Exercises with Lecture 10 of Topology in Physics (UvA/Mastermath 2018)

17 April 2018

This is the sheet of exercises corresponding to the material covered in the tenth lecture of the 17rd of April. It is recommended that you make all exercises on the sheet even though only the exercises with a $\star$ are graded and will count towards the final grade. The homework should be handed in before the next lecture, which is on the 24th of April, by (in order of preference):

1 E-mailing the pdf-output of a LaTeX file to n.dekleijn@uva.nl;

2 E-mailing a scanned copy of a hand-written file to n.dekleijn@uva.nl;

3 Depositing a hard-copy of the pdf-output of a LaTeX file in my mailbox (Niek de Kleijn) at Science Park 107, building F, floor 3;

4 Depositing a hand-written file in my mailbox (Niek de Kleijn) at Science Park 107, building F, floor 3;

5 Giving it to one of the teachers in person (at the beginning of the lecture).

You will receive comments on all the exercises you hand in (not just the homework) and we advise you to make use of this option.

Exercises

$\star$ Exercise 1: Adjoint operators

a. Let $V = \mathbb{C}^n$ and $W = \mathbb{C}^m$ be vector spaces with the usual inner product $\langle a, b\rangle = \overline{a}^T b$. An $m \times n$ matrix A with complex entries can be viewed as an operator from V to W. Show that its adjoint operator $A^\dagger$ is given by the Hermitian conjugate matrix.

b. Let M be an m-dimensional (real) manifold – closed, oriented and without boudary – equipped with a Riemannian metric, and consider the exterior derivative

$$d : \Omega^{r-1}(M) \to \Omega^r(M). \tag{1}$$

Recall the inner product on any $\Omega^s(M)$ is defined as $(\omega, \eta) = \int \omega \wedge \star\eta$. Show that the adjoint operator to d is

$$d^\dagger = (-1)^{mr+m+1} \star d \star \,. \tag{2}$$

You can use the fact that the Hodge star squares to $(\star)^2 = (-1)^{s(m-s)}$ when acting on $\Omega^s(M)$.

c. Let $D : X \to Y$ be a Fredholm operator, where X and Y are (possibly infinite-dimensional) vector spaces. Show that

$$\text{coker } D \cong \ker D^\dagger . \tag{3}$$

Hint: Decompose $Y = \text{Im}(D) \oplus Z$, where Z is orthogonal to $\text{Im}(D)$. You may assume that $\text{Im}(D)$ is a closed subspace. Show that Z is isomorphic to both coker D and $\ker D^\dagger$.

d. What is the adjoint of the operator $D^\dagger D$? Assuming D maps X to itself, what is the index of the "generalized Laplacian" $\Delta_D = D^\dagger D + DD^\dagger$?

Exercise 2: Why "elliptical"?

Let D be a second order differential operator acting on (real) functions on $\mathbb{R}^2$.

a. Show that the symbol of D can be written in the form

$$\sigma(D; \xi^1, \xi^2) = \alpha(\xi^1)^2 + 2\beta\xi^1\xi^2 + \gamma(\xi^2)^2 \tag{4}$$

b. When does the equation $\sigma(D; \xi^1, \xi^2) = 1$ describe an ellipse, when a hyperbola, and when a straight line in the (ξ^1, ξ^2)-plane? Hint: write this equation in terms of two vectors and a matrix and diagonalize the matrix.

c. Argue that D is an elliptical operator if and only if $\sigma(D; \xi^1, \xi^2) = 1$ describes an ellipse.

Exercise 3: Characteristic classes of dual bundles

Let L be a complex line bundle over an m-dimensional manifold M, and L^* its dual line bundle. (That is: the fibers of L^* are maps from the fibers of L into $\mathbb{C}$.

a. Show that $L \otimes L^*$ is a trivial line bundle. Hint: there are (at least) two ways to do this – you can construct transition functions on $L \otimes L^*$, or you can try to explicitly show that $L \otimes L^*$ has a nonvanishing section and argue that this implies triviality.

b. Use the previous result to argue that $c_1(L^*) = -c_1(L)$.

c. As in the lectures, denote the "diagonal two-forms" one obtains from diagonalizing the curvature two-form $\mathcal{R}$ on $TM^{\mathbb{C}}$ by $x_i(TM^{\mathbb{C}})$, and similarly define $x_i(T^*M^{\mathbb{C}})$. Using the splitting principle, argue that $x_i(T^*M^{\mathbb{C}}) = -x_i(TM^{\mathbb{C}})$. (A detailed proof is not required, a simple few-line argument suffices.)