Exercises with Lecture 11 of Topology in Physics (UvA/Mastermath 2018)

April 23, 2018

This is the sheet of exercises corresponding to the material covered in the eleventh lecture of the 24th of April. It is recommended that you make all exercises on the sheet even though only the exercises with a $\star$ are graded and will count towards the final grade. The homework should be handed in before the next lecture, by (in order of preference):

1 E-mailing the pdf-output of a LaTeX file to n.dekleijn@uva.nl;

2 E-mailing a scanned copy of a hand-written file to n.dekleijn@uva.nl;

3 Depositing a hard-copy of the pdf-output of a LaTeX file in my mailbox (Niek de Kleijn) at Science Park 107, building F, floor 3;

4 Depositing a hand-written file in my mailbox (Niek de Kleijn) at Science Park 107, building F, floor 3;

5 Giving it to one of the teachers in person (at the beginning of the lecture).

You will receive comments on all the exercises you hand in (not just the homework) and we advise you to make use of this option.

Exercises

Exercise 1: Supersymmetry

Let $\mathcal{H} = \mathcal{H}^+ \oplus \mathcal{H}^-$ be a $\mathbb{Z}_2$-graded Hilbert space.

a) Show that the supertrace vanishes on supercommutators:

$$\mathrm{Tr}_s([A, B]) = 0.$$

b) Let $H = Q^2$ be the Hamiltonian of a supersymmetric quantum mechanical system, and denote by $\mathcal{H}_n$ its n-th eigenspace with eigenvalue λ_n, where $\mathcal{H}_0$ corresponds to the zero modes, i.e., $\lambda_0 = 0$. Show that $\mathcal{H}_n = \mathcal{H}_n^+ \oplus \mathcal{H}_n^-$ and that for $n > 0$, $\mathcal{H}_n^+ \cong \mathcal{H}_n^-$.

c) Use *b)* to show that the *Witten index*

$$\mathrm{Tr}_s(e^{-\beta H}),$$

equals the number $\dim \mathcal{H}_0^+ - \dim \mathcal{H}_0^-$. Also, show the β-independence by taking the derivative w.r.t. β.

$\star$ Exercise 2: The symbol of a differential operator

Let D be a differential operator of degree k on a manifold M acting on functions (i.e., not sections of a vector bundle). In local coordinates $x = (x^1, \ldots, x^n)$ we have

$$D = \sum_{|I| \leq k} c_I(x) \frac{\partial}{\partial x^I}$$

a) Show that the *symbol* of D,

$$\sigma(D)(x, \xi) := \sum_{|I|=k} c_I(x) \xi_I,$$

is coordinate invariant if we interpret ξ_i, $i = 1, \ldots, n$ as coordinates on the cotangent bundle T^*M induced by x^i: $\alpha = \sum_i \xi_i dx^i$ for $\alpha \in T^*M$. In other words: the symbol $\sigma(D)$ is a smooth function on T^*M!

b) Show that the *full symbol*

$$\sigma_f(D)(x, \xi) := \sum_{|I| \leq k} c_I(x) \frac{\partial}{\partial x^I}$$

is coordinate dependent. This is the reason to only consider the top order part, which is sometimes called simply *symbol*, but also often *principal symbol*.

c) Show that the symbol can alternatively be found by the formula

$$\sigma(D)(x, \xi) = \frac{1}{k!} \left(D(f^k) \right) (x)$$

where $f \in C^\infty(M)$ such that $f(x) = 0$ and $d_x f = \xi$. Note that this expression is inherently coordinate independent.

d) Show that for a differential operator $D : \Gamma(E) \to \Gamma(F)$ acting on sections of vector bundles not much changes and the symbol is a section of the vector bundle $\pi^*\mathrm{Hom}(E, F)$ over T^*M.

e) Compute the symbol of a Dirac operator and show that over a *Riemannian* manifold a Dirac operator is always elliptic.

Exercise 3: The heat kernel of the harmonic oscillator

Consider the Hamiltonian of the harmonic oscillator

$$H_x = -\frac{d^2}{dx^2} + a^2 x^2, \quad a > 0,$$

acting on $L^2(\mathbb{R})$. The associated heat operator e^{-tH} can be represented by a smooth function $k_t(x,y),\ t > 0,\ x, y \in \mathbb{R}$ satisfying

- $k_t(x,y)$ is symmetric in x and y,
- $k_t(x,y)$ satisfies the heat equation

$$\frac{\partial k}{\partial t} + H_x k = 0,$$

- initial conditions are given by

$$\lim_{t \downarrow 0} k_t(x,y) = \delta(x-y).$$

Using the most general Gaussian function

$$k_t(x,y) = \exp(\alpha_t x^2 + \beta_t xy + \alpha_t y^2 + \gamma_t),$$

as ansatz, write down a system of ODE's for the coefficient functions α, β, γ. Solve these equations to show that

$$k_t(x,y) = (4\pi t)^{-1/2} \left(\frac{2at}{\sinh 2at}\right)^{1/2} \exp\left(-\frac{1}{4t}\left[\frac{2at}{\tanh 2at}(x^2+y^2) - \frac{2at}{\sinh 2at}(2xy)\right]\right).$$