Exercises with Lecture 3 of Topology in Physics (UvA/Mastermath 2018)

February 19, 2018

This is the sheet of exercises corresponding to the material covered in the third lecture of the 20th of February. It is recommended that you make all exercises on the sheet even though only the exercises with a $\star$ are graded and will count towards the final grade. The homework should be handed in before the next lecture, which is on the 27th of February, by (in order of preference):

1 E-mailing the pdf-output of a LaTeX file to n.dekleijn@uva.nl;

2 E-mailing a scanned copy of a hand-written file to n.dekleijn@uva.nl;

3 Depositing a hard-copy of the pdf-output of a LaTeX file in my mailbox (Niek de Kleijn) at Science Park 107, building F, floor 3;

4 Depositing a hand-written file in my mailbox (Niek de Kleijn) at Science Park 107, building F, floor 3;

5 Giving it to one of the teachers in person (at the beginning of the lecture).

You will receive comments on all the exercises you hand in (not just the homework) and we advise you to make use of this option.

Exercises

$\star$ Exercise 1: Maxwell theory and de Rham cohomology.

The advantage of formulating Maxwell's theory in terms of differential forms is that it now makes sense on any manifold M, not even 4-dimensional! For this we consider the first few terms of the de Rham complex:

$$\Omega^0(M) \xrightarrow{d} \Omega^1(M) \xrightarrow{d} \Omega^2(M) \xrightarrow{d} \dots$$

As we have seen, the electric and magnetic fields are gathered in a two-form $F \in \Omega^2(M)$, which the homogeneous Maxwell equations require to be closed: $dF = 0$.

i) Assume that $H^2_{\mathrm{dR}}(M) = H^1_{\mathrm{dR}}(M) = 0$.

a) Show that for any field strength F there is a potential $A \in \Omega^1(M)$ such that $F = dA$.

b) Show that the "configuration space" of fields satisfying the homogeneous Maxwell equations ($dF = 0$) is given by the quotient $\Omega^1(M)/d\Omega^0(M)$.

Elements in $d\Omega^0(M)$ are called "gauge transformations". From now on we will drop the assumption that $H^2_{\mathrm{dR}}(M) = H^1_{\mathrm{dR}}(M) = 0$. We write Ω^k_{cl} for the space of *closed* k-forms. Recall that in this notation we have

$$H^k_{\mathrm{dR}}(M) := \frac{\mathrm{Ker}\left(d\colon \Omega^k(M) \to \Omega^{k+1}(M)\right)}{\mathrm{Im}\left(d\colon \Omega^{k-1}(M) \to \Omega^k(M)\right)} = \Omega^k_{\mathrm{cl}}(M)/d\Omega^{k-1}(M)$$

ii) Show that there is a sequence of maps

$$0 \to H^1_{\mathrm{dR}}(M) \xrightarrow{f_1} \Omega^1(M)/d\Omega^0(M) \xrightarrow{f_2} \Omega^2_{\mathrm{cl}}(M) \xrightarrow{f_3} H^2_{\mathrm{dR}}(M) \longrightarrow 0, \tag{*}$$

such that the image of any map is the kernel of the following map (this includes the image and kernels of the zero maps). We call this property of the sequence *exactness.*

Minkowski space $\mathbb{R}^{1,3}$ is topologically trivial, so $H^1_{\mathrm{dR}}(\mathbb{R}^{1,3}) = 0 = H^2_{\mathrm{dR}}(\mathbb{R}^{1,3})$, and the exactness of the sequence (*) amounts to the identification of $\Omega^2_{\mathrm{cl}}(\mathbb{R}^{1,3})$ with $\Omega^1(\mathbb{R}^{1,3})/d\Omega^0(\mathbb{R}^{1,3})$. In other words: we may equally well describe the electromagnetic field using the potential A, as long as we make sure that we use "gauge invariant" observables, i.e., functions $A \mapsto O(A)$ that are invariant under shifts by $d\Omega^0(\mathbb{R}^{1,3})$: $O(A + d\Lambda) = O(A)$. For a topologically nontrivial manifold M (already $M = \mathbb{R}^3 \times S^1$ is an example) we no longer have $\Omega^2_{\mathrm{cl}}(M) \cong \Omega^1(M)/d\Omega^0(M)$, as the sequence (*) shows. One of the lessons from Quantum Mechanics, as witnessed for example by the Aharonov–Bohm effect is that the potential A, is more fundamental than the field strength F! Therefore, it is better to describe Maxwell theory as an action functional on the space of "fields" $A \in \Omega^1(M)$.

iii) For $\gamma\colon S^1 \to M$ a smooth closed curve we consider the function

$$O_\gamma\colon A \mapsto \int_\gamma A$$

from $\Omega^1(M)$ to $\mathbb{R}$.

a) Show that O_γ is a gauge invariant observable for all γ.

b) Show that the potential A defines a cohomology class if there is no field, i.e. $F = 0$.

c) Suppose we can measure the observables O_γ for all (homotopy classes of) curves γ. Use the de Rham theorem to show that we can measure the cohomology class defined by A.

The Aharonov-Bohm experiment sets up exactly the situation described above. Namely it sets up an experiment where one can measure the observables O_γ in a region of space with no magnetic field, thus allowing to measure the cohomology class of the underlying potential (this is a quantum mechanical effect).

iv) Show that the action functional

$$S(A) = \frac{1}{2}||dA||^2 = \frac{1}{2}\int_M dA \wedge \star dA$$

is gauge invariant and variation leads to the vacuum Maxwell equation $d \star F = 0$. (You actually may have done this already last week...)

$\star$ Exercise 2: Computing $H^\bullet_{dR}(S^n)$

Definition 1 (Homotopy Equivalence)**.**
A (smooth) homotopy equivalence between two manifolds M and N is given by a pair of smooth maps

$$f\colon M \longrightarrow N \quad \text{and} \quad g\colon N \longrightarrow M$$

such that $f \circ g$ is smoothly homotopic to Id_N and $g \circ f$ is smoothly homotopic to Id_M.

Note that homotopy equivalence defines an equivalence relation on smooth manifolds, which we denote $\sim_h$.

i) Show that $N \sim_h M$ implies that $H^\bullet_{\mathrm{dR}}(N) \simeq H^\bullet_{\mathrm{dR}}(M)$.

ii) How do we use the result of i) in the Poincaré lemma?

iii) Using the definition of $H^\bullet_{\mathrm{dR}}$ in terms of differential forms show that $H^\bullet_{\mathrm{dR}}(M \coprod N) \simeq H^\bullet_{\mathrm{dR}}(M) \oplus H^\bullet_{\mathrm{dR}}(N)$.

iv) You will now compute the cohomology of the n-sphere by decomposing it into two opens sets and applying the Mayer–Vietoris sequence.

a) Use the description of H^0_{dR} (or the definition) to show that we have $H^0_{\mathrm{dR}}(S^0) = \mathbb{R}^2$ and $H^0_{\mathrm{dR}}(S^n) = \mathbb{R}$ for $n > 0$.

b) Find two open subsets U and V of S^n such that $U \cap V \sim_h S^{n-1}$ (also show why they are homotopy equivalent).

c) Apply the Mayer–Vietoris sequence to find that $H^1_{\mathrm{dR}}(S^n) = \mathbb{R}^{\delta_{1n}}$.

d) Apply the Mayer–Vietoris sequence and the result of c) to compute the comohology of S^n for any $n \geq 0$ as $H^k_{\mathrm{dR}}(S^n) = \mathbb{R}^{\delta_{k0}+\delta_{kn}}$.

Exercise 3: Computing $H^\bullet_{dR}(\mathbb{T}^2)$

In this exercise we will compute the cohomology of the 2-torus $\mathbb{T}^2$. We consider the flat model of the 2-torus as the space $\mathbb{R}^2/\mathbb{Z}^2$, i.e. we consider the plane and identify points (x_1, y_1) and (x_2, y_2) if $x_1 - x_2$ and $y_1 - y_2$ are both integers.

i) Show that $\mathbb{T}^2$ is given by considering the square $[0,1] \times [0,1] \subset \mathbb{R}^2$ and identifying the points $(0,t)$ with $(1,t)$ for $t \in [0,1]$ as well as identifying the points $(s,0)$ with $(s,1)$ for $s \in [0,1]$.

i) [**Bonus**] What does this model of $\mathbb{T}^2$ have to do with snake?

ii) Compute the cohomology of $\mathbb{T}^2$ by decomposing it into two open subsets U_{outer} and U_{middle} such that you already know the cohomology of U_{middle}, U_{outer} and $U_{middle} \cap U_{outer}$ and applying the Mayer–Vietoris sequence.

Exercise 4: The Hodge–Maxwell Theorem

In this exercise we will define the Hodge $\star$ starting from a general (pesudo-Riemannian) metric g on the oriented manifold M without using coordinates. Recall that g allows us to define a notion of volume on the manifold M. The volume of the submanifold B is given as the integral $\int_B \mathrm{vol}$. In coordinates vol is given by the formula

$$\mathrm{vol} = \sqrt{|g|} dx^1 \wedge \ldots \wedge dx^n,$$

where

$$|g| = \left| \sum_{i_1,\ldots,i_n=1}^{n} \varepsilon_{i_1\ldots i_n} g_{1i_1} \cdots g_{ni_n} \right|$$

denotes the absolute value of the determinant of g and the dx^i form a positively oriented basis. Recall that the coordinate transformation $x^i \to y^i$ is called positive if $\mathrm{Det} \frac{\partial \mathrm{x}^\mathrm{i}}{\partial \mathrm{y}^\mathrm{j}}$ (the Jacobian determinant) is positive.

i) Show that the formula for vol above defines an n-form ω. Do this by performing a (positive) coordinate transformation.

ii) The metric g is given by a symmetric non-degenerate bilinear pairing on the tangent spaces

$$(v, w) \mapsto g_{\mu\nu}(x) v^\mu v^\nu,$$

for $v, w \in T_x M$. Show that we get a $C^\infty(M)$-bilinear pairing

$$\mathfrak{X}(M) \times \mathfrak{X}(M) \longrightarrow C^\infty(M),$$

where $\mathfrak{X}(M)$ denotes vector fields.

Note that similarly the maps

$$(\alpha, \beta) \mapsto g^{\mu_1\nu_1}(x) \dots g^{\mu_p\nu_p}(x)\alpha_{\mu_1\dots\mu_p}\beta_{\nu_1\dots\nu_p}$$

define a $C^\infty(M)$-bilinear pairing on $\Omega^p(M)$. In fact this is the pairing $\langle \alpha, \beta \rangle$ mentioned in the lecture notes.

iii) Assume that M is compact and show that

$$(\alpha, \beta) = \int_M \langle \alpha, \beta \rangle \omega$$

defines an $\mathbb{R}$-bilinear, symmetric pairing on $\Omega^p(M)$.

iv) Consider $\beta \in \Omega^p(M)$, define $\star\beta$ as the $n-p$ form satifying

$$\alpha \wedge \star\beta = \langle \alpha, \beta \rangle \omega$$

for all $\alpha \in \Omega^p(M)$ and show that this definition coincides with the coordinate expression given in the lectures.

Hint: Show first that $\star\beta$ is uniquely defined. To do this note that if a form is 0 around every point, then it vanishes globally.

v) Show that the adjoint d^* of the exterior derivative d is given by the formula $\star d \star$.