LECTURE 1: CALCULUS ON MANIFOLDS

1. Manifolds: a refresher

We start by briefly recalling the definition of a manifold. Informally, a manifold is a space that "locally looks like $\mathbb{R}^n$". This vague statement is formalized by the notion of an atlas: Let M be set.

Definition 1.1. A *smooth atlas* on M is given by a collection of pairs $\{(U_\alpha, x_\alpha)\}_{\alpha\in I}$, where I is some indexing set, with:

i) $U_\alpha \subset M, \alpha \in I$ are subsets that cover M: $M = \bigcup_\alpha U_\alpha$,

ii) $x_\alpha\colon U_\alpha \to \mathbb{R}^n, \alpha \in I$ are bijections $U_\alpha \cong x_\alpha(U_\alpha) \subset \mathbb{R}^n$ (called "charts") ,

iii) For all $\alpha, \beta \in I$, the maps

$$x_\beta \circ x_\alpha^{-1}\colon x_\alpha(U_\alpha \cap U_\beta) \xrightarrow{\cong} x_\beta(U_\alpha \cap U_\beta), \tag{1}$$

are *smooth* (i.e., C^∞).

Remark 1.2. We can define the notion of a *complex or holomorphic* atlas in a similar way: this time the charts (U_α, z_α) map $z_\alpha\colon U_\alpha \to \mathbb{C}^n$ and are such that the transition maps $z_\beta \circ z_\alpha^{-1}\colon \varphi_\alpha(U_\alpha \cap U_\beta) \to \mathbb{C}^n$ are holomorphic. (We say that $f\colon U \to \mathbb{C}$, $U \subset \mathbb{C}^n$ is holomorphic if, in standard coordinates $(z^1, \ldots, z^n) \in U \subset \mathbb{C}^n$, the functions given by $z^i \mapsto f(z^1, \ldots, z^i, \ldots, z^n)$, keeping the other z^j, $j \neq i$ fixed, are holomorphic functions of one variable, i.e., satisfy the Cauchy–Riemann equations:

$$\frac{\partial f_1}{\partial x^i} = \frac{\partial f_2}{\partial y^i}, \quad \frac{\partial f_2}{\partial x^i} = -\frac{\partial f_1}{\partial y^i}, \qquad f = f_1 + \sqrt{-1}f_2,\ z^i = x^i + \sqrt{-1}y^i$$

Definition 1.3. A *smooth* manifold is a set equipped with a *smooth* atlas. A *complex* manifold is a set equipped with a *complex* atlas.

Mathematical Remark 1.4. This definition is not quite precise. There are two mathematical objections to this definition, in the sense that the definition above is not quite what we want.

i) An atlas $\{(U_\alpha, x_\alpha)\}_{\alpha\in I}$ on M induces a *topology* by declaring a set U to be open if and only if $x_\alpha(U \cap U_\alpha) \subset \mathbb{R}^n$ is open for all $\alpha \in I$. To avoid pathological behavior, we have to assume this topology to be Hausdorff and second countable. This excludes for example the possibility to turn $\mathbb{R}^n$ itself into a k-dimensional manifold for $k < n$. In the usual mathematical definition, one starts with a topological space (Haussdorf and second countable) and defines an atlas as above,

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assuming in addition that $U_\alpha \subset M$ are *open*. It can be checked that the induced atlas-topology agrees with the original one.

ii) The mathematical definition uses the notion of a *maximal atlas*: We say that two atlases $\{(U_\alpha, x_\alpha)\}_{\alpha\in I}$ and $\{(U_{\alpha'}, x_{\alpha'})\}_{\alpha'\in J}$ are compatible if the collection $\{(U_\alpha, \varphi_\alpha)\}_{\alpha\in I} \coprod \{(U_{\alpha'}, \varphi_{\alpha'})\}_{\alpha'\in J}$ is still an atlas. (For this one needs to check condition *iii*) above.) Being compatible defines an equivalence relation, and a maximal atlas is, by definition, an equivalence class of charts.

Remark 1.5. A complex manifold is in particular a smooth manifold, because of the fact that a holomorphic function is smooth. For a local complex chart $(z^1_\alpha, \dots z^n_\alpha)$ the underlying smooth chart is given as $(x^1_\alpha, y^1_\alpha, \dots, x^n_\alpha, y^n_\alpha)$ or $(x^1_\alpha, \dots, x^n_\alpha, y^1_\alpha, \dots, y^n_\alpha)$ by writing $z^i_\alpha = x^i_\alpha + \sqrt{-1} y^i_\alpha$ (the order of coordinates may matter when one assigns an *orientation*). Our focus in this course is on smooth manifolds, but it is convenient to have the concept of a complex manifold at hand in some cases.

Given an atlas $\{(U_\alpha, x_\alpha)\}_{\alpha\in J}$ for a manifold M, we can write out $x_\alpha \colon U_\alpha \to \mathbb{R}^n$ in coordinates:

$$x_\alpha(x) := (x^1_\alpha(x), \dots, x^n_\alpha(x)), \quad x \in U_\alpha.$$

If $x \in U_\alpha \cap U_\beta$ we have two charts around x and the local coordinates are related by

$$(x_\beta \circ x_\alpha^{-1})(x^1_\alpha(x), \dots, x^n_\alpha(x)) = (x^1_\beta(x), \dots, x^n_\beta(x)) \tag{2}$$

Remark 1.6 ("clutching and pasting")**.** A slightly different point of view on manifolds is given by focusing on the *transition functions* $\varphi_{\alpha\beta} := x_\alpha \circ x_\beta^{-1}$, which are by definition local diffeomorphisms on $\mathbb{R}^n$. We now think of M as consisting of pieces $\tilde{U}_\alpha := x_\alpha(U_\alpha) \subset \mathbb{R}^n$ which are glued together using the transition functions $\varphi_{\alpha\beta}$:

$$M \cong \coprod_{\alpha\in I} \tilde{U}_\alpha \Big/ \sim, \tag{3}$$

where $x \sim y$ means that $\varphi_{\alpha\beta}(x) = y$, for $x \in \tilde{U}_\alpha$ and $y \in \tilde{U}_\beta$. The quotient makes sense because the transition functions $\{\varphi_{\alpha\beta}\}_{\alpha,\beta\in I}$ satisfy the following properties ensuring that $\sim$ defines an equivalence relation:

$$\varphi_{\alpha\alpha} = \mathrm{id} \quad \text{(Reflexivity)} \tag{4a}$$

$$\varphi_{\beta\alpha} = \varphi_{\alpha\beta}^{-1} \quad \text{(Symmetry)} \tag{4b}$$

$$\varphi_{\alpha\beta} \circ \varphi_{\beta\gamma} = \varphi_{\alpha\gamma} \quad \text{(Transitivity)} \tag{4c}$$

Conversely, given a bunch of open subsets $U_\alpha \subset \mathbb{R}^n$, together with local diffeomorphisms $\varphi_{\alpha\beta} \colon U \to U'$ with $U \subset U_\alpha$ and $U' \subset U_\beta$, satisfying the three properties above, we can define a smooth manifold structure on M defined by (3).

Example 1.7 (Projective spaces)**.** Consider $\mathbb{P}^n = \mathbb{CP}^n$, the space of all one-dimensional lines in $\mathbb{C}^{n+1}$. We denote the usual homogeneous coordinates by $[z^0, \ldots, z^n] \in \mathbb{CP}^n$. The standard manifold charts are given by

$$U_i = \{[z^0, \ldots, z^n] \in \mathbb{CP}^n,\ z^i \neq 0\},$$

with coordinate charts $\varphi_i : U_i \to \mathbb{C}^n$, given by

$$\varphi_i([z^0, \ldots, z^n]) = \left(\frac{z^0}{z^i}, \ldots, \frac{z^{i-1}}{z^i}, \frac{z^{i+1}}{z^i}, \ldots, \frac{z^n}{z^i}\right). \tag{5}$$

The transition maps $\varphi_{ij} = \varphi_i \circ \varphi_j^{-1}$ are given by $\varphi_{ij}(z^0, \ldots, z^n) = (z^i)^{-1} \cdot (z^0, \ldots, z^n)$, where we have identified $\varphi_j(U_j)$ with the affine hyperplane $\{(z^0, \ldots, z^n),\ z^j = 1\} \subset \mathbb{C}^{n+1}$.

The main idea behind the definition of a manifold is that we can use the local structure on M, as being equal to $\mathbb{R}^n$, to introduce key concepts from analysis such as smooth functions, mappings and vector fields on M. All we have to do is phrase the definition in terms of local charts, and check that the definition is invariant under change of coordinates (2).

As a simple example, define a function f on M to be smooth if for each chart $(U_\alpha, \varphi_\alpha)$ the function $f_\alpha := f \circ \varphi_\alpha^{-1}$ is a smooth function on $\mathbb{R}^n$. By the chain rule for derivatives, this definition is independent of the choice of local chart, and therefore makes sense on the manifold M. The space of smooth functions on M is denoted by $C^\infty(M)$. From the point of view of Remark 1.6, a smooth function $f \in C^\infty(M)$ is given by a collection of smooth functions $\{f_\alpha \in C^\infty(U_\alpha)\}_{\alpha \in I}$ on pieces of $\mathbb{R}^n$, that agree on overlaps:

$$f_\alpha(x) = f_\beta(\varphi_{\alpha\beta}(x)), \quad \text{for all } x \in U_\alpha \cap U_\beta.$$

Another, more general example is given by the notion of a *smooth map between manifolds*: A map $f : M \to N$ is said to be smooth if, for atlases (U_α, x_α) for M and (V_β, y_β) of N, the composition

$$y_\beta \circ f \circ x_\alpha^{-1} \colon U_\alpha \to V_\beta,$$

is smooth. By the chain rule, this notion of smoothness is independent of local coordinates.

2. Tangent bundle and vector fields

Recall that for an open subset $U \subset \mathbb{R}^n$, its tangent bundle is defined to be $TU := U \times \mathbb{R}^n$. If we write $x \in U$ in coordinates $x = (x^1, \ldots, x^n)$, the tangent space $T_xU = \mathbb{R}^n$ to U at x has the basis $\{\partial/\partial x^1, \ldots, \partial/\partial x^n\}$. Let $\varphi \colon U \to V$ with $V \subset \mathbb{R}^n$ be a diffeomorphism sending $x \in U$ to $\varphi(x) = y(x) = (y^1, \ldots, y^n) \in V$. Its tangent map acts on the tangent space by the Jacobi matrix:

$$T_x\varphi\left(\frac{\partial}{\partial x^i}\right) = \sum_j \frac{\partial y^j}{\partial x^i}(x)\frac{\partial}{\partial y^j} \in T_{\varphi(x)}V. \tag{6}$$

Varying the basepoint $x \in U$, these matrices together form the tangent mapping $T\varphi : TU \to TV$, and by the chain rule for Jacobi matrices we see that

$$T(\psi \circ \varphi) = T\psi \circ T\varphi, \tag{7}$$

where $\psi\colon V \to W$ is another diffeomorphism.

Suppose now that we are given a manifold structure on M provided by an atlas $\{(U_\alpha, x_\alpha)\}_{\alpha \in I}$, with associated "gluing data" $\{\tilde{U}_\alpha := x_\alpha(U_\alpha) \subset \mathbb{R}^n\}$ with transition functions $\varphi_{\alpha\beta} := x_\beta \circ x_\alpha^{-1}$ satisfying the conditions (4a)–(4c). Then the subsets

$$\{T\tilde{U}_\alpha = \tilde{U}_\alpha \times \mathbb{R}^n \subset \mathbb{R}^{2n}\},$$

together with the local diffeomorphisms $\psi_{\alpha\beta} := T\varphi_{\alpha\beta}\colon T\tilde{U}_\alpha \to T\tilde{U}_\beta$, satisfy the same "cocycle conditions" (4a)–(4c) by the chain rule (7). It therefore defines another $2n$-dimensional manifold, called the tangent bundle TM of M. It comes equipped with a canonical projection $\pi\colon TM \to M$, and the fiber $\pi^{-1}(x) = T_xM$ is called the tangent space of M at x. In the "gluing picture" of Remark 1.6 we have

$$TM := \coprod_\alpha (\tilde{U}_\alpha \times \mathbb{R}^n) \Big/ \sim,$$

and we therefore see that, given a smooth map $f\colon M \to N$, the collection

$$\coprod_{\alpha,\beta} T(y_\beta \circ f \circ x_\alpha^{-1})\colon \coprod_\alpha T(x_\alpha(U_\alpha)) \to \coprod_\beta T(y_\beta(V_\beta)),$$

where $\{(V_\beta, y_\beta)\}$ is an atlas for N, descends to the quotient to define a smooth map $Tf\colon TM \to TN$.

Smooth sections of the projection π, i.e. smooth maps $X\colon M \to TM$ satisfying $\pi \circ X = \mathrm{id}_M$, are called *vector fields*. In local coordinates $(x_\alpha^1, \ldots, x_\alpha^n)$ on U_α a vector field can be written as

$$X = \sum_i X_\alpha^i(x)\frac{\partial}{\partial x_\alpha^i},$$

where the "coefficients" $X_\alpha^i(x)$ are smooth functions of $x \in U_\alpha$. When $x \in U_\alpha \cap U_\beta$ and we change to coordinates $(x_\beta^1, \ldots, x_\beta^n)$, we see from (6) that

$$X = \sum_i X_\beta^i(x)\frac{\partial}{\partial x_\beta^i}, \quad \text{with } X_\beta^i = \sum_j X_\alpha^j \frac{\partial x_\beta^i}{\partial x_\alpha^j}.$$

This explains the physicists' point of view on vector fields: for them, a vector field is given by a vector of functions $X_\alpha^i(x)$ in local coordinates, which transforms as above under coordinate changes. We write $\mathfrak{X}(M)$ for the vector space of all vector fields on M

The following properties are easy to check:

- A diffeomorphism $f\colon M \to N$ induces a push forward map $f_*\colon \mathfrak{X}(M) \to \mathfrak{X}(N)$ by the formula

$$f_*X(y) := T_{f^{-1}(y)}(X)$$

- A vector field $X \in \mathfrak{X}(M)$ acts on $C^\infty(M)$ by taking "directional derivatives" $f \mapsto X(f)$. Once again in local coordinates

$$X(f)(x) := \sum_i X^i_\alpha(x)\frac{\partial f}{\partial x^i}. \tag{8}$$

- There is a "Lie bracket" of vector fields given in local coordinates

$$[X,Y] = \sum_{i,j=1}^n \left(X^i_\alpha(x)\frac{\partial Y^j(x)}{\partial x^i_\alpha} - Y^i_\alpha(x)\frac{\partial X^j(x)}{\partial x^i_\alpha}\right)\frac{\partial}{\partial x^j_\alpha}$$

3. Cotangent bundle and differential forms

The cotangent bundle is dual to the tangent bundle. For $U \subset \mathbb{R}^n$ we define the cotangent space T^*_xU, $x \in U$ as the vector space with basis $\{dx^i\}_{i=1}^n$ dual to the basis $\{\partial/\partial x^i\}_{i=1}^n$ of T_xU:

$$dx^i\left(\frac{\partial}{\partial x^j}\right) = \delta^i{}_j.$$

This duality implies, by the rule (6), that a diffeomorphism $\varphi\colon U \to V$ with $V \subset \mathbb{R}^n$ sending $x \in U$ to $\varphi(x) = y(x) = (y^1,\dots,y^n) \in V$ sends

$$T^*_y\varphi(dy^i) = \sum_{j=1}^n \frac{\partial y^i}{\partial x^j}dx^j, \tag{9}$$

i.e. the covectors dx^i transform according to the *inverse* of the Jacobi matrix. Again, the pieces $T^*\tilde{U}_\alpha$ together with the inverse of the Jacobi matrices of the transition functions $T^*\varphi_\alpha$ satisfy the conditions (4a)–(4c) and define a manifold called the *cotangent bundle* T^*M. Again there is an obvious smooth projection $\pi\colon T^*M \to M$ and sections are called *differential 1-forms* (these are sometimes called covector fields). By definition a differential 1-form θ maps a point $x \in M$ to a linear map $T_xM \to \mathbb{R}$. In local coordinates, θ can be written as

$$\theta = \sum_\alpha \theta^\alpha_i(x)dx^i.$$

Changing to another chart φ_β, equation (9) implies the transformation rule

$$\theta^\beta_i = \sum_j \frac{\partial x^j_\alpha}{\partial x^i_\beta}\theta^\alpha_j.$$

We shall write $\Omega^1(M)$ for the vector space of all differential 1-forms on M. In higher degrees, a k-form ω maps a point $x \in M$ to an antisymmetric linear k-form on T_xM, i.e. an element in $\bigwedge^k T^*_xM$. To define what it means for ω to be smooth, we have to define a manifold structure on $\bigwedge T^*M$. In brief: locally for $U \subset \mathbb{R}^n$, a basis of $\bigwedge^k T^*_x U$ is given by $\{dx^{i_i} \wedge \ldots \wedge dx^{i_k}\}$ so that ω can be written in local coordinates as

$$\omega = \sum_{i_1,\ldots,i_k} \omega^\alpha_{i_1,\ldots,i_k} dx^{i_1} \wedge \ldots \wedge dx^{i_k},$$

where $\omega^\alpha_{i_1,\ldots,i_k}$ is a collection of smooth functions on U_α, which are *antisymmetric* under permutations of the k-indices $i_1, \ldots, i_k$. The transformation rules under changes of local coordinates are given by

$$\omega^\beta_{i_1,\ldots,i_k} = \sum_{j_1,\ldots,j_k} \frac{\partial x^{j_1}_\alpha}{\partial x^{i_1}_\beta} \cdots \frac{\partial x^{j_k}_\alpha}{\partial x^{i_k}_\beta} \omega^\alpha_{j_1,\ldots,j_k} \tag{10}$$

Again, the following properties are easy to derive:

- a smooth map $f\colon M \to N$ induces a pull-back map $f^*\colon \Omega^k(N) \to \Omega^k(M)$ defined as

$$f^*\omega(x)(V_1, \ldots, V_k) := \omega(f(x))(T_x f(V_1), \ldots, T_x f(V_k)), \quad \text{for } x \in M,\ V_1, \ldots, V_k \in T_x M.$$

- By definition, there is a pairing

$$\Omega^1(M) \times \mathfrak{X}(M) \to \mathbb{R}, \quad (\theta, X) \mapsto \theta(X).$$

- The total derivative of a function $f \in C^\infty(M)$, written in local coordinates

$$df|_{U_\alpha} = \sum_i \frac{\partial f}{\partial x^i} dx^i,$$

  defines a 1-form $df \in \Omega^1(M)$. This is consistent with (or can be derived from) the action (8) of vector fields on functions: in other words, we can now write $X(f) := df(X)$.
- The formula

$$d\omega|_{U_\alpha} := \sum_{i,i_1,\ldots,i_k} \frac{\partial \omega^\alpha_{i_1,\ldots,i_k}}{\partial x^i} dx^i \wedge dx^{i_1} \wedge \ldots \wedge dx^{i_k},$$

  defines an operator $d\colon \Omega^k(M) \to \Omega^{k+1}(M)$, called the *exterior derivative*. There is a coordinate independent formula for this derivative as follows

$$\begin{aligned} d\omega(X_0, \ldots, X_k) = &\sum_{i=0}^{p} (-1)^i X_i\big(\omega(X_0, \ldots, \hat{X}_i, \ldots, X_k)\big) \\ &+ \sum_{i<j} (-1)^{i+j} \omega([X_i, X_j], X_0, \ldots, \hat{X}_i, \ldots, \hat{X}_j, \ldots, X_k), \end{aligned} \tag{11}$$

  where the hat means that we omit that term from the argument. It is not difficult to prove that $dd\omega = 0$ for all $\omega \in \Omega^k(M)$, because we can change the order in which we take partial derivatives.

- For any vector field $X \in \mathfrak{X}(M)$ there is an operator $\iota_X \colon \Omega^k(M) \to \Omega^{k-1}(M)$ called contraction with X and defined as

$$(\iota_X\omega) := \omega(x)(X, -, \dots, -) \colon \bigwedge^{k-1} T_xM \to \mathbb{R}.$$

- *Cartan's magic formula*

$$L_X := \iota_X \circ d + d \circ \iota_X : \Omega^k(M) \to \Omega^k(M)$$

defines an action of vector fields on k-forms. This extends the action of $\mathfrak{X}(M)$ on $C^\infty(M) = \Omega^0(M)$.

- Using a partition of unity, the integral $\int_M \alpha$ of an n-form over an n-dimensional manifold is well defined, i.e., independent of the choice of local coordinates. This is because the transformation rule (10) for an n-form is exactly given by multiplication with the Jacobian (i.e., the determinant of the Jacobi matrix) which appears in the change of coordinates of multidimensional integrals. When M is a manifold with boundary ∂M, Stokes' theorem asserts that

$$\int_M d\beta = \int_{\partial M} \beta, \quad \text{for } \beta \in \Omega^{k-1}(M).$$

4. Calculus on complex manifolds

When the manifold M is complex, the differential calculus on M is a bit richer when we complexify the tangent bundle. Let us first again consider the local situation $U \subset \mathbb{C}^n$. Using the coordinates $z^i = x^i + \sqrt{-1}y^i$ with $i = 1, \dots, n$, a real basis for the tangent space T_zU is given by $\{\partial/\partial x^i, \partial/\partial y^i\}_{i=1}^n$. On $\mathbb{C}^n$, we can also use the complex coordinates $(z^i, \bar{z}^i)$, so it is convenient to introduce the complex basis

$$\frac{\partial}{\partial z^i} = \frac{1}{2}\left(\frac{\partial}{\partial x^i} - \sqrt{-1}\frac{\partial}{\partial y^i}\right), \quad \frac{\partial}{\partial \bar{z}^i} = \frac{1}{2}\left(\frac{\partial}{\partial x^i} + \sqrt{-1}\frac{\partial}{\partial y^i}\right).$$

The complex tangent space is then defined as $T_z^{\mathbb{C}}U = \mathrm{span}_{\mathbb{C}}\{\partial/\partial z^i, \partial/\partial \bar{z}^i, i = 1, \dots, n\}$. In a similar way we define the *complex tangent bundle* $T^{\mathbb{C}}M$: this is a complex manifold, just like the tangent bundle to a smooth manifold is a smooth manifold in its own right.

In the notation above the Cauchy–Riemann equations are given by the simple equation $\partial f/\partial \bar{z} = 0$. The transition functions $\varphi \colon z^i \mapsto w^i(z^1, \dots, z^n)$ of the complex manifold are by definition holomorphic, so $\partial w/\partial \bar{z}^i = 0$ and therefore

$$T_z\varphi\left(\frac{\partial}{\partial z^i}\right) = \sum_{j=1}^n \frac{\partial w^j}{\partial z^i}\frac{\partial}{\partial w^j}, \quad T_z\varphi\left(\frac{\partial}{\partial \bar{z}^i}\right) = \sum_{j=1}^n \frac{\partial \bar{w}^j}{\partial \bar{z}^i}\frac{\partial}{\partial \bar{w}^j}$$

It follows that the transition functions for the complex tangent bundle $T^{\mathbb{C}}M$ have the form

$$\begin{pmatrix} \partial w^j/\partial z^i & 0 \\ 0 & \partial \bar{w}^j/\partial \bar{z}^i \end{pmatrix}.$$

Because of this special shape of the transition matrix, with off-diagonal terms in this 2×2 matrix equal to zero, the complex tangent bundle splits as

$$T^{\mathbb{C}}M = T^{(1,0)}M \oplus T^{(0,1)}M,$$

with $T_z^{(1,0)}M$ the subspace spanned by $\partial/\partial z^i$ and $T^{(0,1)}M$ spanned by $\partial/\partial \bar{z}^i$ in a local complex chart $(z^1, \ldots, z^n)$. Dually this leads to a decomposition of the space of complex differential 1-forms (these are sections of the complex cotangent bundle)

$$\Omega^1_{\mathbb{C}}(M) = \Omega^{(1,0)}(M) \oplus \Omega^{(0,1)}(M),$$

where $\alpha \in \Omega^{(1,0)}(M)$ when in local holomorphic coordinates $z = (z^1, \ldots, z^n)$ can be written as $\alpha = \sum_i \alpha_i(z, \bar{z}) dz^i$ (no $d\bar{z}^i$'s) and $\beta \in \Omega^{(0,1)}(M)$ when $\beta = \sum_i \beta_i(z, \bar{z}) d\bar{z}^i$ (no dz^i's). Going over to higher degree differential forms, we get

$$\Omega^k_{\mathbb{C}}(M) = \bigoplus_{p+q=k} \Omega^{(p,q)}(M),$$

with $\alpha \in \Omega^{(p,q)}(M)$ if locally, in some holomorphic coordinate system $z = (z^1, \ldots, z^n)$ we have

$$\alpha = \sum_{i_1,\ldots,i_p,j_1,\ldots,j_q} \alpha_{i_1,\ldots,i_p,j_1,\ldots,j_q}(z, \bar{z}) dz^{i_1} \wedge \ldots \wedge dz^{i_p} \wedge d\bar{z}^{j_1} \wedge \ldots \wedge d\bar{z}^{j_q}$$

The exterior differential

$$d\alpha = \sum_{i,i_1,\ldots,i_p,j_1,\ldots,j_q} \left(\frac{\partial \alpha_{i_1,\ldots,i_p,j_1,\ldots,j_q}}{\partial z^i} dz^i + \frac{\partial \alpha_{i_1,\ldots,i_p,j_1,\ldots,j_q}}{\partial \bar{z}^j} d\bar{z}^i \right) \wedge dz^{i_1} \wedge \ldots \wedge dz^{i_p} \wedge d\bar{z}^{j_1} \wedge \ldots \wedge d\bar{z}^{j_q},$$

accordingly splits as $d = \partial + \bar{\partial}$, where $\partial : \Omega^{(p,q)}(M) \to \Omega^{(p+1,q)}(M)$ and $\bar{\partial} : \Omega^{(p,q)}(M) \to \Omega^{(p,q+1)}(M)$. The fundamental property $d \circ d = 0$ of the exterior differential now amounts to

$$\partial \circ \partial = 0, \quad \partial \circ \bar{\partial} + \bar{\partial} \circ \partial = 0, \quad \bar{\partial} \circ \bar{\partial} = 0.$$