Exercises with Lecture 8 of Topology in Physics (UvA/Mastermath 2018)

3 April 2018

This is the sheet of exercises corresponding to the material covered in the eighth lecture of the 3rd of April. It is recommended that you make all exercises on the sheet even though only the exercises with a $\star$ are graded and will count towards the final grade. The homework should be handed in before the next lecture, which is on the 10th of April, by (in order of preference):

1 E-mailing the pdf-output of a LaTeX file to n.dekleijn@uva.nl;

2 E-mailing a scanned copy of a hand-written file to n.dekleijn@uva.nl;

3 Depositing a hard-copy of the pdf-output of a LaTeX file in my mailbox (Niek de Kleijn) at Science Park 107, building F, floor 3;

4 Depositing a hand-written file in my mailbox (Niek de Kleijn) at Science Park 107, building F, floor 3;

5 Giving it to one of the teachers in person (at the beginning of the lecture).

You will receive comments on all the exercises you hand in (not just the homework) and we advise you to make use of this option.

Exercises

$\star$ Exercise 1: The holonomy group

Let $E \to M$ be a vector bundle with some fixed connection, and consider the set of all holonomies for loops $\gamma : [0,1] \to M$ that start and end at a given point $x \in M$:

$$\Gamma_x = \{G_\gamma | \gamma(0) = \gamma(1) = x\} \tag{1}$$

To be mathematically precise, assume that we are considering loops γ that are continuous and piecewise smooth (meaning that the loops may have occasional "corners" but are otherwise smooth).

a. Show that Γ_x is in fact a group.

b. Show that if M is connected, $\Gamma_x \cong \Gamma_y$ (as groups, not just sets) for any two $x, y \in M$.

The upshot is that for connected M, one can speak of *the* holonomy group of a bundle with connection; it turns out that this group often gives an interesting piece of data about the structure of the bundle.

⋆ Exercise 2: Chern-Simons theory

a. Show that the equation of motion for Chern-Simons theory is $F = 0$. That is, the "classical" solutions to Chern-Simons theory are the *flat* (zero curvature) gauge fields.

b. In Chern-Simons theory (on a trivial patch U), make a gauge transormation $A \to g^{-1}Ag + g^{-1}dg$. Show that the Chern-Simons Lagrangian changes as follows:

$$\delta L_{CS} = -d\mathrm{Tr}\,\left(dg \cdot g^{-1}A\right) - \frac{1}{3}\mathrm{Tr}\,\left(g^{-1}dg \wedge g^{-1}dg \wedge g^{-1}dg\right) \qquad (2)$$

c. Use the above result to argue that if g is a map from a 3-manifold Σ without boundary into a Lie group, then

$$\frac{1}{24\pi^2}\int_\Sigma \mathrm{Tr}\,\left(g^{-1}dg \wedge g^{-1}dg \wedge g^{-1}dg\right) \qquad (3)$$

is always an integer.

Exercise 3: Grassmann variables

If you are not familiar with Grassmann variables yet, this exercise contains some simple problem to get acquainted with those objects. In all that follows, θ_i $(1 \leq i \leq N)$ are Grassmann variables.

a. Show that the derivatives $\frac{d}{d\theta_i}$ and $\frac{d}{d\theta_j}$ anticommute.

b. Show that the anticommutator of $\frac{d}{d\theta_i}$ and θ_j equals δ_{ij} (the Kronecker delta).

c. If $f(\theta)$ and $g(\theta)$ each have given parity – that is, f contains only terms with either an even or an odd number of Grassmann variables, but not both, and similarly for g – give a Leibniz-rule-like expression for $\frac{d}{d\theta_i}(f(\theta)g(\theta))$.

d. As explained in the lecture, definite integration on a Grassmann algebra G should be implemented by a map $I : G \to G$ which is linear and satisfies the rules

1. $DI = 0$
2. $ID = 0$
3. $D(A) = 0 \implies I(BA) = I(B)A$

where D is the operator $\frac{d}{d\theta_i}$. Show that this means that $I = cD$ for some number c.