EXERCISE: CONCRETE RIEMANN SURFACES

Exercise 1. Let $F : \mathbb{C}^2 \to \mathbb{C}$ be a function of two complex variables. We say that F is holomorphic if the maps $z \mapsto F(z,w)$ for fixed w, and $w \mapsto F(z,w)$ for fixed z are holomorphic. For such a holomorphic function F, we are interested in the zero locus

$$S := \{(z,w) \in \mathbb{C}^2,\ F(z,w) = 0\}.$$

a) Show that if

$$\frac{\partial F}{\partial w}(z_0, w_0) \neq 0$$

at a point $(z_0, w_0) \in S$, the equation $F(z,w) = 0$ determines a holomorphic function $z \mapsto w(z)$ in a neighborhood of z_0. (*Hint:* prove that the usual implicit function theorem determines a holomorphic function in this case.)

b) Show that if also

$$\frac{\partial F}{\partial z}(z_0, w_0) \neq 0,$$

the equation $F(z,w) = 0$ determines a holomorphic map $w \mapsto z(w)$ which is the inverse of the map in a).

c) Show that if

$$\left(\frac{\partial F}{\partial z}, \frac{\partial F}{\partial w}\right)(z,w) \neq 0,$$

for all $(z,w) \in S$, the set S can be given the structure of a (possibly disconnected) Riemann surface.

Exercise 2. Consider the function

$$F(z,w) := w^2 - (z^2 - 1)(z^2 - \alpha^2), \quad \alpha > 1.$$

a) Show that the zero locus $S := \{(z,w) \in \mathbb{C}^2,\ F(z,w) = 0\}$ of F is a Riemann surface.

b) Show that the projection p onto the z-plane is a 2-sheeted cover. Determine the branch points and give the local form around these points.

c) Denote by f the projection onto the w-coordinate. Show that the triple (S, p, f) is the algebraic function defined by the polynomial

$$P(T) = T^2 - (z^2 - 1)(z^2 - \alpha^2)$$

d) Show that dp is a holomorphic differential on S. For what $k \geq 1$ is the differential dp/f^k holomorphic?