Exercises with Lecture 9 of Topology in Physics (UvA/Mastermath 2018)

April 10, 2018

This is the sheet of exercises corresponding to the material covered in the ninth lecture of the 10th of April. It is recommended that you make all exercises on the sheet even though only the exercises with a $\star$ are graded and will count towards the final grade. The homework should be handed in before the next lecture, by (in order of preference):

1 E-mailing the pdf-output of a LaTeX file to n.dekleijn@uva.nl;

2 E-mailing a scanned copy of a hand-written file to n.dekleijn@uva.nl;

3 Depositing a hard-copy of the pdf-output of a LaTeX file in my mailbox (Niek de Kleijn) at Science Park 107, building F, floor 3;

4 Depositing a hand-written file in my mailbox (Niek de Kleijn) at Science Park 107, building F, floor 3;

5 Giving it to one of the teachers in person (at the beginning of the lecture).

You will receive comments on all the exercises you hand in (not just the homework) and we advise you to make use of this option.

Exercises

Exercise 1: The Pfaffian

Let us write G_n for the Grassmann algebra on n-variables $\theta^i,\ i = 1, \dots, n$. Define the *Berezin integral* $T : G_n \to \mathbb{R} \subset G_n$ by

$$T(\theta^1 \cdots \theta^n) := 1,$$

while T vanishes on products of degree $\leq n - 1$.

a) Show that T equals $\int d\theta^1 \cdots \int d\theta^n = I_n \circ I_{n-1} \circ \ldots \circ I_1$ where I_i denotes the integral over θ_i.

b) Suppose that n is even. Given a skew-symmetric $n \times n$-matrix A, define its *Pfaffian* by

$$\mathrm{Pf}(A) := T\left(\exp \frac{1}{2}\sum_{i,j} A_{ij}\theta^i\theta^j\right),$$

where exp is defined in G_n by its power series (which terminates after finitely many terms). Show that

$$\mathrm{Pf}(A)^2 = \det(A).$$

(*Hint: Recall from the previous lecture (notes) how the integrals over θ_i behave under substitution of variables.*)

c) Show that the Pfaffian defines a GL^+-invariant polynomial of degree $n/2$, i.e. show that

$$\mathrm{Pf}(gAg^{-1}) = \mathrm{Pf}(A)$$

for all $g \in GL(n, \mathbb{R})$ such that $\det(g) > 0$.

Remark 1. *Because of property c) above, one can use the Pfaffian to define a characteristic class of an even-dimensional oriented manifold M, called the* Euler class, *as follows: The curvature R of a riemannian metric on M is a skew-symmetric 2-form, so we can apply the Chern–Weil construction to define the cohomology class*

$$e(M) := [\mathrm{Pf}(R)] \in H^{\dim(M)}_{\mathrm{dR}}(M).$$

⋆ Exercise 2: Clifford algebras and Grassmann variables

The Clifford algebra and Grassmann variables may look similar, they are not the same: notice that in the Clifford algebra we have $\psi_i^2 = \pm 1$, whereas in the Grassmann algebra we have $\theta_i^2 = 0$. There is a relation however between the two, and the purpose of this exercise is to explore this connection. We will consider the general Clifford algebra $\mathrm{Cliff}_{p,q}$ and put $n := p + q$

a) Show that both $\mathrm{Cliff}_{p,q}$ and the Grassmann algebra on n-variables are of dimension 2^n.

b) In the Grassmann algebra on n-variables $\theta_i,\ i = 1, \dots, n$ introduce the operators

$$\hat{\psi}_i := \theta_i \pm \frac{d}{d\theta_i},$$

with the $-$-sign for $i = 1, \dots, p$ and $+$ for $i = p + 1, \dots, n$. Show that the $\hat{\psi}_i$ satisfy the commutation relations of the Clifford algebra $\mathrm{Cliff}_{p,q}$.

Exercise 3: Chirality

Consider the Clifford algebra $\mathrm{Cliff}_{p,q}$ and write $n := p+q$. Define the volume element

$$\tau := \psi_1 \cdots \psi_n.$$

a) Show that

$$\tau^2 = (-1)^{\frac{n(n-1)}{2}+p}, \quad \psi_i \tau = (-1)^{n-1} \tau \psi_i$$

b) Suppose that $\tau^2 = -1$ (for example in $\mathrm{Cliff}_{3,1}$). Show that

$$\pi^{\pm} := \frac{1 \pm i\tau}{2}$$

satisfy

$$\pi^+ + \pi^- = 1, \quad [\pi^+, \pi^-] = 0, \quad (\pi^{\pm})^2 = \pi^{\pm}$$

⋆ Exercise 4: The Euler–Dirac operator

In this exercise we turn to geometry. Let (M, g) be a compact Riemannian manifold. Using the Riemannian metric, we can identify TM with T^*M. Taking sections, this means that we can map vector fields to differential 1-forms and vice versa: we write $\tilde{X}$ for the 1-form associated to a vector field X. In local coordinates we have $\tilde{X}_j = X^i g_{ij}$. We write $\mathrm{Cliff}(TM)$ for the bundle of Clifford algebras. We consider the vector bundle $\bigwedge T^*M$, sections of this bundle are differential forms of arbitrary degree. The Riemannian metric on TM induces a metric on this bundle by the formula

$$\langle \alpha, \beta \rangle := \sum_{\substack{i_1,\dots,i_k \\ j_i,\dots,j_k}} g^{i_1 j_1} \cdots g^{i_k j_k} \alpha_{i_1 \dots i_k} \beta_{j_1 \dots j_k}, \quad \text{for } \alpha, \beta \in \Omega^k(M).$$

Often it is useful to consider an orthonormal frame $\eta^1, \dots, \eta^n$ for $\Omega^1(M)$. Writing $\alpha = \alpha_{i_1 \dots i_k} \eta^{i_1} \wedge \ldots \wedge \eta^{i_k}$ and similar for β we obtain

$$\langle \alpha, \beta \rangle = \sum_{i_1,\dots,i_k} \alpha_{i_1 \dots i_k} \beta_{i_1 \dots i_k}.$$

a) Given a vector field $X \in \mathfrak{X}(M)$, consider the operators

$$\iota_X \alpha, \quad \tilde{X} \wedge \alpha, \quad \text{for } \alpha \in \Omega^k(M)$$

Prove that these operators are adjoint to each other, i.e. show that

$$\langle \iota_X \alpha, \beta \rangle = \langle \alpha, \tilde{X} \wedge \beta \rangle$$

for all $\alpha \in \Omega^k(M)$ and $\beta \in \Omega^{k-1}(M)$.

b) Use the previous exercise to prove that $\bigwedge T^*M$, equipped with the Levi-Civita connection and the action

$$\psi(X)\alpha := \tilde{X} \wedge \alpha - \iota_X \alpha, \quad \text{for } \alpha \in \Omega^k(M),$$

is a Clifford bundle.