EXERCISE SHEET 3 NONCOMMUTATIVE GEOMETRY

Exercise 1. In this exercise we consider the Weyl algebra A_1: This is the associative algebra generated by two elements p and q subject to the relations

$$[q,q] = 0 = [p,p], \quad [p,q] = 1.$$

a) Show that A_1 is isomorphic to the algebra of differential operators with polynomial coefficients acting on the polynomial algebra $\mathbb{C}[x]$.

b) Show that assigning to an element its degree as a differential operator, turns A_1 into a filtered algebra: $A_1 = \bigcup_k F_k(A_1)$ with

$$F_0(A_1) \subset F_1(A_1) \subset \dots$$

and $F_k \cdot F_l \subset F_{k+l}$. Show that the associated graded algebra

$$\mathrm{Gr}(A_1) = \bigoplus_{k \geq 0} F_{k+1}(A_1)/F_k(A_1),$$

is a commutative algebra isomorphic to $\mathbb{C}[q,p]$.

c) Let $\Omega_i := \bigwedge^i \mathbb{C}^2$. Show that the chain

$$0 \longrightarrow \Omega_2 \otimes A_1^e \xrightarrow{\partial_2} \Omega_1 \otimes A_1^e \xrightarrow{\partial_1} A_1^e \otimes \Omega_0 \xrightarrow{m} A_1 \longrightarrow 0,$$

with m the multiplication, and

$$\partial_1(e_i \otimes a \otimes b) = \begin{cases} a \cdot q \otimes b - a \otimes q \cdot b & i = 1 \\ a \cdot p \otimes b - a \otimes p \cdot b & i = 2. \end{cases}$$

$$\partial_2(e_1 \wedge e_2 \otimes a \otimes b) = e_2 \otimes (a \cdot q \otimes b - a \otimes q \cdot b) - e_1 \otimes (a \cdot p \otimes b - a \otimes p \cdot b),$$

forms a chain complex in the category of A_1 bimodules. Show that this complex is filtered, and that the associated graded complex is the Koszul complex of $\mathbb{C}[q,p]$. (This observation can be used to prove that the complex above is a projective resolution in the category of A_1 bimodules. Can you find an argument?)

d) Take the tensor product $\otimes_{A_1^e} A_1$ and write down the resulting complex computing the Hochschild homology. Prove that this complex is isomorphic to the algebraic de Rham complex of $\mathbb{C}[q,p]$.

e) Determine the cyclic and periodic cyclic homology of A_1.

Exercise 2. Let A be a unital algebra and $\mathfrak{g}$ a Lie algebra acting on A by derivations. Suppose that there exists a trace $\tau : A \to \mathbb{K}$ which is invariant in the sense that

$$\tau(X(a)) = 0, \quad \text{for all } a \in A,\ X \in \mathfrak{g}.$$

a) Consider the Lie algebra homology chain complex $C_\bullet^{\mathrm{Lie}}(\mathfrak{g}, \mathbb{K})$ of the trivial $\mathfrak{g}$-module $\mathbb{K}$ equipped with the differential ∂_{Lie}. Show that the map $C_k^{\mathrm{Lie}}(\mathfrak{g}, \mathbb{K}) \to C^k(A)$ defined by

$$\varphi_c(a_0, \ldots, a_k) := \sum_{\sigma \in S_k} (-1)^\sigma \tau(a_0 X_{\sigma(1)}(a_1) \cdots X_{\sigma(k)}(a_k))$$

commutes b with the zero operator and B with ∂_{Lie}. Conclude that there exists a map

$$H_\bullet^{\mathrm{Lie}}(\mathfrak{g}; \mathbb{K}) \to HC^\bullet(A).$$

b) What happens if $\mathfrak{g}$ is abelian?

c) Show, with a) and b), that the map

$$\varphi(f_0, \ldots, f_n) := \int_{\mathbb{R}^n} f_0 df_1 \wedge \ldots \wedge df_n$$

defines a cyclic cocycle on $C_c^\infty(\mathbb{R}^n)$.

Exercise 3. We consider (again) the noncomtative torus $\mathcal{A}_\theta$: this is the algebra of Laurent polynomials of the form

$$\sum_{m,n \in \mathbb{Z}} \alpha_{m,n} U^m V^n$$

with generators U and V satisfying the relation

$$UV = e^{2\pi i \theta} VU.$$

Show that the Lie algebra $\mathbb{R}^2$ acts on $\mathcal{A}_\theta$ by

$$X_1(U) = U, \quad X_1(V) = 0$$
$$X_2(U) = 0, \quad X_2(V) = V.$$

Show that this defines an action by derivations and that the trace τ is invariant. Write down a maximal set of cyclic cocycles in degree 0, 1 and 2. (It can be shown that these cocycles generate the whole periodic cyclic cohomology.)

Exercise 4. The algebra of formal pseudodifferential operators on the circle Ψ_1 is given by elements

$$a(x, \partial_x) := \sum_{n=-\infty}^{N} a_i(x) \partial_x^n,$$

with $a_n(x)$ a Laurent polynomial in x and $\partial_x := d/dx$. The algebra structure is defined by the relation $[\partial_x, x] = 1$.

i) Write out the product between two elements $a, b \in \Psi_1$. Show that there is an inclusion $A_1 \hookrightarrow \Psi_1$.

ii) Ψ_1 is formally obtained from A_1 by inverting ∂_x. Construct $(x + \partial_x)^{-1} \in \Psi_1$.

iii) Show that the functional

$$\tau(a) := \frac{1}{2\pi i} \int_{S^1} a_{-1}(x) dx,$$

defines a trace on Ψ_1. (This trace is called the Adler–Manin trace.)

iv) The formal series

$$\log \partial := -\sum_{n=1}^{\infty} \frac{(1-\partial)^n}{n}$$

is not an element of Ψ_1. However, show that

$$[\log \partial, a] = \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \partial_x^k a(x, \partial) \partial^{-k},$$

so that $a \mapsto [\log \partial, a]$ defines a derivation on Ψ_1.

v) Show that

$$\varphi(a, b) := \tau\left(a[\log \partial, b]\right),$$

defines a cyclic 1-cocycle. Compute the pairing $\langle [(x + \partial_x)], \varphi \rangle$, where $[(x + \partial_x)] \in K_1^{alg}(\Psi_1)$ is the class of the invertible element $(x + \partial_x) \in \Psi_1$.

Exercise 5. We start by recalling the following: for a group G and a representation V, its *group homology* $H_\bullet(G; V)$ is the homology of the chain complex

$$C_0(G; V) \xleftarrow{\partial} C_1(G; V) \xleftarrow{\partial} C_2(G; V) \xleftarrow{\partial} \ldots,$$

where $C_k(G; V) = V \otimes \mathbb{C}[G^{\times k}]$ and $\partial : C_k(G; V) \to C_{k-1}(G; V)$ is given by

$$\begin{aligned}\partial(v, g_1, \ldots, g_k) =& (g_1 v, g_2, \ldots, g_k + \sum_{i=1}^{k-1} (-1)^i (v, g_1, \ldots, g_i g_{i+1}, \ldots, g_k) \\ &+ (-1)^k (v, g_1, \ldots, g_{n-1}).\end{aligned}$$

Now we can start with the exercise: let Γ be a discrete group and consider its group algebra $\mathbb{C}\Gamma$.

i) We can identify the Hochschild chains $C_k(\mathbb{C}\Gamma) \cong \mathbb{C}[G^{\times(k+1)}]$, the space of functions on $\Gamma^{\times(k+1)}$ with finite support. Let $C \subset \Gamma$ be a conjugacy class. Show that the subspaces $B_k(\Gamma, C)$ given by the $\mathbb{C}$-linear span of elements $(\gamma_0, \ldots, \gamma_k)$ satisfying

$$\gamma_0\gamma_1 \cdots \gamma_k \in C,$$

forms a sub complex of the Hochschild complex, so that we have

$$C_\bullet(\mathbb{C}\Gamma) = \bigoplus_C B_\bullet(\Gamma, C).$$

Finally show that each $B_\bullet(\Gamma, C)$ is in fact a mixed sub complex. What is B?

ii) Let $C = e$. Show that there exists an isomorphism $C_k(\Gamma; \mathbb{C}) cong B_k(\Gamma, e)$ leading to an inclusion

$$H_\bullet(\Gamma; \mathbb{C}) \subset HH_\bullet(\mathbb{C}\Gamma).$$

iii) Next, we consider an arbitrary conjugacy class C passing through $\gamma \in C$. Let $Z(\gamma$ be the centralizer of γ. Define an inclusion of the Hochschild complex

$$C_\bullet(Z(\gamma); \mathbb{C}) \hookrightarrow B_\bullet(\Gamma, C).$$

Can you find a chain homotopy to prove that this inclusion is a quasi-isomorphism? (May be tricky!) This shows that

$$HH_\bullet(\mathbb{C}\Gamma) \cong \bigoplus_{C_\gamma} H_\bullet(Z(\gamma); \mathbb{C}).$$

The dual theory of group homology is group *co*homology. It is constructed as the cohomology of the complex

$$\ldots \xrightarrow{\delta} C^k(G; V) \xrightarrow{\delta} C^{k+1}(G; V) \xrightarrow{\delta} \ldots,$$

where $C^k(G; V) := \mathrm{Map}(G^{\times k}, V)$, and

$$\begin{aligned}(\delta f)(g_1, \ldots, g_{k+1}) :=& g_1 f(g_2, \ldots, g_{k+1}) + \sum_{i=1}^{k} (-1)^i f(g_1, \ldots, g_i g_{i+1}, \ldots, g_{k+1}) \\ &+ (-1)^{k+1} f(g_1, \ldots, g_k).\end{aligned}$$

In fact, we can assume that our cochains are normalized in the sense that $f(g_1, \ldots, g_k) = 0$ if $g_i = e,\ i = 1, \ldots k$ or $g_1 \cdots g_k = 1$.

iv) Consider now cyclic cohomology. Find a map
$$C^k(\Gamma;\mathbb{C}) \to C^k(\mathbb{C}\Gamma),$$
inducing a map $H^k(\Gamma;\mathbb{C}) \to HC^k(\mathbb{C}\Gamma)$.

v) Let Γ now be a finite group, and V a representation of Γ. Show that the element
$$e_V(\gamma) = \frac{\dim(V)}{|\Gamma|}\mathrm{Trace}(\gamma)$$
defines an idempotent in $\mathbb{C}\Gamma$, hence a class in $K_0(\mathbb{C}\Gamma)$. Compute the pairing $\langle [e_V], \tau\rangle$.

vi) Can you construct more traces associated with conjugacy classes in Γ? Evaluate the pairing with $[e_V] \in K_0(\mathbb{C}\Gamma)$.

Exercise 6. Consider the spectral triple given by $\mathcal{A} = C^\infty(S^1)$, $\mathcal{H} = L^2(S^1)$ and $D = -i/d\theta$ where we have identified S^1 with $\mathbb{R}/2\pi\mathbb{Z}$ and $\theta \in \mathbb{R}$ is the natural coordinate. Show that the arc length between two points on S^1 is given by Connes' formula
$$d(p,q) = \sup\{|f(p) - f(q)|,\ f \in \mathcal{A},\ \|[D,f]\| \leq 1\}.$$