EXERCISE SHEET 2 NONCOMMUTATIVE GEOMETRY

Exercise 1 (A Lemma needed to prove half-exactness of K_0)**.** Let A be a unital C^*-algebra and $I \subset A$ an ideal. Let $u \in A/I$ be a unitary element.

a) Use the functional calculus to show that there exists an element $a \in A$ with $\|a\| \leq 1$ with $\pi(a) = u$: pick any lift b and show that $a := bf(b^*b)$ with $f(x) = \min(1, |x|^{-1/2})$ does the job.
b) Show that the matrix

$$v = \begin{pmatrix} a & -(1-a^*a)^{1/2} \\ (1-a^*a)^{1/2} & a^* \end{pmatrix} \text{ is unitary and } \pi(v) = \begin{pmatrix} u & 0 \\ 0 & u^* \end{pmatrix}.$$

Exercise 2 (Half-exactness of K_0)**.** Let

$$0 \longrightarrow I \longrightarrow A \xrightarrow{\pi} A/I \longrightarrow 0$$

be a short exact sequence of C^*-algebras. We want to prove that the induced sequence

$$K_0(I) \longrightarrow K_0(A) \xrightarrow{\pi_*} K_0(A/I)$$

is exact in the middle. We use the standard picture of $K_0(A)$ to represent a general element as $[p] - [p_n]$ with p a projector in $M_\infty(\tilde{A})$, $p_n = \operatorname{diag}(1, \ldots, 1, 0, \ldots)$, and $p \sim p_n \bmod M_\infty(A)$.

a) Suppose that $[p] - [p_n] \in \ker(\pi_*)$. Show that there exists a $k \in \mathbb{N}$ and a unitary $u \in U(M_{n+k}(\widetilde{A/I}))$ such that

$$u \begin{pmatrix} \pi(p) & 0 \\ 0 & p_k \end{pmatrix} u^* = \begin{pmatrix} p_n & 0 \\ 0 & p_k \end{pmatrix}$$

b) Use Exercise 1 to lift the unitary $\operatorname{diag}(u, u^*)$ to v. With this element, define

$$f := v \begin{pmatrix} p & 0 & 0 \\ 0 & p_k & 0 \\ 0 & 0 & 0 \end{pmatrix} v^* \in M_{2(n+k)}(\widetilde{A}).$$

Show that $f \in M_{2(n+k)}(\tilde{I})$
c) Prove half-exactness of K_0.

Exercise 3. Let A be a C^*-algebra. A *trace* on A is a linear map $\tau : A \to \mathbb{C}$ with the property

$$\tau(ab) = \tau(ba), \quad \text{for all } a, b \in A.$$

Remark that τ is not a homomorphism of algebras. However, show that by extending τ to matrix algebras over A in the obvious way, one gets a map $K_0(A) \to \mathbb{C}$ by taking the trace of idempotents. (This is a first example of the map induced by a *cyclic cocycle*.)

Exercise 4 (The noncommutative torus)**.** Let $\theta \in (0,1)$ be a fixed real number, and consider the Hilbert space $\mathcal{H} := L^2(\mathbb{T}^2)$, where $\mathbb{T}^2 = S^1 \times S^1$ equipped with the Haar measure. Define operators u and v by

$$(u\xi)(z_1, z_2) := z_1 \xi(z_1, z_2), \quad (v\xi)(z_1, z_2) := z_2 \xi(e^{2\pi i\theta} z_1, z_2).$$

a) Show that u and v are unitary operators and satisfy

$$vu = e^{2\pi i\theta} uv.$$

b) Define A_θ to be the sub C^*-algebra of $B(\mathcal{H})$ generated by u and v. Denote by $\mathcal{A}_\theta$ the subset of Laurent polynomials of the form

$$\sum_{m,n \in \mathbb{Z}} \alpha_{m,n} u^m v^n,$$

with only finitely many terms nonzero. Show that $\mathcal{A}_\theta$ is a dense $*$-subalgebra of A_θ.

c) Let $\xi_0 \in \mathcal{H}$ be the unit vector given by the constant function 1 on $S^1 \times S^1$, and define $\tau(a) := \langle \xi_0, a\xi_0 \rangle$, $a \in A_\theta$. Show that

$$\tau\left(\sum_{m,n \in \mathbb{Z}} \alpha_{m,n} u^m v^n n\right) = \alpha_{0,0}.$$

Show that τ defines a trace on $\mathcal{A}_\theta$.

We now start looking for projectors in this C^*-algebra. Let $\varphi : S^1 \to S^1$ be the map given by rotating over an angle $2\pi\theta$. Let $f, g \in C(S^1; \mathbb{R})$ and define

$$p = f(u)v^* + g(u) + vf(u) \in A_\theta.$$

d) Show that $p^* = p$ and by approximating f and g with Laurent polynomials show that

$$\tau(p) = \int_{S^1} g(z)dz.$$

e) Show that $vg(u) = (g \circ \varphi)(u)v$.

f) Show that $p^2 = p$ if and only if

$$f(f \circ \varphi) = 0, \quad f(g + g \circ \varphi^{-1}) = f, \quad g = g^2 + f^2 + (f \circ \varphi)^2.$$

g) Choose $0 < \epsilon \ll \theta < \theta + \epsilon \leq 1$ and define

$$g(e^{2\pi i t}) = \begin{cases} \epsilon^{-1} t & 0 \leq t \leq \epsilon \\ 1 & \epsilon \leq t \leq \theta \\ \epsilon^{-1}(\theta + \epsilon - t) & \theta \leq t \leq \theta + \epsilon \\ 0 & \theta + \epsilon \leq t \leq 1. \end{cases}$$

Show that there exists an f such that p is a projector and that $\tau(p) = \theta$.

Remark: This is not the end of the story: Pimsner and Voiculescu proved in 1980 that the trace τ induces an isomorphism $K_0(A_\theta) \cong \mathbb{Z} + \theta\mathbb{Z}$.

Exercise 5. In this exercise, we shall prove the Toeplitz index theorem.

a) First consider the case of the function $f(z) = z$. Use the standard orthonormal basis in the Hardy subspace to prove that the Toeplitz operator T_z is equivalent to the shift operator on $\ell^2(\mathbb{N})$. Compute its index.
b) Do the same for the operator T_{z^n}.
c) It can be proved that the set of Fredholm operators is open inside the bounded operators, and that the index is continuous (and therefore locally constant). Prove the Toeplitz index theorem by connecting a general T_f to T_{z^n} for some n via a path.