LECTURE 3: COHOMOLOGY OF MANIFOLDS

A good reference for this lecture is e.g. [1].

1. The de Rham complex

As before, we let M be a manifold. We now consider the system of differential forms (of arbitrary degree) together with the exterior differential:

$$C^\infty(M) \xrightarrow{d} \Omega^1(M) \xrightarrow{d} \Omega^2(M) \xrightarrow{d} \dots \tag{1}$$

A differential form $\alpha \in \Omega^k(M)$ for which $d\alpha = 0$, is called *closed*. If there exists a form $\beta \in \Omega^{k-1}(M)$ such that $d\beta = \alpha$, α is called exact. We have seen that $d \circ d = 0$, so being exact implies being closed. With this property, the system (1) is an example of a *cochain complex*: we have $\mathrm{Im}\{d\colon \Omega^{k-1}(M) \to \Omega^k(M)\} \subset \ker\{d\colon \Omega^k(M) \to \Omega^{k+1}(M)\}$. The de Rham *cohomology groups* measure to what extend closedness fails to imply exactness:

$$H^k_{dR}(M) := \ker\{d\colon \Omega^k(M) \to \Omega^{k+1}(M)\}/\mathrm{Im}\{d\colon \Omega^{k-1}(M) \to \Omega^k(M)\}.$$

The main point of *de Rham's theorem* (see Theorem 3.2 below) is that these groups are *topological invariants* of the underlying topological space. For now, let us collect a few properties of these groups.[1]

- The assignment $M \mapsto H^\bullet_{\mathrm{dR}}(M)$ associates a (graded) vector space to a manifold.
- The wedge product of differential forms induces a product
  $$\wedge\colon H^p_{\mathrm{dR}}(M) \times H^q_{\mathrm{dR}}(M) \to H^{p+q}_{\mathrm{dR}}(M),$$
  because $d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^{|\alpha|}\alpha \wedge d\beta$.
- A smooth map $f\colon M \to N$ induces a map $f^*\colon H^\bullet_{\mathrm{dR}}(N) \to H^\bullet_{\mathrm{dR}}(M)$, because of the fact that the pull-back of differential forms is compatible with the exterior differential: $f^* d\alpha = d f^*\alpha$ for all $\alpha \in \Omega^k(N)$. This map is compatible with the wedge product.

The following property is less straightforward, but all the more important:

Theorem 1.1 (Homotopy invariance of de Rham cohomology)**.** *Let $f_0, f_1\colon M \to N$ be two smooth maps that are smoothly homotopic. Then they induce the same map on the level of de Rham cohomology groups:*

$$[f_0^*] = [f_1^*]\colon H^\bullet_{\mathrm{dR}}(N) \to H^\bullet_{\mathrm{dR}}(M).$$

Date: February 26, 2018.

[1] For the mathematically minded: These three properties can be rephrased by saying that de Rham cohomology defines a contravariant functor from the category of smooth manifolds (with morphisms given by smooth maps) to the category of graded algebras.

Proof. The fact that f_0 and f_1 are smoothly homotopic means that there exists a smooth map $F\colon M \times [0,1] \to N$ with $F(x,0) = f_0(x)$ and $F(x,1) = f_1(x)$ for all $x \in M$. With this homotopy F we shall construct a map $H\colon \Omega^k(M) \to \Omega^{k-1}(N)$ satisfying[2]

$$(\star) \qquad f_0^*\alpha - f_1^*\alpha = (d \circ H + H \circ d)\alpha, \quad \text{for all } \alpha \in \Omega^k(N).$$

This property implies that indeed $[f_0^*] = [f_1^*]$.

To construct H, observe that a k-form on $M \times [0,1]$ decomposes as

$$\beta = \beta^0 + dt \wedge \beta^1, \quad \beta \in \Omega^k(M \times [0,1]),$$

where t is the coordinate on $[0,1]$, $\beta^0 = \sum \beta^0_{i_1,\dots,i_k}(x,t)dx^{i_1} \wedge \ldots \wedge dx^{i_k}$ (in local coordinates) a k-form which does not contain dt and $\beta^1 = \sum \beta^1_{i_1,\dots,i_{k-1}}(x,t)dx^{i_1} \wedge \ldots \wedge dx^{i_{k-1}}$ a $k-1$ form. We can define the *fiber integral* along the projection $M \times [0,1] \to M$ by integrating the dt-component over $[0,1]$:

$$\int_0^1 (\iota_{\partial/\partial t}\beta)dt = \int_0^1 \beta^1 dt.$$

This defines a map $\int_{[0,1]}\colon \Omega^k(M \times [0,1]) \to \Omega^{k-1}(M)$ and Stokes' theorem gives the property

$$d\int_{[0,1]} \beta + \int_{[0,1]} d\beta = \beta|_{M\times\{1\}} - \beta|_{M\times\{0\}}.$$

(This is easily seen using the fact that the exterior derivative on $M \times [0,1]$ is given by $d_t + d$, where d_t is the derivative in the t variable and d the exterior derivative on M.)

With the fiber integral, we now define H by

$$H(\alpha) = \int_{[0,1]} F^*\alpha, \quad \alpha \in \Omega^k(N).$$

The property ($\star$) now follows from the the above version of Stokes' theorem together with the fact that the exterior derivative is compatible with the pull-back along F. $\square$

An important Corollary of this theorem is the Poincaré Lemma: recall that a domain $U \in \mathbb{R}^n$ is called *star-shaped* if there is a point $x_0 \in U$ such that for any other point $x \in U$, the straight line $tx + (1-t)x_0$ connecting x_0 and x is in U. For example $\mathbb{R}^n$ itself is star-shaped.

Corollary 1.2 (Poincaré Lemma)**.** *Let $U \subset \mathbb{R}^n$ be a star-shaped domain. Then:*

$$H^\bullet_{\mathrm{dR}}(U) = \begin{cases} \mathbb{R} & \bullet = 0 \\ 0 & \bullet > 0. \end{cases}$$

In the end this is a statement about solutions to certain systems of PDE's: Given a k-form $\alpha \in \Omega^k(M)$ on a manifold M, a necessary condition for the equation $\alpha = d\beta$ to have a solution $\beta \in \Omega^{k-1}(M)$, is that α is closed: $d\alpha = 0$. The Poincaré Lemma says that for M a star-shaped domain in $\mathbb{R}^n$, this condition is also sufficient: any closed form

[2] In homological algebra, H is called a *chain homotopy* between $[f_0^*]$ and $[f_1^*]$.

is always exact. Since any point $x \in M$ in a manifold has a neighborhood that is star-shaped, this means that *locally* any closed form on a manifold is also exact. The answer to the question whether this is *globally* true however depends on the global topology of M.

2. Singular (co)homology of manifolds

Recall that the standard k-dimensional simplex $\Delta^k \subset \mathbb{R}^{k+1}$ is defined as the convex subset satisfying the equation

$$\Delta^k := \{(t_0, \ldots . t_k) \in \mathbb{R}^{k+1}, \sum_{i=0}^{k} t_i = 1, \ t_i \geq 0.\}$$

The boundary of Δ^k consists of $k+1$ copies of the $(k-1)$-dimensional simplex by putting $t_i = 0$, $i = 0, \ldots, k$. We write $d_i : \Delta^{k-1} \hookrightarrow \Delta^k$, $i = 0, \ldots, k$ for the corresponding inclusion. A *smooth singular k-simplex* is a smooth map $\sigma : \Delta^k \to M$, where smooth means that we can extend σ to a small open neighborhood of Δ^k in $\mathbb{R}^{k+1}$. We write $S_k^\infty(M)$ for the vector space (over $\mathbb{R}$) spanned by all smooth singular k-simplices. So an element in $S_k^\infty(M)$ is given by a finite sum $\sum_i \lambda_i \sigma_i$ with $\lambda_i \in \mathbb{R}$ and σ_i smooth singular k-simplices. There is an operator $\partial : S_k^\infty(M) \to S_{k-1}^\infty(M)$ given on simplices by

$$\partial\sigma := \sum_{i=0}^{k} (-1)^i \sigma \circ d_i,$$

i.e. this operator restricts a smooth singular k-simplex σ, a map from Δ^k to M, to its $k+1$ boundary faces equal to Δ^{k-1}, with a sign. Exactly because of this sign, one checks that $\partial \circ \partial = 0$, i.e. the system

$$\ldots \xrightarrow{\partial} S_2^\infty(M) \xrightarrow{\partial} S_1^\infty(M) \xrightarrow{\partial} S_0^\infty(M)$$

forms a *chain complex*.[3] This time, for a chain complex, we should take its *homology*:

$$H_k^{\text{sing}}(M, \mathbb{R}) := \ker\{\partial \colon S_k^\infty(M) \to S_{k-1}^\infty(M)\}/\text{Im}\{\partial \colon S_{k+1}^\infty(M) \to S_k^\infty(M)\}.$$

To get an idea what these groups measure, consider $H_0^{\text{sing}}(M, \mathbb{R})$: a singular 0-simplex $\sigma : \Delta^0 \to M$ is just a point in M, and any such simplex is automatically closed since $S_{-1}^\infty(M, \mathbb{R}) = 0$. If two points $x, y \in M$ are in the same path connected component of M, any path $\gamma \colon [0,1] \to M$ from $\gamma(0) = x$ to $\gamma(1) = y$ defines a 1-simplex $\gamma \colon \Delta^1 \to M$ such that $\partial\gamma = x - y$, showing that they induce the same homology class. In other words: $H_0^{\text{sing}}(M, \mathbb{R})$ measures the number of path connected components of M.

Example 2.1 (The fundamental class of an oriented manifold)**.** Let M be a smooth n-dimensional manifold. It was proved by Whitehead in 1940 that M can be triangulated: we can write M as a finite union of smooth singular n-simplices $\sigma_i \colon \Delta^n \to M$ for $i =$

[3]The difference between a *chain* complex and a *cochain* complex is that in the former the differential has degree -1, whereas in the latter it has degree $+1$.

$1, \ldots, p$ such that any boundary face of σ_i is a boundary face of exactly one other simplex $\sigma_j,\ j \neq i$. Consider the combination

$$\sum_{i=1}^{p} \pm\sigma_i \in S_n^\infty(M, \mathbb{R}).$$

If we can consistently put the $\pm$-signs so that each boundary face appears with both a $+$ and a $-$ sign when taking ∂ of this expression, we get an n-cycle and therefore a class in $H_n^{\text{sing}}(M, \mathbb{R})$. This can be done exactly when M is *orientable* and the resulting class of an oriented manifold M is called the *fundamental class*, written $[M] \in H_n^{\text{sing}}(M, \mathbb{R})$.

If we want to have a *cochain* complex, we should take the dual:

$$S^k_\infty(M) := \mathrm{Hom}_{\mathbb{R}}(S_k^\infty(M), \mathbb{R}).$$

Now the differential ∂ dualizes to a degree *increasing* operator $d\colon S^k_\infty(M) \to S^{k+1}_\infty(M)$ by

$$d\varphi(\sigma) := \varphi(\partial\sigma), \quad \varphi \in S^k_\infty(M),\ \sigma \in S^\infty_{k+1}(M).$$

Clearly $d \circ d = 0$, so that we have a cochain complex, and its cohomology

$$H^k_{\text{sing}}(M, \mathbb{R}) := \ker\{d\colon S^k_\infty(M) \to S^{k+1}_\infty(M)\}/\mathrm{Im}\{d\colon S^{k-1}_\infty(M) \to S^k_\infty(M)\}$$

3. The de Rham theorem

Given a k-form on M and a smooth singular k-simplex $\sigma\colon \Delta^k \to M$ we can integrate:

$$\langle \alpha, \sigma \rangle := \int_{\Delta^k} \sigma^*\alpha. \tag{2}$$

Notice that it is important that we use *smooth* singular simplices to be able to pull-back the differential form to Δ^k. Stokes' theorem now gives us:

Lemma 3.1. *For $\alpha \in \Omega^{k-1}(M)$ and $\sigma \in S_k^\infty(M)$ we have the equality*

$$\langle d\alpha, \sigma \rangle = \langle \alpha, \partial\sigma \rangle.$$

We can therefore reinterpret the pairing (2) as a map

$$\Psi : \Omega^\bullet(M) \longrightarrow S^\bullet_\infty(M)$$

satisfying $d \circ \Psi = \Psi \circ d$. Such a map is called a *morphism of cochain complexes*. The fact that Ψ is compatible with the differentials on both sides implies that it induces a map on cohomology:

$$[\Psi] : H^\bullet_{\text{dR}}(M) \longrightarrow H^\bullet_{\text{sing}}(M, \mathbb{R}).$$

Theorem 3.2 (de Rham's theorem)**.** *The map $[\Psi]$ is an isomorphism.*

We will not give the full proof of the theorem, but only sketch the main idea. An important ingredient in the proof is the following crucial property satisfied by de Rham cohomology:

Theorem 3.3 (Mayer–Vietoris)**.** *Suppose that $M = U \cup V$ is covered by two open subsets. Then there exists a long exact sequence*[4]

$$\ldots \longrightarrow H^k_{\mathrm{dR}}(U \cup V) \longrightarrow H^k_{\mathrm{dR}}(U) \oplus H^k_{\mathrm{dR}}(V) \longrightarrow H^k_{\mathrm{dR}}(U \cap V) \longrightarrow H^{k+1}_{\mathrm{dR}}(U \cup V) \longrightarrow \ldots$$

Proof. Given U and V, we have maps on the level of differential forms

$$\begin{array}{ccccc} \Omega^k(U \cup V) & \to & \Omega^k(U) \oplus \Omega^k(V) & \to & \Omega^k(U \cap V) \\ \alpha & \mapsto & (\alpha|_U, \alpha|_V) & & \\ & & (\beta_U, \beta_V) & \mapsto & \beta_U|_{U \cap V} - \beta_V|_{U \cap V} \end{array}$$

Remark that the composition of these maps is zero, and that $(\beta_U, \beta_V) \in \Omega^k(U) \oplus \Omega^k(V)$ mapping to zero in $\Omega^k(U \cap V)$ means it comes from a form $\beta \in \Omega^k(U \cup V)$. Applying cohomology, we get

$$H^k_{\mathrm{dR}}(U \cup V) \to H^k_{\mathrm{dR}}(U) \oplus H^k(V)_{\mathrm{dR}} \to H^k_{\mathrm{dR}}(U \cap V).$$

Let us now construct a map $H^k(U \cap V) \to H^{k+1}(U \cup V)$. For this we choose a function $\chi_U \in C^\infty(U)$ which is ≤ 1 on U and equal to 1 on $U \backslash (U \cap V)$. Then $\chi_V := 1 - \chi_U$ is equal to 1 on $V \backslash (U \cap V)$ and we have $\chi_U + \chi_V = 1$. Given a closed differential form $\omega \in \Omega^k(U \cap V)$, let

$$(d\chi_U \wedge \omega, d\chi_V \wedge \omega) \in \Omega^k(U) \oplus \Omega^k(V).$$

Then on $U \cap V$ we have

$$\begin{aligned} d\chi_U \wedge \omega - d\chi_V \wedge \omega &= d(\chi_U - \chi_V) \wedge \omega \\ &= d(1) \wedge \omega \\ &= 0, \end{aligned}$$

and therefore these two forms glue together to a closed form of degree $k+1$ on $U \cup V$. We will skip the proof that the sequence is exact. ☐

Remark 3.4. Those who know a bit of homological algebra will recognize the snake Lemma in the proof above: The core of the argument is to show that the sequence

$$0 \to \Omega^\bullet(U \cup V) \to \Omega^\bullet(U) \oplus \Omega^\bullet(V) \to \Omega^\bullet(U \cap V) \to 0$$

is exact. This short exact sequence of complexes induces the long exact sequence in cohomology. Remark that the choice of the function χ_U is irrelevant: choosing another χ'_U results in a closed differential form which differs from the one constructed above by an exact form. (Try to prove this!)

[4] A *long exact sequence* is a complex with zero cohomology. In other words: each composition of maps is zero and the kernel of each map equals the image of the map preceding it.

The proof of de Rham's Theorem now amounts to proving first that singular cohomology $H^\bullet_{\mathrm{sing}}(M, \mathbb{R})$ satisfies the same properties as the properties of de Rham cohomology we have just outlined: functoriality, homotopy invariance and the existence of Mayer–Vietoris sequences. With that, by choosing an open cover of the manifold by open sets that are homeomorphic to a star-shaped domain, the proof is reduced to proving the de Rham isomorphism for such domains, which is done by the Poincaré Lemma. For the full details, see [2, §V.9].

Example 3.5. The Mayer–Vietoris property, together with homotopy invariance is extremely useful in computations. As an example let us compute the cohomology of $\mathbb{P}^n$. The inclusion $\mathbb{C}^n \hookrightarrow \mathbb{C}^{n+1}$ given by $(z^0, \ldots, z^{n-1}) \mapsto (z^0, \ldots, z^{n-1}, 0)$ induces an inclusion $\mathbb{P}^{n-1} \subset \mathbb{P}^n$. The complement $U := \mathbb{P}^n \backslash \mathbb{P}^{n-1}$ is isomorphic to $\mathbb{C}^n$ via the map

$$(z^0, \ldots, z^n) \mapsto (\frac{z^0}{z^n}, \ldots, \frac{z^{n-1}}{z^n}).$$

On the other hand, define $V := \mathbb{P}^n \backslash \{[0, \ldots, 0, 1]\}$. Then $U \cap V \cong \mathbb{C}^n \backslash \{0\} \sim S^{2n-1}$, and the map $F : V \times [0,1] \to V$ defined by

$$F([z^0, \ldots, z^n], t) = [z^0, \ldots, z^{n-1}, tz^n],$$

defines a contraction $V \sim \mathbb{P}^{n-1}$. The Mayer–Vietoris sequence, together with the homotopy invariance of de Rham cohomology leads to the exact sequence

$$\ldots \longrightarrow H^k_{\mathrm{dR}}(\mathbb{P}^n) \longrightarrow H^k_{\mathrm{dR}}(\mathbb{P}^{n-1}) \longrightarrow H^k_{\mathrm{dR}}(S^{2n-1}) \longrightarrow H^{k+1}_{\mathrm{dR}}(\mathbb{P}^n) \longrightarrow \ldots$$

Because

$$H^k_{\mathrm{dR}}(S^{2n-1}) = \begin{cases} \mathbb{R} & k = 2n-1 \\ 0 & k \neq 2n-1 \end{cases},$$

and $H^{2n-1}_{\mathrm{dR}}(\mathbb{P}^{n-1}) = 0$ (recall that $\mathbb{P}^{n-1}$ is $2n-2$-dimensional), the sequence above breaks up into

$$\begin{aligned} 0 \to H^k_{\mathrm{dR}}(\mathbb{P}^n) \to H^k_{\mathrm{dR}}(\mathbb{P}^{n-1}) \to 0, \quad &\text{for } k < 2n-1 \\ 0 \to H^{2n-1}_{\mathrm{dR}}(\mathbb{P}^{n-1}) \to 0 \\ 0 \to \mathbb{R} \to H^{2n}_{\mathrm{dR}}(\mathbb{P}^n) \to 0 \end{aligned}$$

From this we see, by induction that for $0 \leq k \leq 2n$:

$$H^k_{\mathrm{dR}}(\mathbb{P}^n) = \begin{cases} \mathbb{R} & k = \text{even} \\ 0 & k = \text{odd.} \end{cases}$$

References

[1] Raoul Bott and Loring W. Tu. *Differential forms in algebraic topology*, volume 82 of *Graduate Texts in Mathematics*. Springer-Verlag, New York-Berlin, 1982.

[2] Glen E. Bredon. *Topology and geometry*, volume 139 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1997. Corrected third printing of the 1993 original.