R}^3$. Recall the definition of the group Spin(3):

$$\mathrm{Spin}\ (3) := \{\psi(v_1)\psi(v_2)\ldots\psi(v_{2k}) \mid \|v_i\| = 1, k \in \mathbb{N}\},$$

where the $\|\cdot\|$ denotes the Euclidean norm.

3 *pts* c. Show that $\exp(\theta\psi_i\psi_j) \in \mathrm{Spin}\ (3)$ for all for all $i, j = 1, 2, 3,\ i \neq j$ and $\theta \in \mathbb{R}$.

3 *pts* d. Denote by G the Lie group given by the exponents of elements of the Lie algebra in (a), i.e. the Lie algebra generated by the J_i. Consider the group homomorphism $\rho\colon G \to SO(3)$ given on the generators by

$$\rho(\exp(\theta J_1)) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(\theta) & -\sin(\theta) \\ 0 & \sin(\theta) & \cos(\theta) \end{pmatrix}, \quad \rho(\exp(\theta J_2)) = \begin{pmatrix} \cos(\theta) & 0 & \sin(\theta) \\ 0 & 1 & 0 \\ -\sin(\theta) & 0 & \cos(\theta) \end{pmatrix}$$

and

$$\rho(\exp(\theta J_3)) = \begin{pmatrix} \cos(\theta) & -\sin(\theta) & 0 \\ \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Argue that this map is $2:1$ (two to one).