EXERCISE SHEET 1 NONCOMMUTATIVE GEOMETRY

Exercise 1. Let A be a commutative C^*-algebra with unit. We consider the description of $\mathrm{Spec}(A)$ as the set of maximal ideals $I \subset A$. For any subset $F \subset \mathrm{Spec}(A)$ define

$$\overline{F} := \{J \in \mathrm{Spec}(A), \bigcap_{I \in F} I \subset J\}.$$

Show that this corresponds to taking the closure in the w^*-topology on $\mathrm{Spec}(A)$.

Exercise 2 (Gelfand–Naimark for non-unital algebras)**.** Use the unitization procedure to prove that any (not necesarily unital) C^*-algebra is of the form $C_0(X)$ for some locally compact Hausdorff space X. What is the categorical version of this version of the Gelfand–Naimark theorem? (Pay special attention to the morphisms in the categories.)

Exercise 3. Denote by $\ell^1(\mathbb{Z})$ the Banach space of double infinite sums

$$a := \sum_{n \in \mathbb{Z}} a_n,\ a_n \in \mathbb{C} \quad \text{with} \quad \sum_{n \in \mathbb{Z}} |a_n| < \infty.$$

a) Show that the product and involution defined by

$$(a * b)_n = \sum_{k \in \mathbb{Z}} a_k b_{n-k}, \quad a^* := \sum_{n \in \mathbb{Z}} \bar{a}_{-n},$$

equip $\ell^1(\mathbb{Z})$ with the structure of a Banach algebra with involution. What is the unit? Does the C^*-identity hold true?

b) Show that $\ell^1(\mathbb{Z})$ acts on $\mathcal{H} = \ell^2(\mathbb{Z})$ by bounded operators, i.e., construct a morphism $\rho : \ell^1(\mathbb{Z}) \to B(\mathcal{H})$ of Banach algebras with involution, and define $C^*_r(\mathbb{Z})$ as the smallest C^*-algebra containing $\ell^1(\mathbb{Z})$.

c) Let $E \in \ell^1(\mathbb{Z})$ be the element with $a_n = \delta_{1,n}$. Show that E is unitary, and compute its spectrum $\mathrm{sp}(E)$.

d) Determine the spectrum of the algebra $C^*_r(\mathbb{Z})$.

e) Write down explicitly the Gelfand transform. Looks familiar?

Exercise 4.

a) Show that for a selfadjoint element $a \in A$ (A unital), $\mathrm{Sp}(a) \subset \mathbb{R}$. We say that $a \geq 0$, if $\mathrm{sp}(a) \subset \mathbb{R}_{\geq 0}$. Show that $\sqrt{a}$ exists when $a \geq 0$. Hint for the first part: consider the elements $a \pm iv2 \in A$, $v > 0$. Show that $\|a \pm iv\| \leq \|a\|^2 + v^2$, and therefore

$$\mathrm{sp}(a) \subset \{z \in \mathbb{C}, |z \pm iv|^2 \leq \|a\|^2 + v^2\}.$$

b) Show that the relation

$$a \geq b \Leftrightarrow a - b \geq 0,$$

defines a partial ordering on A.

c) Show that $a^* a \leq \|a\|^2 1$ for all $a \in A$. Use this to prove the inequality

$$b^* a^* ab \leq \|a\|^2 b^* b, \quad \text{for all } a, b \in A.$$

Exercise 5 (The GNS construction)**.** A *state* on a unital C^*-algebra A is a linear functional $\sigma : A \to \mathbb{C}$ satisfying

$$\sigma(1) = 1, \quad \sigma(a^* a) \geq 0, \quad \text{for all } a \in A.$$

a) Show that the sesquilinear form

$$\langle a, b \rangle := \sigma(a^* b), \quad a, b \in A,$$

satisfies all the axioms for an inner product on A, except for positive definiteness. Show that the Cauchy–Schwarz inequality holds true:

$$|\langle a, b \rangle|^2 \leq \langle a, a \rangle \langle b, b \rangle, \quad \text{for all } a, b \in A.$$

b) Show that the set

$$N := \{a \in A, \ \langle a, a \rangle = 0\}$$

forms a linear subspace of A, and that $\langle \ , \ \rangle$ induces an inner product on A/N. Denote by $\mathcal{H}$ its Hilbert space completion.

c) Use 4c) to define a representation of A on $\mathcal{H}$ by bounded operators. This is called the *GNS representation.*

d) For $A = C_r^*(\mathbb{Z})$, show that the map $f \mapsto f(0)$ defines a state on A. What is the associated GNS representation?