Exercises with Lecture 7 of Topology in Physics (UvA/Mastermath 2018)

March 20, 2018

This is the sheet of exercises corresponding to the material covered in the seventh lecture of the 20th of March. It is recommended that you make all exercises on the sheet even though only the exercises with a $\star$ are graded and will count towards the final grade. The homework should be handed in before the next lecture, by (in order of preference):

1 E-mailing the pdf-output of a LaTeX file to n.dekleijn@uva.nl;

2 E-mailing a scanned copy of a hand-written file to n.dekleijn@uva.nl;

3 Depositing a hard-copy of the pdf-output of a LaTeX file in my mailbox (Niek de Kleijn) at Science Park 107, building F, floor 3;

4 Depositing a hand-written file in my mailbox (Niek de Kleijn) at Science Park 107, building F, floor 3;

5 Giving it to one of the teachers in person (at the beginning of the lecture).

You will receive comments on all the exercises you hand in (not just the homework) and we advise you to make use of this option.

Exercises

Exrecise 1: The gauge group

In the lectures we have seen that gauge theories, such as Yang–Mills theory, are described using principal G-bundles for some fixed compact Lie group G. These theories are called gauge theories because they have a large symmetry group, called the *gauge group*. The aim of this exercise is to better understand the structure of this gauge group.

Fix a principal G-bundle $\pi\colon P \to M$. The gauge group is defined to be

$$\mathcal{G}(P) := \left\{ \psi\colon P \to P, \text{ satisfying } \begin{matrix} \psi(pg) = \psi(p)g \\ \pi(\psi(p)) = \pi(p) \end{matrix} \text{ for all } p \in P,\ g \in G \right\},$$

where all ψ are assumed to be smooth.

a) Show that $\mathcal{G}(P)$ is indeed a group. When showing that $\mathcal{G}(P)$ contains inverses you may assume that they are smooth.

b) Suppose that $P = M \times G$ is the trivial bundle. Show that the equation

$$\psi(m, g) = (m, \varphi(m)g) \quad \text{for all } (m, g) \in P,$$

defines, for any $\psi \in \mathcal{G}(P)$, a function $\varphi : M \to G$. Explain that this correspondence shows that $\mathcal{G}(P) \cong C^\infty(M, G)$. What is the group structure on $C^\infty(M, G)$?

c) Let A be a connection form on P, i.e. $A \in \Omega^1(P, \mathfrak{g})$ is a Lie algebra valued 1-form that satisfies

$$\begin{aligned} \iota_{\xi_P} A &= \xi, \quad \text{for all } \xi \in \mathfrak{g}, \\ R_g^* A &= \mathrm{Ad}_{g^{-1}}(A), \quad \text{for all } g \in G. \end{aligned}$$

Show that for $\psi \in \mathcal{G}(P)$, the pull-back $\psi^* A$ is another connection 1-form.

d) When P is trivial, it has a global section $s : M \to P$ and we can use this section to pull the connection form A back to a $\mathfrak{g}$-valued 1-form $\alpha := s^* A \in \Omega^1(M, \mathfrak{g})$. A computation shows that the action of $\mathcal{G}(P)$ on the space of connections is given by

$$\varphi \cdot \alpha = \varphi \alpha \varphi^{-1} + (d\varphi)\varphi^{-1}, \tag{1}$$

using the notation of b). Show by an explicit computation that this defines an action of $\mathcal{G}(P)$: $\varphi_1 \cdot (\varphi_2 \cdot \alpha) = (\varphi_1 \varphi_2) \cdot \alpha$.

e) Show that the curvature $F(\alpha) := d\alpha + \alpha \wedge \alpha$ satisfies

$$F(\varphi \cdot \alpha) = \varphi F(\alpha) \varphi^{-1}.$$

Use this to show that the Yang–Mills action

$$S_{YM}(\alpha) := \int_M \mathrm{Tr}(F(\alpha) \wedge \star F(\alpha))$$

is invariant under the action of the gauge group.

Remark

The configuration space of pure Yang–Mills theory is given by the space of all connections $\mathcal{C}(P)$ on the principal G-bundle P. The Yang–Mills action has a huge symmetry group, the gauge group $\mathcal{G}(P)$ so that the physically relevant configuration space is actually the quotient $\mathcal{C}(P)/\mathcal{G}(P)$. When $P = M \times G$ is trivial $\mathcal{C}(P) = \Omega^1(M, \mathfrak{g})$ with the action of $\mathcal{G}(P) = C^\infty(M, G)$ given by (1). When P is not trivial, these spaces look slightly different, taking into account the twisting of the bundle P: in the general case we have $\mathcal{C}(P) = \Omega^1(M, \mathrm{ad}(P))$ and $\mathcal{G}(P) = \Gamma^\infty(M, \mathrm{Ad}(P))$ where $\mathrm{ad}(P)$ and $\mathrm{Ad}(P)$ are bundles associated to P:

$$\begin{aligned}\mathrm{Ad}(P) &:= P \times_G G = (P \times G)/\{(ph, g) = (p, hgh^{-1}),\ p \in P,\ g, h \in G\} \\ \mathrm{ad}(P) &:= P \times_G \mathfrak{g} = (P \times \mathfrak{g})/\{(ph, \xi) = (p, \mathrm{Ad}_h(\xi)),\ p \in P,\ h \in G, \xi \in \mathfrak{g}\}.\end{aligned}$$

(Remark that $\mathrm{ad}(P)$ is a vector bundle, whereas $\mathrm{Ad}(P)$ is only a fiber bundle with typical fiber G.)

$\star$ Exercise 2: The Chern character

a) Show that the restriction of an invariant symmetric polynomial

$$P : \mathrm{Mat}_r(\mathbb{C}) \times \ldots \times \mathrm{Mat}_r(\mathbb{C}) \to \mathbb{C}$$

to the subset diagonalizable matrices, defines a symmetric polynomial[1] in r variables. With this, relate the Chern classes to the elementary symmetric functions

$$\sigma_k(\lambda_1, \ldots, \lambda_r) := \sum_{1 \le i_1 \le \ldots \le i_k \le r} \lambda_{i_1} \cdots \lambda_{i_r}, \quad k = 1, \ldots, r.$$

b) It is an algebraic fact that any symmetric polynomial can be written as a linear sum of products of the σ_k's. Show that any characteristic class obtained from the Chern–Weil formalism can be expressed in terms of Chern classes by using this fact, the splitting principle and the results of a).

c) Consider the polynomial functions P_k on $\mathrm{Mat}_r(\mathbb{C})$ ($r \times r$-matrices) defined by the expansion

$$\mathrm{Tr}(e^{tA}) = P_0(A) + tP_1(A) + t^2 P_2(A) + \ldots, \quad A \in Mat_r(\mathbb{C}).$$

Show that the P_k are invariant, and therefore define characteristic classes $\mathrm{ch}_k(E) \in H^{2k}_{\mathrm{dR}}(M)$ of a vector bundle $E \to M$. Express ch_1 and ch_2 in terms of Chern classes. Have you seen ch_2 before?

d) The *Chern character* is defined as the sum

$$\mathrm{ch}(E) := \sum_{k \ge 0} \mathrm{ch}_k(E) \in H^\bullet_{\mathrm{dR}}(M).$$

Why is this a finite sum? Show that the Chern character satisfies $\mathrm{ch}(E \oplus F) = \mathrm{ch}(E) + \mathrm{ch}(F)$.

[1] A symmetric polynomial is a polynomial function Q of r-variable that is invariant under permutations of the variables: $Q(\lambda_1, \ldots, \lambda_r) = Q(\lambda_{\tau(1)}, \ldots, \lambda_{\tau(r)}),\ \forall \tau \in S_r$.

Exercise 3: The Chern Simons form

a) If you view the trace on matrices as an invariant polynomial via $(A, B) \mapsto \mathrm{Tr}(AB)$, what is the associated characteristic class?

b) Explain why the characteristic class in a) is, up to a normalization of $4\pi^2$, an integral cohomology class.

c) Suppose that $E \to M$ is a trivial vector bundle and let $\nabla = d + A$ be a connection. Show that the transgression form $L(d, d + A)$ is exactly the Chern–Simons form $\mathrm{Tr}(A \wedge dA + \frac{2}{3} A \wedge A \wedge A)$ discussed before.

Exercise 4: The Euler Class

In this exercise we will consider oriented real vector bundles (and in the end also complex ones). One way to characterize an orientation on a real rank n vector bundle $\pi\colon E \to M$ over the manifold M is to say that relative to some open cover $\{U_\alpha\}$ we may pick the transition functions denoted by $\phi_{\alpha\beta}\colon U_{\alpha\beta} \to GL(n, \mathbb{R})$ to have values in the subgroup $SO(n)$ of $GL(n, \mathbb{R})$. So we will now assume that $\phi_{\alpha\beta}\colon U_{\alpha\beta} \to SO(n)$, i.e. we will consider oriented vector bundles. This means in particular that we can choose point-wise linearly independent sections $e_{\alpha,i}\colon U_\alpha \to E|_{U_\alpha}$ for $i = 1, \dots, n$ for any α such that $\phi_{\alpha\beta}(x)_i^j e_{\beta,j}(x) = e_{\alpha,i}(x)$ for all $x \in U_\alpha \cap U_\beta$. Given $x \in M$ and any α such that $x \in U_\alpha$ we can now consider polar coordinates $(r_\alpha, \theta_{\alpha,1}, \dots, \theta_{\alpha,n-1})$ on $E_x \backslash \{0\}$ by treating the $e_{\alpha,i}$ as standard coordinates. Note also that this system of coordinates varies smoothly with x.

i) Show that, if $x \in U_{\alpha\beta}$, then $r_\alpha(x) = r_\beta(x)$.

For simplicity's sake let us set $n = 2$ from now on.

ii) Argue that we may pick functions $\varphi_{\alpha\beta} \in C^\infty(U_{\alpha\beta})$ such that

$$\theta_\beta = \theta_\alpha + \pi^* \varphi_{\alpha\beta}$$

holds on $E|_{U_{\alpha\beta}} \backslash 0(U_{\alpha\beta})$ for 0 the zero section.

We note that

$$d\varphi_{\alpha\beta} - d\varphi_{\alpha\gamma} + d\varphi_{\beta\gamma} = 0$$

on the triple intersections $U_{\alpha\beta\gamma}$ (as a bonus exercise prove this).

iv) Show that there exist one-forms $\xi_\alpha \in \Omega^1(U_\alpha)$ such that

$$\frac{1}{2\pi} d\varphi_{\alpha\beta} = \xi_\beta - \xi_\alpha.$$

HINT: consider $\frac{1}{2\pi} \sum_\gamma \rho_\gamma \varphi_{\alpha\gamma}$ *for* $\{\rho_\gamma\}$ *a partition of unity subordinate to* $\{U_\gamma\}$.

v) Show that the two-forms $d\xi_\alpha$ define a class $e(E) \in H^2_{\text{dR}}(M)$ that is independent of the choice of $\xi'_\alpha s$.

The class $e(E)$ is called the Euler class of the oriented vector bundle E. Given a rank 1 complex vector bundle V it is given by transition functions $\phi_{\alpha\beta}$ with values in $U(1)$ and by considering the isomorphism $SO(2) \simeq U(1)$ given by

$$e^{i\phi} \mapsto \begin{pmatrix} \cos(\phi) & -\sin(\phi) \\ \sin(\phi) & \cos(\phi) \end{pmatrix}$$

it gives rise to a rank 2 oriented real vector bundle $V_{\mathbb{R}}$.

vii) Show that $e(V_{\mathbb{R}}) = c_1(V)$.

Finally let us compute such a class. Consider the two-sphere S^2 and note that it is isomorphic to the complex manifold $\mathbb{P}^1$. This means that the tangent bundle $TS^2 = V_{\mathbb{R}}$ for V a rank 1 complex bundle.

viii) Show that

$$\int_{S^2} e(TS^2) = \chi(S^2)$$

where $\chi(M) := \sum_{i=0}^{\infty}(-1)^i \text{Dim } H^i_{\text{dR}}(M)$ denotes the so-called *Euler characteristic.*

HINT: Split the integral into a sum of integrals over the north and south hemispheres (keep orientation in mind). Consider the usual cover of S^2 given by U_N and U_S by deleting the south and north poles respectively. To determine φ_{NS} consider orthonormal vector fields e^1_N, e^2_N on U_N and e^1_S, e^2_S on U_S for the usual Riemannian metric.