Exercises with Lecture 12 of Topology in Physics (UvA/Mastermath 2018)

1 May 2018

This is the sheet of exercises corresponding to the material covered in the twelfth lecture of the 1st of May. It is recommended that you make all exercises on the sheet even though only the exercises with a $\star$ are graded and will count towards the final grade. The homework should be handed in before the next lecture, which is on the 8th of May, by (in order of preference):

1 E-mailing the pdf-output of a LaTeX file to n.dekleijn@uva.nl;

2 E-mailing a scanned copy of a hand-written file to n.dekleijn@uva.nl;

3 Depositing a hard-copy of the pdf-output of a LaTeX file in my mailbox (Niek de Kleijn) at Science Park 107, building F, floor 3;

4 Depositing a hand-written file in my mailbox (Niek de Kleijn) at Science Park 107, building F, floor 3;

5 Giving it to one of the teachers in person (at the beginning of the lecture).

You will receive comments on all the exercises you hand in (not just the homework) and we advise you to make use of this option.

Exercises

Exercise 1: More on the heat kernel

Let us consider the heat kernel for an elliptic operator $\Delta : \mathcal{H} \to \mathcal{H}$, where $\mathcal{H}$ is a Hilbert space (say over $\mathbb{R}^n$) of quantum states. We can now express the heat kernel in coordinate basis as

$$h(x, y, t) = \left\langle x \middle| e^{-t\Delta} \middle| y \right\rangle , \tag{1}$$

where $|x\rangle$ and $|y\rangle$ are position eigenstates.

a. Show that this function satisfies the heat equation

$$\left(\frac{\partial}{\partial t} + \Delta_x\right) h(x,y,t) = 0, \tag{2}$$

where Δ_x indicates that we act with Δ on the x-variable.

The reason for the name "heat kernel" is that we can use the above function as a "kernel" to generate *any* solution to the heat equation. It is a standard result in the theory of differential equations that such a solution $u(x,t)$ is completely determined by its initial conditions $u(x,0) \equiv u_B(x)$.

b. Show that the function

$$u(x,y) = \int_{\mathbb{R}^n} h(x,y,t) u_B(y) dy \tag{3}$$

satisfies the heat equation and has the initial conditions $u_B(x)$.

$\star$ Exercise 2: An alternative way to compute the index

We have seen in the lectures that the index of an operator D can be computed using the operator $e^{-t\Delta}$. Here, we investigate an alternative but similar computation. Assume we have an elliptic, Fredholm operator $D : \Gamma(M,E) \to \Gamma(M,F)$ and intruduce its adjoint operator $D^\dagger$ and the Laplacians $\Delta_E = D^\dagger D$ and $\Delta_F = DD^\dagger$. Moreover, we introduce the function

$$I_E(s) = \mathrm{Tr}\,\left(\frac{s}{\Delta_E + s}\right) \tag{4}$$

and similarly define $I_F(s)$.

a. Show that for $s > 0$, $I_E(s) - I_F(s)$ is independent of s.

b. Show that $I_E(s) - I_F(s) = \mathrm{ind}(D)$.

You may assume without proof in this exercise that all traces involved exist and are finite numbers. This does require the condition that $s > 0$, though! (Why?)

Exercise 3: The fermionic harmonic oscillator

Introduce operators c and $c^\dagger$ that have the anticommutation relation

$$\{c, c^\dagger\} = 1. \tag{5}$$

Define $|0\rangle$ by $c|0\rangle = 0$ and $|1\rangle = c^\dagger|0\rangle$. Note that we do *not* impose that $\{c,c\} = \{c^\dagger, c^\dagger\} = 0$ yet.

a. Introduce a state $|2\rangle$ and compute its norm.

b. Argue from (a) that it is natural to impose $\{c, c\} = \{c^\dagger, c^\dagger\} = 0$. What happens if we don't?

As in the lecture, we now introduce the coherent state $|\theta\rangle = |0\rangle + \theta|1\rangle$.

c. Show that the completeness relation can now be written as

$$I = \int d\theta^* d\theta \; |\theta\rangle\langle\theta| e^{-\theta^*\theta} \tag{6}$$

d. Similarly, show that the trace of $e^{-\beta H}$ can be written as

$$\mathrm{Tr}\; e^{-\beta H} = \int d\theta^* d\theta \; e^{-\theta^*\theta} \langle -\theta| e^{-\beta H} |\theta\rangle. \tag{7}$$

That is, we need *anti-periodic* boundary conditions to define the trace as a Grassmann integral.

$\star$ Exercise 4: A supersymmetric Lagrangian

As in the lecture, we study the Lagrangian

$$L = \frac{1}{2}\dot{x}^i \dot{x}_i + \frac{i}{2}\psi^i \dot{\psi}_i \tag{8}$$

a. Show that the transformation $\delta x^i = i\epsilon\psi^i$, $\delta\psi^i = -\epsilon\dot{x}^i$ changes the Lagrangian by a total t-derivative.

b. Using the canonical commutation relations between coordinates and momenta, show that the transformation in (a) is generated by the operator $Q = \psi^i \dot{x}_i$.