Exercises with Lecture 5 of Topology in Physics (UvA/Mastermath 2018)

March 7, 2018

This is the sheet of exercises corresponding to the material covered in the third lecture of the 6th of March. It is recommended that you make all exercises on the sheet even though only the exercises with a $\star$ are graded and will count towards the final grade. The homework should be handed in before the next lecture, which is on the 13th of March, by (in order of preference):

1 E-mailing the pdf-output of a LaTeX file to n.dekleijn@uva.nl;

2 E-mailing a scanned copy of a hand-written file to n.dekleijn@uva.nl;

3 Depositing a hard-copy of the pdf-output of a LaTeX file in my mailbox (Niek de Kleijn) at Science Park 107, building F, floor 3;

4 Depositing a hand-written file in my mailbox (Niek de Kleijn) at Science Park 107, building F, floor 3;

5 Giving it to one of the teachers in person (at the beginning of the lecture).

You will receive comments on all the exercises you hand in (not just the homework) and we advise you to make use of this option.

Exercises

$\star$ Exercise 1: The Hopf fibration

We consider the 3-sphere defined as

$$S^3 := \{(z_1, z_2 \in \mathbb{C}^2,\ |z_1|^2 + |z_2|^2 = 1\}$$

and recall the definition of the complex projective line $\mathbb{P}^1$ better known as the Riemann sphere

$$\mathbb{P}^1 := \left(\mathbb{C}^2 \backslash \{(0,0)\}\right) /_{\mathbb{C}^\times}$$

where $\mathbb{C}^\times := \mathbb{C}\backslash\{0\}$ acts by scalar multiplication. Note that S^3 is given by all pairs of complex numbers satisfying a certain equation, while $\mathbb{P}^1$ is given by pairs of complex numbers (z_1, z_1) (not both 0) up to a certain equivalence, namely $(z_1, z_2) \sim (w_1, w_2)$ if there is $0 \neq \lambda \in \mathbb{C}$ such that $\lambda z_1 = w_1$ and $\lambda z_2 = w_2$.

i) The group $U(1) \cong S^1$ acts on S^3 by

$$(z_1, z_2) \cdot e^{i\theta} := (z_1 e^{i\theta}, z_2 e^{i\theta})$$

Find a smooth map $S^3/U(1) \longrightarrow \mathbb{P}^1$ that allows for a smooth inverse, i.e. show that $S^3/U(1) \cong \mathbb{P}^1$.

We may compose the map of *i*) with the quotient map $S^3 \to S^3/U(1)$ to obtain a map $\pi\colon S^3 \to \mathbb{P}^1$. Recall from the lecture notes of lecture 1 that we had the atlas of $\mathbb{P}^1$ given by the charts

$$U := \{[(z_1, z_2)] \in \mathbb{P}^1 \mid z_1 \neq 0\}$$

and

$$V := \{[(z_1, z_2)] \in \mathbb{P}^1 \mid z_2 \neq 0\}.$$

ii) Find sections $U \to \pi^{-1}(U)$ and $V \to \pi^{-1}(V)$.

iii) Compute the transition function $\varphi_{UV} : U \cap V \to U(1)$.

(BONUS) Consider the standard (defining) representation of $U(1)$ on $\mathbb{C}$:

$$e^{i\theta} \cdot z = e^{i\theta} z,$$

and consider the line bundle associated to the Hopf fibration above. Show that this line bundle agrees with the tautological line bundle over $\mathbb{P}^1$.

Exercise 2: The Hopf invariant

We consider a smooth map $f : S^{2n-1} \to S^n$. Let $\omega \in \Omega^n(S^n)$ be the volume form

$$\omega := \sum_{i=1}^{n} (-1)^{i+1} dx^1 \wedge \ldots \wedge \widehat{dx^i} \wedge \ldots \wedge dx^{n+1}$$

i) Show that $f^*\omega$ is exact: $f^*\omega = d\alpha$ for some $\alpha \in \Omega^{n-1}(S^{2n-1})$.

ii) Show that the integral

$$H(f) := \int_{S^{2n-1}} \alpha \wedge d\alpha$$

is independent of the choice of "potential" α: it only depends on the map f.

iii) Show that the integral above is zero for odd n.

iv) Now you will show that the Hopf invariant $H(f)$ is a homotopy invariant. So consider two maps $f_i\colon S^{2n-1} \to S^n$ for $i = 0, 1$ and a homotopy $F\colon S^{2n-1} \times [0, 1] \to S^n$ between them. Note that this means that if $\iota_i\colon S^{2n-1} \to S^{2n-1} \times [0, 1]$ denotes the inclusion at an endpoint for $i = 0, 1$ respectively, then $F \circ \iota_i = f_i$.

   a) Show that $F^*\omega = d\alpha$ for some $\alpha \in \Omega^{n-1}(S^{2n-1} \times [0, 1])$.

   b) Show that $f_i^*\omega = d\alpha_i$ for $\alpha_i = \iota_i^*\alpha$ the restriction of α to the endpoint $S^{2n-1} \times \{i\}$ for $i = 0, 1$. Conclude that we may use α_i to compute $H(f_i)$.

   c) Show that $d\alpha \wedge d\alpha = 0$.

   d) Show that $H(f_0) = H(f_1)$.
   hint: Stokes' theorem

v) Since $\mathbb{P}^1 \simeq S^2$ we can consider $H(\pi)$ where π is the map from exercise 1. It turns out that $H(\pi) \neq 0$. Show that this means that we cannot extend the map π to a map $\mathbb{R}^4 \to \mathbb{R}^3$.

vi) Show that f $n = 1$ then $H(f)$ equals the winding number of the map $f\colon S^1 \to S^1$.

(hard BONUS) Show that $H(\pi) = 1$.

($\star$) Exercise 3: Cocycles and Representations

In this exercise we will consider how one uses representations of general linear groups in order to perform constructions from linear algebra like the direct sum and tensor product. For this purpose recall that a rank n vector bundle $E \to M$ and a cover $\cup_\alpha U_\alpha = M$ yield a cocycle

$$\varphi_{\alpha\beta}\colon U_\alpha \cap U_\beta \longrightarrow GL(n)$$

and we may reconstruct the vector bundle E given the cocycle. Now suppose $\phi\colon GL(n) \to GL(m)$ is a group homomorphism, i.e. a representation of $GL(n)$ on the m-dimensional standard vector space. In particular $\phi(I) = I$, $\phi(A^{-1}) = \phi(A)^{-1}$ and $\phi(AB) = \phi(A)\phi(B)$ and so $\phi \circ \varphi_{\alpha\beta}$ satisfies the cocycle conditions. Thus the representation ϕ coupled with the vector bundle E define another (rank m) vector bundle S!

i) Recall that $E \oplus E$ is the vector bundle with fiber $E_x \oplus E_x$. Give the representation $GL(n) \to GL(2n)$ that gives rise to this vector bundle.

ii) Similarly $E \otimes E$ is the vector bundle with fiber $E_x \otimes E_x$. Give the representation $GL(n) \to GL(n^2)$ that gives rise to this vector bundle.

iii) Again $\bigwedge^n E$ is the vector bundle with fiber $E_x \wedge E_x \wedge \ldots \wedge E_x$ (n-times). Give the representation $GL(n) \to GL(1)$ that gives rise to this vector bundle.

The rest of this exercise does not need to be handed in as homework

iv) The vector bundle E^* is the vector bundle with fibers E_x^*. Give the representation $GL(n) \to GL(n)$ that gives rise to this vector bundle.

Given another rank k vector bundle F with corresponding cocycle denoted by $\psi_{\alpha\beta} \colon U_{\alpha\beta} \to GL(k)$ we find that a representation of the product $\phi \colon GL(n) \times G(k) \to GL(m)$ gives rise to the cocycle

$$\phi \circ (\varphi_{\alpha\beta}, \psi_{\alpha\beta}) \colon U_{\alpha\beta} \longrightarrow GL(m)$$

and thus a rank m vector bundle S.

i) Recall that $E \oplus F$ is the vector bundle with fiber $E_x \oplus F_x$. Give the representation $GL(n) \to GL(n+k)$ that gives rise to this vector bundle.

ii) Similarly $E \otimes F$ is the vector bundle with fiber $E_x \otimes F_x$. Give the representation $GL(n) \to GL(nm)$ that gives rise to this vector bundle.

iii) Again $Hom(E, F)$ is the vector bundle with fiber $Hom(E_x, F_x)$. Give the representation $GL(n) \to GL(nm)$ that gives rise to this vector bundle. What does this have to do with the vector bundle $E^* \otimes F$?

Exercise 4: Non-trivial subbundle

In this exercise we will consider the relation between projection valued functions and vector bundles. In the proces we will construct a non-trivial subbundle of a trivial bundle. In fact on a compact manifold every bundle is a subbundle of a trivial one since any vector bundle E in that case admits a *complementary* bundle F such that $E \oplus F$ is trivial. We will consider the base space $M = \mathbb{T}^2 = \mathbb{R}^2/(2\pi\mathbb{Z})^2$, the 2-dimensional torus, and we will denote the rank 2 trivial bundle $\mathbb{T}^2 \times \mathbb{C}^2$ by $C^2 \to \mathbb{T}^2$.

i) Show that the sections $\Gamma^\infty(\mathbb{T}^2; C^2)$ are given by columns $\begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix}$ of smooth functions $\eta_i \in C^\infty(\mathbb{T}^2)$.

ii) Show that the map $D \colon \Gamma^\infty(\mathbb{T}^2; C^2) \to \Omega^1(\mathbb{T}^2; C^2)$ given by

$$D \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} d\eta_1 \\ d\eta_2 \end{pmatrix}$$

defines a connection on C^2.

iii) Show that the curvature $F(D)$ vanishes identically.

Suppose f, g and h are smooth functions on the circle such that

$$f - f^2 = g^2 + h^2 \quad \text{and} \quad gh = 0.$$

iv) Consider the smooth function $p\colon \mathbb{T}^2 \to M_2(\mathbb{C})$ given by

$$p(\theta, \phi) = \begin{pmatrix} f(\theta) & g(\theta) + h(\theta)e^{i\phi} \\ g(\theta) + h(\theta)e^{-i\phi} & 1 - f(\theta) \end{pmatrix}$$

and show that $p^2 = p$.

Now we note p defines a map $P\colon C^2 \to C^2$ that is linear in the fibers and since $p^2 = p$ the image Im P defines a rank 1 vector bundle $N^1 \to \mathbb{T}^2$.

v) Show that map $\nabla\colon \Gamma^\infty(\mathbb{T}^2; N^1) \to \Omega^1(\mathbb{T}^2; N^1)$ given by

$$\nabla P\eta = PDP\eta$$

defines a connection on N^1.

Recall that we may extend the definition of ∇ to a map

$$\nabla\colon \Omega^k(\mathbb{T}^2; N^1) \longrightarrow \Omega^{k+1}(\mathbb{T}^2; N^1)$$

in the obvious way, i.e. by the formula $PD = \nabla$ and noting that D is extended as the entrywise exterior derivative. Then the curvature $F(\nabla)$ is defined by the equation $\nabla^2 P\eta = F(\nabla) \wedge P\eta$.

vi) Show that $F(\nabla) = pdp \wedge dp$

In a following lecture we will see that $Tr(F(\nabla))$ is a closed 2-form whose class is called the *first Chern class* (up to a factor). We will also see that this class only depends on the isomorphism class of the vector bundle. Moreover we will see that the first Chern class of a trivial vector bundle always vanishes. In this case we may compute that $Tr(F(\nabla))$ is not exact by integrating it over $\mathbb{T}^2$ and therefore N^1 is a non-trivial subbundle of the trivial bundle C^2. As a last (very hard) exercise you can try to show that $Tr(F(\nabla))$ is not exact by assuming the following 4 conditions on f, g and h

(1) $0 \leq f(\theta) \leq 1$ for all θ;

(2) $f(0) = 1$ and $f(\pi) = 0$;

(3) $g(\theta) = \sqrt{f - f^2}$ and $h(\theta) = 0$ for all $\theta \in [0, \pi]$ and finally

(4) $g(\theta) = 0$ and $h(\theta) = \sqrt{f - f^2}$ for all $\theta \in [\pi, 2\pi]$.