# Exercises with Lecture 6 of Topology in Physics (UvA/Mastermath 2018)

13 March 2018

This is the sheet of exercises corresponding to the material covered in the sixth lecture of the 13th of March. It is recommended that you make all exercises on the sheet even though only the exercises with a $\star$ are graded and will count towards the final grade. The homework should be handed in before the next lecture, which is on the 20th of March, by (in order of preference):

**1** E-mailing the pdf-output of a LaTeX file to n.dekleijn@uva.nl;

**2** E-mailing a scanned copy of a hand-written file to n.dekleijn@uva.nl;

**3** Depositing a hard-copy of the pdf-output of a LaTeX file in my mailbox (Niek de Kleijn) at Science Park 107, building F, floor 3;

**4** Depositing a hand-written file in my mailbox (Niek de Kleijn) at Science Park 107, building F, floor 3;

**5** Giving it to one of the teachers in person (at the beginning of the lecture).

**You will receive comments on all the exercises you hand in (not just the homework) and we advise you to make use of this option.**

## Exercises

### $\star$Exercise 1: Triviality of principal bundles

Let $P$ be a principal bundle with base space $M$, fiber a Lie group $G$, and projection $\pi : P \to M$. Assume we are also given a (global) section $\sigma \in \Gamma(M, P)$.

a. Using $\sigma$, construct a map from $M \times G$ to $P$ that is one-to-one, i.e. it is both injective and surjective (and show that it has those properties, of course).

The above result is a rather remarkable theorem: *any* principal bundle that has a global section, is isomorphic to the trivial bundle!

b. There are of course many ways to argue that the Möbius strip is not homeomorphic to the cylinder. Give such an argument using the theorem we obtained above. (Think of the Möbius strip and the cylinder as real vector bundles over a circle, and consider associated principal bundles.)

## Exercise 2: Classifying instantons in Yang-Mills theory

In this exercise, we fill in some of the gaps in the discussion of instantons from the lecture.

In the lecture, we have described $S^4$ as $\mathbb{R}^4 \cup \{\infty\}$ and divided it into two hemispheres, $U_N$ with $|x| \leq R + \epsilon$ and $U_S$ with $|x| \geq R - \epsilon$. Recall that to describe instantons, on the southern hemisphere, we may choose a trivial Yang-Mills gauge potential, $A_S(x) = 0$.

a. Show that the map $g_n : x \mapsto r^{-n}(x^i \sigma_i + x^4 I)^n$, with $I$ the $2 \times 2$ identity matrix and $\sigma_i$ the Pauli matrices, gives a map from the boundary of $U_N$ to $SU(2)$ for every integer $n$. Here, $r^2 = (x^1)^2 + (x^2)^2 + (x^3)^2 + (x^4)^2$.

The map $g_n$ can be smoothly extended over $U_N$ (convince yourself of this!) On the northern hemisphere of $S^4$, we now introduce the Yang-Mills gauge potential $A_N(x) = g_n^{-1} dg_n$. Since this gauge potential is pure gauge, $A_S$ and $A_N$ together give a well-defined connection on an $SU(2)$ principal bundle over $S^4$.

In an exercise from a previous lecture, you have shown that Tr $F \wedge F$ can be written as $dCS$ where $CS$ is the Chern-Simons form $CS = \text{Tr } (A \wedge dA + \frac{2}{3} A \wedge A \wedge A)$.

b. Use the Chern-Simons form to show that

$$\int_{S^4} \text{Tr } F \wedge F = -\frac{1}{3} \int_{S^3} \text{Tr } A_N \wedge A_N \wedge A_N. \qquad (1)$$

where $S^3$ is the 3-sphere defined by $|x| = R$.

c. Argue (for the $n = 1$ case for simplicity) that Tr $A_N \wedge A_N \wedge A_N$ is constant a constant multiple of the volume form on $S^3$.

The above results were used in the lecture to argue that $\int_{S^4} \text{Tr } F \wedge F$ is a discrete invariant that can be used to classify the "winding number" of different instanton configurations.

## $\star$Exercise 3: The Berry connection

We consider a quantum mechanical system that can be described by one "slow moving coordinate" $X$ and one "fast moving coordinate" $x$. The hamiltonian operator for this system is

$$H = \frac{\hat{p}^2}{2m} + \frac{\hat{P}^2}{2M} + V(x, X) \tag{2}$$

where $\hat{p} = -i\partial_x$ is the momentum operator for $x$, and $\hat{P}$ is defined similarly for $X$. The fact that $X$ is "slow moving" means that wave functions $\psi(x, X)$ vary much faster when we change $x$ then they do when we change $X$: to first approximation (in some parameter that we will not specify) we can think of these wave functions as being constant in $X$. Therefore, it is interesting to ignore the $\hat{P}$-term in the hamiltonian at first, and consider the hamiltonian

$$h(X) = \frac{\hat{p}^2}{2m} + V(x, X) \tag{3}$$

where $X$ now has become a parameter, and wave functions depend on $x$ only. Assume that $h(X)$ has a discrete set of energy eigenstates $|n, X\rangle$ and corresponding eigenvalues $\epsilon_n(X)$, and that all of these depend smoothly on $X$, so that in particular eigenvectors for different $n$ never cross when varying $X$.

Of course, the true energy eigenstates of the hamiltonian (2) *do* depend on $X$ as well, but it can be shown that to a good approximation for fixed $X$ they equal $|n, X\rangle$ up to an $X$-dependent normalization:

$$\Psi_n(x, X) = \Phi_n(X)|n, X\rangle \tag{4}$$

We call the energy eigenvalues corresponding to these states $E_n$.

Act with $H$ on $\Psi_n(x, X)$, and require that the result equals $E_n\Psi_n(x, X)$. Use this to derive a differential equation for $\Phi_n(X)$ that contains the Berry connection

$$A_n(X) = \langle n, X|\partial_X|n, X\rangle, \tag{5}$$

where you may assume without proof that $\langle n, X|\partial_X|m, X\rangle = 0$ for $n \neq m$. (This assumption is known as the Born-Oppenheimer approximation.) In this way, show that $\Phi_n(X)$ is an energy eigenstate for the effective hamiltonian

$$H_{eff} = \frac{1}{2M}\left(\hat{P} - iA_n(X)\right)^2 + \epsilon_n(X). \tag{6}$$

The upshot of this exercise is that one can "integrate out" the fast degree of freedom, resulting in a *covariant* derivative in the hamiltonian for the slow degree of freedom, where the connection involved is precisely the Berry connection.