EXERCISE: PROJECTIVE CURVES

In these exercises we are going to have a closer look at the Riemann surfaces defined by polynomial equations. Let us first recall the result of the first exercise sheet: consider the subset $C \subset \mathbb{C}^2$ defined as the zero locus of a polynomial $p \in \mathbb{C}[z_1, z_2]$ in two variables

$$C := \{(z_1, z_2) \in \mathbb{C}^2,\ p(z_1, z_2) = 0\}.$$

This is called an *affine curve*. In the first exercise sheet you have proved that when

$$(\star) \qquad C \cap \{\frac{\partial p}{\partial z_1} = 0\} \cap \{\frac{\partial p}{\partial z_2} = 0\} = \emptyset,$$

C carries the structure of a Riemann surface. Remark that C is noncompact.

Exercise 1. Let $P \in \mathbb{C}[z_0, z_1, z_2]$ be a *homogeneous* polynomial in three variables, and consider its zero locus in $\mathbb{P}^2$:

$$\hat{C} := \{(z_0 : z_1 : z_2) \in \mathbb{P}^2,\ P(z_0, z_1, z_2) = 0\}.$$

a) Show that $\hat{C}$ is a compact subset of $\mathbb{P}^2$. (Use that $\mathbb{P}^2$ itself is compact.)

b) A *singular point* of $\hat{C}$ is a point $(a : b : c) \in \hat{C}$ such that

$$\frac{\partial P}{\partial z_0}(a, b, c) = \frac{\partial P}{\partial z_1}(a, b, c) = \frac{\partial P}{\partial z_2}(a, b, c) = 0.$$

Show that if $\hat{C}$ has no singular points, it is a Riemann surface. (Such a Riemann surface is called a smooth projective curve.)

Exercise 2. Consider the affine curve C defined by the polynomial

$$p(z_1, z_2) = z_2^3 - (z_1^3 - z_1).$$

a) By considering the unique homogeneous polynomial $P(z_0, z_1, z_2)$ of degree 3 such that $P(1, z_1, z_2) = p(z_1, z_2)$, show that C has a canonical compactification to a compact Riemann surface $\hat{C} \subset \mathbb{P}^2$.

b) Denote by $\pi : \hat{C} \to \mathbb{P}^1$ the map $\pi(z_0 : z_1 : z_2) = (z_0 : z_1)$ and use Riemann–Hurwitz to compute the genus of $\hat{C}$.

c) Write down a basis for the holomorphic 1-forms on $\hat{C}$.