LECTURE 11: DIRAC OPERATORS AND SUPERSYMMETRY

1. The index of elliptic differential operators

Let E and F be vector bundles over a manifold M. A *differential operator* of degree k is an operator $D : \Gamma^\infty(M, E) \to \Gamma^\infty(M, F)$ with the property that it can be written as

$$D = \sum_{|I| \leq k} c_I(x) \frac{\partial^{|I|}}{\partial x^I},$$

in local coordinates $(x^1, \ldots, x^n) : U \to \mathbb{R}^n$ and local trivializations $E|_U \cong \mathbb{C}^p$, $F|_U \cong \mathbb{C}^q$, and where $c_I(x)$ is an $q \times p$-matrix valued function on U.

The *principal symbol*[1] of such a differential operator is given locally by

$$\sigma_k(D)(x, \xi) := \sum_{|I|=k} c_I(x) \xi^I. \tag{1}$$

We will see in the exercises that this defines a section of the vector bundle $\pi^*\mathrm{Hom}(E, F)$ over T^*M, where $\pi : T^*M \to M$ is the canonical projection. This gives a coordinate-independent meaning to the symbol. With this, the condition of being elliptic means that the section $\sigma(D)$ takes values in invertible operators outside $M \subset T^*M$. The importance of ellipticity for index theory is given by the following:

Theorem 1.1. *An elliptic differential operator on a compact manifold is Fredholm.*

As we have seen this implies that D has a well-defined index

$$\mathrm{index}(D) := \dim \ker(D) - \dim \ker(D^*) \in \mathbb{Z}.$$

Recall that the *Dirac operator* of a Clifford bundle (S, ∇^S) is defined as the composition

$$\Gamma(M, S) \xrightarrow{\nabla^S} \Gamma^\infty(M, T^*M \otimes S) \xrightarrow{T^*M \overset{g}{\cong} TM} \Gamma^\infty(M, TM \otimes S) \xrightarrow{\substack{\text{Clifford} \\ \text{multiplication}}} \Gamma^\infty(M, S).$$

This means that locally in a local orthonormal system of vector fields $\{e_i\}_{i=1}^n$ on $U \subset M$. Then the definition of the Dirac operator above amounts to the local formula

$$Ds = \sum_{i=1}^n \psi(e_i) \nabla^S_{e_i} s. \tag{2}$$

Locally we can write the connection as $\nabla^S = d + A$, so we see that the Dirac operator is a differential operator of degree 1. We will see in an exercise that the Dirac operator on a riemannian manifold is elliptic, so it looks like a good candidate for index theory. But

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[1]Quite often we simply write symbol, omitting the principal.

here is a small disappointment: we can always choose a hermitian metric $h_x : S_x \times S_x \to \mathbb{C}$ on S with the following properties:

- h is compatible with the connection ∇^S:
$$dh(s_, s_2) = h(\nabla^S(s_1), s_2) + h(s_1, \nabla^S(s_2)), \quad \text{for all local sections } s_1, s_2.$$
- Clifford multiplication is skew-adjoint:
$$h(\psi(X)s_1, s_2) = -h(s_1, \psi(X)s_2), \quad \text{for all } X \in \mathfrak{X}(M).$$

With the hermitian metric h we can define the L^2-inner product on sections of S by
$$\langle s_1, s_2 \rangle := \int_M h(s_1, s_2)\Omega_g,$$
where Ω_g is the volume form of the riemannian metric g. We then have:

Lemma 1.2. *The Dirac operator D is selfadjoint, therefore has index* 0.

Proof. We will first show that for two local sections s_1 and s_2, the difference $h(Ds_1, s_2) - h(s_1, Ds_2)$ is a divergence. Using the local expression (2) for D in a synchronous frame at $x \in M$[2] we see that
$$\begin{aligned} h(Ds_1, s_2) - h(s_1, Ds_2) &= \sum_{i=1}^{n} (h(\psi(e_i)\nabla^S_{e_i} s_1, s_2) - h(s_1, \psi(e_i)\nabla^S_{e_i} s_2)) \\ &= \sum_{i=1}^{n} (h(\nabla^S_{e_i}\psi(e_i)s_1, s_2) + h(\psi(e_i)s_1, \nabla^S_{e_i} s_2)) \\ &= \sum_{i=1}^{n} dh(\psi(e_i)s_1, s_2)(e_i) \\ &= d^*\alpha, \end{aligned}$$
where $\alpha \in \Omega^1(M)$ is the 1-form defined as $\alpha(X) := h(\psi(X)s_1, s_2)$ for $X \in \mathfrak{X}(M)$. With this we now have
$$\begin{aligned} \langle Ds_1, s_2 \rangle - \langle s_1, Ds_2 \rangle &= \int_M d^*\alpha\Omega_g \\ &= \pm \int_M d(\star\alpha) \\ &= 0, \end{aligned}$$
since M is closed, i.e., has no boundary. □

This may be dissappointing, but there is a way out. For this we have to introduce a $\mathbb{Z}_2$-grading. For this the framework of supersymmetry is useful, so we first turn to that.

[2]This means that $\nabla^{\mathrm{LC}}_{e_i} e_j = 0$ and $[e_i, e_j] = 0$ at x.

2. Supersymmetric Quantum Mechanics

Quantum Mechanics is usually described by a Hilbert space $\mathcal{H}$ and an algebra of observables $\mathcal{A}$. One of these observables is usually given a special status: the Hamiltonian H controls the time evolution of the system via the Schrödinger equation.

In *supersymmetric Quantum Mechanics* the Hilbert space carries a $\mathbb{Z}_2$-grading: $\mathcal{H} = \mathcal{H}^+ \oplus \mathcal{H}^-$, states in $\mathcal{H}^+$ are called *bosonic* and those in $\mathcal{H}^-$ *fermionic*. Any operator A on $\mathcal{H}$ can therefore be decomposed as

$$A = \begin{pmatrix} A_{++} & A_{+-} \\ A_{-+} & A_{--} \end{pmatrix}. \tag{3}$$

When $A(\mathcal{H}^\pm) \subset \mathcal{H}^\pm$, i.e., only diagonal terms in the decomposition above, we say that A is *even*. When $A(\mathcal{H}^\pm) \subset \mathcal{H}^\mp$, i.e., off-diagonal terms, we say that A is *odd*.

The basic rule of "super-mathematics" is that we do everything analogous to the ungraded world, but "with $\pm$-signs. Accordingly, we define the *supertrace* of an operator A as in (3) as

$$\mathrm{Tr}_s(A) := \mathrm{Tr}(A_{++}) - \mathrm{Tr}_s(A_{--}).$$

If we insist on this supertrace to vanish on commutators,

$$\mathrm{Tr}_s([A, B]) = 0,$$

we must modify the the commutator as follows:

$$[A, B] := A \circ B - (-1)^{|A||B|} B \circ A,$$

where A and B are assumed to be homogeneous, i.e., either even or odd and the degree $|A|$ is 0 for even and 1 for odd A. To compute this super-commutator for general A and B we must decompose them into homogeneous pieces. The formula above for the super-commutator signifies the basic rule about signs in super-mathematics: if a formula involves moving two operators A and B passed each other, we add a sign $(-1)^{|A||B|}$.

The principal assumption in supersymmetric quantum mechanics is now that there exists a selfadjoint odd operator Q on $\mathcal{H}$ such that

$$H = Q^2.$$

Since $Q = Q^*$, this implies that $H \geq 0$, i.e., the spectrum of H is given by eigenvalues $\lambda_n \geq 0$. We shall write $\lambda_0 = 0$ for the lowest eigenvalue, and $\mathcal{H}_n$ for the energy eigenspace corresponding to the eigenvalue λ_n. The $\mathbb{Z}_2$-grading of $\mathcal{H}$ gives $\mathcal{H}_n = \mathcal{H}_n^+ \oplus \mathcal{H}_n^-$, and the operator Q induces an isomorphism $\mathcal{H}_n^+ \cong \mathcal{H}_n^-$ for $n > 0$. We will see in the exercises that it follows from this that the so-called *Witten index*

$$\mathrm{Tr}_s(e^{-\beta H}),$$

which a priori seems to depend on $\beta > 0$, in fact is independent of β and equals

$$\mathrm{Tr}_s(e^{-\beta H}) = \dim \mathcal{H}_0^+ - \dim \mathcal{H}_0^- = \mathrm{index}(Q^+). \tag{4}$$

Example 2.1. Let (M, g) be a riemannian manfiold. Borrowing an idea from general relativity, consider the classical mechanical system described by the Lagrangian

$$L(q) := \frac{1}{2} g(\dot{q}, \dot{q}),$$

describing the motion of a free particle on M. As we all know from general relativity, the Euler–Lagrange equations for this Lagrangian are given by the geodesic equations

$$\nabla^{\mathrm{LC}}_{\dot{q}} \dot{q} = 0, e$$

implying that free particles move along geodesics.

Simple as this system may be, quantization is not completely straightforward. Of course we work on the Hilbert space $\mathcal{H} := L^2(M, \Omega_g)$, but how to quantize the Hamiltonian[3] $h(p) = \frac{1}{2} g(p, p)$? Of course the naive approach is to replace p by $i\frac{\partial}{\partial q}$, but this prescription is highly coordinate dependent! Thinking a bit more, and looking at the definition (1) of the symbol, we are looking at the problem of finding a (differential) operator whose symbol equals h. But then it is clear that there can only be one serious candidate, namely the Laplacian:

$$H = -\frac{1}{2}\Delta_g = \frac{1}{2\sqrt{\det(g)}} \sum_{i,j=1}^{n} \frac{\partial}{\partial x^i} \left(\sqrt{\det(g)} g^{ij} \frac{\partial}{\partial x^j} \right).$$

This quantum mechanical system does not have supersymmetry. One way of getting supersymmetry in the game is to extend the Hilbert space to included differential forms:

$$\mathcal{H} := \bigoplus_{p \geq 0} \Omega^p_{L^2}(M),$$

where $\Omega^p_{L^2}(M)$ denotes the L^2-completion of $\Omega^p(M)$ in the inner product

$$\langle \alpha, \beta \rangle := \int_M \alpha \wedge \star\beta.$$

As Hamiltonian we now take the Laplace–Beltrami operator

$$H := \Delta_g := dd^* + d^* d.$$

The supersymmetry of this model is defined as follows: First of all, we can decompose $\Omega^\bullet(M) = \Omega^{\mathrm{ev}}(M) \oplus \Omega^{\mathrm{odd}}(M)$, which induces a $\mathbb{Z}/2$-grading of the Hilbert space

$$\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-.$$

[3] Remark that mathematically the Lagrangian is function $L : TM \to \mathbb{R}$ on the tangent bundle, whereas the Hamiltonian $h : T^*M \to \mathbb{R}$ lives on the *co*tangent bundle. The two are related by the Legendre transform.

(+ corresponds to even-, and − to odd-degree differential forms.) With respect to this grading we can view the de Rham operator as an unbounded operator

$$d : \mathcal{H}_\pm \to \mathcal{H}_\mp.$$

Together with its adjoint we can now define

$$Q := \begin{pmatrix} 0 & d + d^* \\ d + d^* & 0 \end{pmatrix}.$$

Then we find, by virtue of $d^2 = 0 = (d^*)^2$, that

$$Q^2 = \begin{pmatrix} dd^* + d^*d & 0 \\ 0 & dd^* + d^*d \end{pmatrix} = \Delta_g.$$

Of course, we have seen the complex with $Q = d + d^*$ last week, where it was shown that the index is given by the Euler characteristic. Applying (4), we see that

$$\mathrm{Tr}_s(e^{-\beta\Delta_g}) = \mathrm{index}(Q^+) = \chi(M).$$

We will see in the next section, that this is part of a more general picture where any Dirac operator plays the role of supercharge operator in supersymmetric quantum mechanics.

3. The McKean–Singer formula

In this section we will generalize the last example to the general case of Dirac operators.

3.1. **The square of the Dirac operator.**

Let (S, ∇^S) be a Clifford bundle over a riemannian manifold (M, g). Recall that in Minkowski spacetime the Dirac operator was a square root of the Laplacian. On a general manifold this is not quite the case, but we always have the following:

Proposition 3.1. *Let D be the Dirac operator associated to a Clifford bundle (S, ∇^S). The square of D has the same principal symbol as the Laplacian:*

$$\sigma_2(D^2) = \sigma_2(\Delta_g)$$

Proof. We use the local expression (2) of the Dirac operator to write down the square

$$\begin{aligned}
D^2 s &= \sum_{i,j} \psi(e_i)\nabla_{e_i}(\psi(e_j)\nabla_{e_j}(s)) \\
&= \sum_{i,j} (\psi(e_i)\psi(e_j)\nabla_{e_i}\nabla_{e_j}(s) + \psi(e_i)\psi(\nabla^{\mathrm{LC}}_{e_i} e_j)\nabla_{e_j}(s)) \\
&= -\sum_i \nabla^2_{e_i}(s) + \sum_{i<j} \psi(e_i)\psi(e_j)(\nabla^S_{e_i}\nabla^S_{e_j} - \nabla^S_{e_j}\nabla^S_{e_i})(s) + \sum_{i,j} \psi(e_i)\psi(\nabla^{\mathrm{LC}}_{e_i} e_j)\nabla_{e_j}(s)) \\
&= -\sum_i \nabla^2_{e_i}(s) + \sum_{i<j} \psi(e_i)\psi(e_j)F(\nabla^S)(e_i,e_j)(s) + \sum_{i,j} \psi(e_i)\psi(\nabla^{\mathrm{LC}}_{e_i} e_j)\nabla_{e_j}(s)) \\
&= -\sum_i \nabla^2_{e_i}(s) + \text{lower order terms.}
\end{aligned}$$

In a local trivialization we have $\nabla^S = d + A$, and the result follows. □

In general, differential operators with symbol equal to that of the Laplace operators are sometimes called *generalized Laplacians*.

3.2. **The Dirac operator and supersymmetry.** Our main example of a supersymmetric quantum mechanical system is given by the Dirac operator on an even dimensional manifold. When the dimension of M is $n = 2m$, the *chirality operator*

$$\Gamma := i^m \tau = i^m \psi_1 \cdots \psi_{2m},$$

squares to 1 and anti-commutes with $TM \subset \mathrm{Cliff}(TM)$, c.f. Lemma **??**. Therefore, any Clifford bundle S can be decomposed into eigenspaces $S^\pm$ of Γ and the Dirac operator maps sections of $S^\pm$ into $S^\mp$. The Hilbert space $\mathcal{H} := L^2(M, S)$ inherits this $\mathbb{Z}_2$-grading, $\mathcal{H} = \mathcal{H}^+ \oplus \mathcal{H}^-$ and we can write

$$D = \begin{pmatrix} 0 & D^+ \\ D^- & 0 \end{pmatrix}.$$

As we have seen, the square $H := D^2$ is a generalized Laplacian: the first term $(\nabla^S)^*\nabla^S$ in Proposition 3.1 is a positive operator because it can be written as an operator composed with its adjoint. The second term is given by a degree 0 operator, i.e., a matrix valued function, which has bounded spectrum on a compact manifold. We therefore see that, although maybe not positive, the "Hamiltonian" H has spectrum bounded from below, and this is sufficient for the interpretation of the Dirac operator as a supersymmetric quantum mechanical system.

3.3. **The heat kernel.** Let H be a a positive selfadjoint operator on a Hilbert space $\mathcal{H}$. We assume H has a complete set of eigenstates $|\psi_n\rangle$ and that the eigenvalues λ_n go sufficiently fast to ∞ as $n \to \infty$ for the series

$$e^{-\beta H} := \sum_n e^{-\beta\lambda_n} |\psi_n\rangle \langle\psi_n|$$

to converge to a (bounded) operator, called the *heat operator* of H. Notice that the heat operator satisfies

$$\frac{\partial}{\partial\beta}e^{-\beta H}+[H,e^{-\beta H}]=0, \tag{5a}$$

$$\lim_{\beta\downarrow 0}e^{-\beta H}=\mathrm{id}_{\mathcal{H}}. \tag{5b}$$

The name heat operator stems from the fact that for any vector $|\psi\rangle\in\mathcal{H}$, the one-parameter family $|\psi(\beta)\rangle := e^{-\beta H}|\psi\rangle$ solves the heat equation with initial value $|\psi\rangle$:

$$\frac{d}{d\beta}|\psi(\beta)\rangle = -H|\psi(\beta)\rangle \tag{6a}$$

$$\lim_{\beta\downarrow 0}|\psi(\beta)\rangle = |\psi\rangle\,. \tag{6b}$$

It is easy to see that the solution to the initial value problem (6a), (6b) is unique: given a solution $|\psi(\beta)\rangle$, we have

$$\begin{aligned}\frac{d}{d\beta}||\psi(\beta)||^2&=\frac{d}{d\beta}\langle\psi(\beta),\psi(\beta)\rangle\\&=-\langle H\psi(\beta),\psi(\beta)\rangle-\langle\psi(\beta),H\psi(\beta)\rangle\\&=-2||Q\psi(\beta)||^2\leq 0,\end{aligned}$$

so that $||\psi(\beta)||\leq||\psi||$. This leads to uniqueness of the initial value problem (6a), (6b). From this it also follows that the heat operator itself is uniquely determined by the equation (5a) and the initial value (5b).

Let us now turn to our main example of supersymmetric quantum mechanics: the Dirac operator. It is a deep mathematical fact that on a compact manifold for any Dirac operator D the generalized Laplacian D^2 satisfies

- D^2 has spectrum bounded from below:
$$\lambda_0<\lambda_1<\ldots<\lambda_n,\quad \lambda_n\to\infty,\text{ as } n\to\infty.$$
- each eigenspace $\mathcal{H}_n$ is finite dimensional and consists of smooth functions $\psi_n(x)$,
- the eigenstates form a complete set: there is a Hilbert space decomposition
$$\mathcal{H}=\bigoplus_n\mathcal{H}_n,$$
- the heat operator defines a function
$$K_\beta(x,y):=\langle x|\,e^{-\beta D^2}\,|y\rangle=\sum_n e^{-\beta\lambda_n}\bar{\psi}_n(x)\psi_n(y)$$
on $M\times M\times(0,\infty)$, which is
  - smooth and symmetric in x and y,

- a solution of the heat equation

$$\frac{\partial}{\partial\beta} K_\beta(x,y) + D_x^2 K_\beta(x,y) = 0$$

- satisfies the initial value condition

$$\lim_{\beta\downarrow 0} K_\beta(x,y) = \delta(x-y).$$

Theorem 3.2 (McKean–Singer)**.** *Let* (S, ∇^S) *be a Clifford bundle over an even dimensional riemannian manifold* (M,g) *The index of the Dirac operator* $D^+ : \Gamma(M,S^+) \to \Gamma(M,S^-)$ *is given by the supertrace of the heat-kernel:*

$$\mathrm{Tr}_s(e^{-\beta D^2}) = \mathrm{index}(D^+).$$