Exercises with Lecture 14 of Topology in Physics (UvA/Mastermath 2018)

15 May 2018

This is the sheet of exercises corresponding to the material covered in the fourteenth lecture of the 15th of May. It is recommended that you make all exercises on the sheet even though only the exercises with a $\star$ are graded and will count towards the final grade. The homework should be handed in before the next lecture, which is on the 22nd of May, by (in order of preference):

1 E-mailing the pdf-output of a LaTeX file to n.dekleijn@uva.nl;

2 E-mailing a scanned copy of a hand-written file to n.dekleijn@uva.nl;

3 Depositing a hard-copy of the pdf-output of a LaTeX file in my mailbox (Niek de Kleijn) at Science Park 107, building F, floor 3;

4 Depositing a hand-written file in my mailbox (Niek de Kleijn) at Science Park 107, building F, floor 3;

5 Giving it to one of the teachers in person (at the beginning of the lecture).

You will receive comments on all the exercises you hand in (not just the homework) and we advise you to make use of this option.

Exercises

Note: This exercise set contains only two new exercises. However, since it has become even more relevant, we have repeated exercise 1 of lecture 9 (on the Pfaffian) for today's exercise class.

$\star$ Exercise 1: Zeta-function regularization

Let $\mathcal{O}$ be an operator acting on a Hilbert space, with a complete set of eigenstates v_n with eigenvalues λ_n ($n = 1, 2, 3, \ldots$). We introduce its *spectral ζ-function* as

$$\zeta_{\mathcal{O}}(s) = \sum_{n=1}^{\infty} (\lambda_n)^{-s}. \tag{1}$$

Beware that this means that we are counting with multiplicities so that eg. $\zeta_{\lambda I_n}(s) = \frac{n}{\lambda^s}$. Note that in the special case where $\lambda_n = n$, this function is the ordinary Riemann zeta function $\zeta(s)$. As is the case for that function, we will assume in what follows that $\zeta_{\mathcal{O}}(s)$ is well-defined for $\mathrm{Re}(s)$ large enough, and that it can then be analytically continued to a meromorphic function on the complex s-plane.

a. Show that

$$\det \mathcal{O} = e^{-\zeta'_{\mathcal{O}}(0)} \tag{2}$$

whenever both sides of this equation are well-defined.

Zeta-function regularization now *defines* $\det \mathcal{O}$ by the right hand side of the above equation (using the analytic contnuation of $\zeta_{\mathcal{O}}(s)$) whenever it is not well-defined directly as a product of the eigenvalues of $\mathcal{O}$.

We are now interested in the situation where

$$\mathcal{O} = -\frac{d^2}{dt^2} \tag{3}$$

where $t \in [0, \beta]$ parameterizes a circle of circumference β. (Note that we are using periodic boundary conditions on the eigenstates of $\mathcal{O}$.) To obtain a nonzero and well-defined result, we remove the "zero mode" (the constant eigenfunction of $\mathcal{O}$) from the Hilbert space.

b. Show that, after the above removal,

$$\zeta_{\mathcal{O}}(s) = 2\left(\frac{\beta}{2\pi}\right)^{2s} \zeta(2s) \tag{4}$$

where the function appearing on the right hand side is the ordinary Riemann ζ-function.

c. Show that

$$\det{}' \mathcal{O} = \beta^2 \tag{5}$$

where the prime on the left hand side indicates the removal of the zero mode. You can use the known values of the Riemann ζ-function and its derivative at the origin: $\zeta(0) = -1/2$ and $\zeta'(0) = -\log(2\pi)/2$.

Exercise 2: Product formula for the sine

Since it is so essential in the proof of the index theorem, we want to prove the product formula for the sine,

$$\frac{\sin x}{x} = \prod_{n=1}^{\infty}\left(1 - \frac{x^2}{\pi^2 n^2}\right) \tag{6}$$

a. Show that the Fourier series for the function $\cos(\alpha x)$ equals

$$\cos(\alpha x) = \frac{\alpha \sin(\pi\alpha)}{\pi} \sum_{n=-\infty}^{\infty} \frac{(-1)^n}{\alpha^2 - n^2} \cos(nx). \tag{7}$$

b. Deduce from the above result that

$$\cot(\pi\alpha) - \frac{1}{\pi\alpha} = \frac{2\alpha}{\pi} \sum_{n=1}^{\infty} \frac{1}{\alpha^2 - n^2} \tag{8}$$

c. Integrate the above formula from $\alpha = 0$ to $\alpha = t$ (you may assume without proof that the sum and integral can be exchanged) and use the result to obtain the product formula for the sine.

⋆ Exercise 3: The Pfaffian

Let us write G_n for the Grassmann algebra on n-variables $\theta^i,\ i = 1, \dots, n$. Define the *Berezin integral* $T : G_n \to \mathbb{R} \subset G_n$ by

$$T(\theta^1 \cdots \theta^n) := 1,$$

while T vanishes on products of degree $\leq n - 1$.

a) Show that T equals $\int d\theta^1 \cdots \int d\theta^n = I_n \circ I_{n-1} \circ \ldots \circ I_1$ where I_i denotes the integral over θ_i.

b) Suppose that n is even. Given a skew-symmetric $n \times n$-matrix A, define its *Pfaffian* by

$$\mathrm{Pf}(A) := T\left(\exp \frac{1}{2} \sum_{i,j} A_{ij} \theta^i \theta^j\right),$$

where exp is defined in G_n by its power series (which terminates after finitely many terms). Show that

$$\mathrm{Pf}(A)^2 = \det(A).$$

(*Hint: Recall from the previous lecture (notes) how the integrals over θ_i behave under substitution of variables.*)

c) Show that the Pfaffian defines a GL^+-invariant polynomial of degree $n/2$, i.e. show that

$$\mathrm{Pf}(gAg^{-1}) = \mathrm{Pf}(A)$$

for all $g \in GL(n, \mathbb{R})$ such that $\det(g) > 0$.

Remark 1. *Because of property c) above, one can use the Pfaffian to define a characteristic class of an even-dimensional oriented manifold M, called the* Euler class*, as follows: The curvature R of a riemannian metric on M is a skew-symmetric 2-form, so we can apply the Chern–Weil construction to define the cohomology class*

$$e(M) := [\mathrm{Pf}(R)] \in H_{\mathrm{dR}}^{\dim(M)}(M).$$