Exercises with Lecture 4 of Topology in Physics (UvA/Mastermath 2018)

27 February 2018

This is the sheet of exercises corresponding to the material covered in the fourth lecture of the 27th of February. It is recommended that you make all exercises on the sheet even though only the exercises with a $\star$ are graded and will count towards the final grade. The homework should be handed in before the next lecture, which is on the 6th of March, by (in order of preference):

1 E-mailing the pdf-output of a LaTeX file to n.dekleijn@uva.nl;

2 E-mailing a scanned copy of a hand-written file to n.dekleijn@uva.nl;

3 Depositing a hard-copy of the pdf-output of a LaTeX file in my mailbox (Niek de Kleijn) at Science Park 107, building F, floor 3;

4 Depositing a hand-written file in my mailbox (Niek de Kleijn) at Science Park 107, building F, floor 3;

5 Giving it to one of the teachers in person (at the beginning of the lecture).

You will receive comments on all the exercises you hand in (not just the homework) and we advise you to make use of this option.

Exercises

$\star$Exercise 1: Gauge fields or field strengts?

Use the exact sequence

$$0 \xrightarrow{f_0} H^1_{\mathrm{dR}}(M) \xrightarrow{f_1} \Omega^1(M)/d\Omega^0(M) \xrightarrow{f_2} \Omega^2_{\mathrm{cl}}(M) \xrightarrow{f_3} H^2_{\mathrm{dR}}(M) \xrightarrow{f_4} 0. \quad (1)$$

to show that in a generic situation where neither of the two cohomology groups is trivial, one has that

$$(\Omega^1(M)/d\Omega^0(M)) \times H^2_{\mathrm{dR}}(M) \cong \Omega^2_{\mathrm{cl}}(M) \times H^1_{\mathrm{dR}}(M) \quad (2)$$

Thus, also in general, 1-cohomology adds field configurations that can not be described by the field strength (as in the Aharonov-Bohm effect), whereas 2-cohomology adds field configurations that can not be described by a single 1-form gauge field (as in the Dirac monopole).

If space-time is $T^2 \times \mathbb{R}^2$ (T^2 is the two-torus), which description contains more information – the description in terms of gauge fields, or the description in terms of field strengths?

Exercise 2: Getting familiar with Yang-Mills theory

a. From the definition of the Yang-Mills field strength, $[D_\mu, D_\nu]\phi^i(x) = -iF^a_{\mu\nu}(x)T_a\phi(x)$, show that $F^a_{\mu\nu}(x) = \partial_\mu A_\nu(x) - \partial_\nu A_\mu(x) - if^a{}_{bc}A^b_\mu(x)A^c_n u(x)$.

b. Show that in form notation, this result can be written as $F = dA - iA \wedge A$.

c. In Maxwell theory, how do the previous two results simplify?

d. Show that the Yang-Mills action is invariant under local Lie group transformations.

e. Show that the Jacobi identity for Lie algebras, $[T_a, [T_b, T_c]] + \text{cycl.} = 0$ implies that $[D_\mu, [D_\nu, D_\rho]] + \text{cycl.} = 0$. ("+ cycl." means that terms with cyclic permutations of the indices must be added in the identity.)

f. Write the result of (e) as an identity for the field strength components, and show that this identity can be rewritten as the Bianchi identity $D_\mu(\star F)^{\mu\nu} = 0$.

⋆Exercise 3: A first encounter with Chern-Simons theory

Just like for Maxwell theory, one might be tempted to write down a different action for the Yang-Mills gauge field, in which no Hodge star appears:

$$S = \int_M \text{Tr } (F \wedge F). \tag{3}$$

Show that the integrand in this action is exact:

$$\text{Tr } (F \wedge F) = d\text{Tr } (A \wedge dA - \frac{2}{3} iA \wedge A \wedge A). \tag{4}$$

If our manifold M has a boundary, $\delta M = B$, use this to rewrite the above action as an integral on the boundary. Also, compute the Euler-Lagrange equation of motion for this action.

The theory described by this action is called *Chern-Simons theory*; it will play an important role in some of the later lectures.