LECTURE 5: VECTOR BUNDLES, CONNECTIONS AND CURVATURE

1. VECTOR BUNDLES

An important class of fiber bundles are given by vector bundles: these are fiber bundles with typical fiber a vector space V:

Definition 1.1. A *vector bundle of rank r* is given by a manifold E together with a smooth map $\pi : E \to M$ and the structure of an r-dimensional vector space on the fibers $E_x := \pi^{-1}(x)$ which is *locally trivial* in the following sense: each $x \in M$ has an open neighborhood U such that there exists a diffeomorphism $\varphi : \pi^{-1}(U) \to U \times \mathbb{C}^r$ making the following diagram commutative

$$\begin{tikzcd} \pi^{-1}(U_x) \arrow[r, "\varphi"] \arrow[d, "\pi"'] & U \times \mathbb{C}^r \arrow[dl, "pr_1"] \\ U & \end{tikzcd}$$

and which is linear over each fiber. A *line bundle* is a vector bundle of rank one.

Remark 1.2. We can consider real or complex vector bundles, depending on whether the fibers are vector spaces over $\mathbb{R}$ or $\mathbb{C}$. With a view on applications in Quantum Mechanics, which always works in a complex Hilbert space, our main focus will be on complex vector bundles.

Example 1.3. Over any manifold M, we always have the trivial vector bundles $M \times \mathbb{R}^r$ (real case) and $M \times \mathbb{C}^r$ (complex). A general vector bundle E need not be of this form: although it is (by definition) *locally* a trivial vector bundle, globally it may not be trivial, but "twisted". The easiest example of a (real) twisted vector bundle is given by the Möbius line bundle over the circle: first we write $S^1 = \mathbb{R}/\mathbb{Z}$, where $\mathbb{Z}$ acts on $\mathbb{R}$ by translations $x \mapsto x + n$. We construct a line bundle L over S^1 by taking the quotient of the trivial line bundle over $\mathbb{R}$:

$$L := (\mathbb{R} \times \mathbb{R})/\mathbb{Z},$$

where we let $\mathbb{Z}$ act on $\mathbb{R} \times \mathbb{R}$ by either

$$(x, y) \mapsto (x + n, y), \quad \text{or} \quad (x, y) \mapsto (x + n, (-1)^n y).$$

In both cases, projection onto the first coordinate defines a smooth map $L \to S^1$ turning L into a line bundle over the circle (check!). In the first case we get the trivial line

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bundle $S^1 \times \mathbb{R}$, in the second case not: this is the Möbius line bundle as it "flips" as we go around the circle once.

Remark 1.4. There is a very concrete point of view on vector bundles using cocycles: Let $M = \bigcup_\alpha U_\alpha$ be a cover of M such that over each U_α there is a trivialization $\varphi_\alpha : \pi^{-1}(U_\alpha) \xrightarrow{\cong} U_\alpha \times \mathbb{C}^r$. (By definition, such a cover exists.) Over $U_\alpha \cap U_\beta$ we have two trivializations:

$$\begin{tikzcd} \pi^{-1}(U_{\alpha\beta}) \arrow[r, "\varphi_\alpha"] \arrow[d, "\varphi_\beta"'] & U_{\alpha\beta} \times \mathbb{C}^r \\ U_{\alpha\beta} \times \mathbb{C}^r \arrow[ur, "\varphi_\alpha \circ \varphi_\beta^{-1}"'] & \end{tikzcd}$$

Since both φ_α and φ_β are compatible with the projection to the base we can write

$$\varphi_\alpha \circ \varphi_\beta^{-1}(x, v) = (x, \varphi_{\alpha\beta}(v)), \quad \text{for } x \in U_{\alpha\beta}, v \in \mathbb{C}^r,$$

with $\varphi_{\alpha\beta} : U_{\alpha\beta} \to GL(r, \mathbb{C})$. We now shift our attention to these "transition functions" $\varphi_{\alpha\beta}$. The following properties are easily derived:

$$\begin{aligned} \varphi_{\alpha\alpha} &= 1 \\ \varphi_{\beta\alpha} &= \varphi_{\alpha\beta}^{-1} \\ \varphi_{\alpha\beta}\varphi_{\gamma\alpha}\varphi_{\beta\gamma} &= 1. \end{aligned}$$

These transition functions completely determine the vector bundle E: Given $\{\varphi_{\alpha\beta}\}_{\alpha,\beta \in I}$ satisfying the three properties above, define

$$E := \left(\coprod_{\alpha \in I} U_\alpha \times \mathbb{C}^r \right) / \sim$$

where

$$(x_\alpha, v) \sim (x_\beta, v) \Longleftrightarrow x_\alpha = x_\beta \in U_\alpha \cap U_\beta,\ \varphi_{\alpha\beta}(v) = w.$$

The properties satisfied by the $\varphi_{\alpha\beta}$ above guarantee that this defines an equivalence relation making the quotient well-defined.

Example 1.5 (The tangent bundle)**.** For any smooth manifold M, its tangent bundle is a real vector bundle: using a coordinate chart we can define local trivializations. The transition cocycles are given by the Jacobian matrices of the changes of coordinates. When the tangent bundle TM is trivial (or rather isomorphic to the trivial vector bundle of rank equal to the dimension n of M), we say that the M is parallelizable. This means that there exist n-vector fields $X_1, \ldots, X_n$ that at each point $x \in M$ form a basis of $T_x M$.

Example 1.6 (The universal line bundle over $\mathbb{P}^n$)**.** Recall the manifold structure of projectve space $\mathbb{P}^n$. Consider the following set:

$$T := \{(v, L) \in \mathbb{C}^{n+1} \times \mathbb{P}^n,\ v \in L\}.$$

There is an obvious projection $T \to \mathbb{P}^n$ projecting onto the second component. Clearly, the fiber $T_L,\ L \in \mathbb{P}^n$ is a vector space of dimension 1. In order to be a line bundle, we have to show local triviality. Over the domain $U_i := \{[z_0, \ldots, z_n],\ z_i \neq 0\}$ of the chart φ_i given in (**??**) there is an bijection

$$\varphi_i : \pi^{-1}(U_i) \xrightarrow{\cong} U_i \times \mathbb{C},$$

given by the fact that any vector v in the line spanned by $(z_0, \ldots, z_n) \in \mathbb{C}^{n+1}$ can be written as

$$v = \lambda(\frac{z_0}{z_i}, \ldots, \frac{z_{i-1}}{z_i}, 1, \frac{z_{i+1}}{z_i}, \ldots \frac{z_n}{z_i}), \quad \lambda \in \mathbb{C}.$$

(Dividing by z_i ensures that v determines λ uniquely.) The map above maps v to λ. This shows that $T \to \mathbb{P}^n$ is indeed a line bundle.

To determine the cocycle $\varphi_{ij} \in C^\infty(U_{ij}, \mathbb{C}^*)$[1] underlying this line bundle we consider the composition

$$U_{ij} \times \mathbb{C} \xrightarrow{\varphi_j^{-1}} \pi^{-1}(U_{ij}) \xrightarrow{\varphi_i} U_{ij} \times \mathbb{C},$$

which maps

$$([z_0, \ldots, z_n], \lambda) \mapsto ([z_0, \ldots, z_n], \frac{z_i}{z_j}\lambda).$$

The cocycle is therefore given by $\varphi_{ij}([z_0, \ldots, z_n]) = \frac{z_i}{z_j}$.

There is a natural way to get vector bundles from principal bundles:

Remark 1.7 (Pull-back of vector bundles)**.** Let $f : M \to N$ be a smooth map, and let $p : E \to N$ be a vector bundle over N. It is easy to see that

$$f^*E = \{(x, e) \in M \times E,\ f(x) = p(e)\}$$

has a canonical vector bundle structure over X.

Remark 1.8 (Linear algebra constructions with vector bundles)**.** Let E and F be vector bundles over M. It is not difficult to show that one can extend the standard constructions from linear algebra to define the following vector bundles over M:

i) the direct sum: $E \oplus F$,
ii) the tensor product $E \otimes F$,
iii) $\mathrm{Hom}(E, F) \cong E^* \otimes F$.

A section of a vector bundle $\pi : E \to X$ is a continuous map $s : M \to E$ such that

$$\pi \circ s = 1,$$

where the 1 on the right hand side means the constant function on X with that value. Denote the space of sections of E by $\Gamma(X, E)$. When E is smooth, one can require section to be be smooth maps as well, and this defines the space of smooth sections $\Gamma^\infty(X, E)$.

[1] Recall that $GL(1, \mathbb{C}) = \mathbb{C}^*$

2. CONNECTIONS AND CURVATURE

Let M be a smooth manifold, and $E \to M$ a smooth vector bundle. We denote by $\Omega^k(M;E)$ the space of differential k-forms on M with values in E:

$$\Omega^k(M;E) := \Gamma^\infty(M, E \otimes \bigwedge^k T^*M)$$

The following definition is fundamental:

Definition 2.1. A *connection* on E is a linear map

$$\nabla : \Gamma^\infty(M;E) \to \Omega^1(M;E),$$

satisfying the Leibniz rule

$$\nabla(fs) = f\nabla(s) + df \otimes s,$$

with $f \in C^\infty(M)$ and $s \in \Gamma^\infty(M;E)$.

In short: a connection on a vector bundle $E \to M$ is a gadget which allows us to take "directional derivatives" of smooth sections of E along vector fields on M. For a vector field $X \in \mathfrak{X}(M)$, we shall write $\nabla_X : \Gamma^\infty(M;E) \to \Gamma^\infty(M;E)$ for this directional derivative: $\nabla_X(s) := \iota_X(\nabla s)$. If we want to stipulate for which bundle exactly ∇ is a connection, we shall write ∇^E.

Lemma 2.2. *The space of connections on a vector bundle E is an affine space modeled on* $\Omega^1(M, \mathrm{End}(E))$.

Proof. Let ∇ and ∇' be two connections on E. It follows form the Leibniz rule that

$$(\nabla - \nabla')fs = f(\nabla - \nabla')s, \quad \text{for all } f \in C^\infty(X),\ s \in \Gamma(M;E).$$

The operator $\nabla - \nabla' : \Gamma^\infty(M;E) \to \Omega^1(M;E)$ is therefore $C^\infty(M)$-linear, and it follows that $\nabla - \nabla' \in \Omega^1(M; \mathrm{End}(E))$ □

Remark 2.3.

i) For a trivial vector bundle $E = M \times \mathbb{C}^r$ we always have the trivial connection given by the de Rham operator d extended to vector valued functions. By the Lemma above, any other connection can be written as $\nabla = d + A$ with $A \in \Omega^1(M, M_r(\mathbb{C}))$ a matrix-valued one-form. ($M_r(\mathbb{C})$ denotes the $r \times r$ matrices with coefficients in $\mathbb{C}$.)

ii) For a general vector bundle, we can write $\nabla = d + A_\alpha$ in a local trivialization over U_α. On the overlap $U_\alpha \cap U_\beta$ of two local trivializations the two one forms A_α and A_β are related by (check!)

$$A_\alpha = \varphi_{\alpha\beta} A_\beta \varphi_{\alpha\beta}^{-1} - \varphi_{\alpha\beta}^{-1} d\varphi_{\alpha\beta}, \tag{1}$$

with $\varphi_{\alpha\beta} : U_{\alpha\beta} \to GL(r, \mathbb{C})$ the transition function. If we adopt the "cocycle point of view" on vector bundles, c.f. Remark 1.4, we can therefore think of a connection on a vector bundle as a collection $\{A_\alpha \in \Omega^1(U_\alpha, M_r(\mathbb{C}))\}_{\alpha\in I}$ of matrix-valued 1-forms, which transform according to (1) under local gauge transformations.

iii) It can be shown by a standard partition of unity argument that a connection always exist on a vector bundle.

Remark 2.4 (Connections and the framebundle[2])**.** Consider a vector bundle $E \to M$ of rank r. Define a space

$$F(E) := \bigcup_{x\in M} F(E_x),$$

where $F(E_x)$ is the space of all bases in E_x: a point $e_x \in F(E_x)$ is a basis $e_x := (e_x^1, \dots, e_x^r)$ of the vector space E_x. Using the local triviality of the vector bundle $E \to M$, one can define a smooth manifold structure on $F(E)$, such that the obvious projection $\pi : F(E) \to M$ is smooth. There is an action of $GL(r,\mathbb{C})$ on $F(E)$ preserving the fibers which moves one basis to another, and we have

$$F(E)/GL(r,\mathbb{C}) \cong M,$$

the isomorphism induced by the projection $\pi : F(E) \to M$.

Let ∇ be a connection on E and $\{A_\alpha \in \Omega^1(U_\alpha, M_r(\mathbb{C}))\}_{\alpha\in I}$ the collection of matrix-valued 1-forms associated to a system of local trivializations. Pulling back these forms to $F(E)$, we obtain a collection of 1-forms $\{\pi^* A_\alpha\}_{\alpha\in I}$ on the cover $\{\pi^{-1}(U_\alpha)\}_{\alpha\in I}$ of $F(E)$. On the overlap $\pi^{-1}(U_\alpha) \cap \pi^{-1}(U_\beta) = \pi^{-1}(U_{\alpha\beta})$ we have by (1):

$$\begin{aligned}\pi^* A_\alpha &= \pi^*(\varphi_{\alpha\beta} A_\beta \varphi_{\alpha\beta}^{-1} - \varphi_{\alpha\beta}^{-1} d\varphi_{\alpha\beta}) \\ &= \pi^* A_\beta.\end{aligned}$$

(The derivation of the last equality is nontrivial.) It follows that the A_α glue together to form a well-defined 1-form $A \in \Omega^1(F(E), \mathfrak{gl}(r,\mathbb{C}))$ with values in the Lie algebra $\mathfrak{gl}(r,\mathbb{C}) = M_r(\mathbb{C})$ of $GL(r,\mathbb{C})$. One can check that this form satisfies the following properties:

$$\iota_{\xi_{\mathrm{Fr}(E)}} A = \xi, \quad \text{for all } \xi \in \mathfrak{gl}(r,\mathbb{C}), \tag{2a}$$

$$R_g^* A = gAg^{-1}, \quad \text{for all } g \in GL(r,\mathbb{C}). \tag{2b}$$

Concluding we see that there are three equivalent ways of thinking about connections:

- as a first order differential operator ∇ on sections,
- as a collection of matrix valued 1-forms $\{A_\alpha \in \Omega^1(U_\alpha, M_r(\mathbb{C}))\}_{\alpha\in I}$, satisfying (1) on overlaps,
- a matrix valued 1-form A on $F(E)$ satisfying (2a) and (2b).

[2] This can be omitted on first reading: we did not cover this in the lecture.

Example 2.5. Consider (again) the universal line bundle over $\mathbb{P}^n$, c.f. Remark 1.6. With the given transition functions $\varphi_{ij}([z_0,\ldots,z_n]) = z_i/z_j$ on U_{ij} we are therefore looking for a collection of complex 1-forms $\{A_i \in \Omega^1(U_i,\mathbb{C})\}_{i=0}^n$ related to each other by

$$(A_i - A_j)|_{U_{ij}} = (d\varphi_{ij})\varphi_{ij}^{-1} = d\log\left(\frac{z_i}{z_j}\right) = \frac{dz_i}{z_i} - \frac{dz_j}{z_j}$$

A solution to this system of equations is given by

$$A_i := \partial \log\left(\sum_{\ell=0}^n \frac{|z_\ell|^2}{|z_i|^2}\right).$$

This is because we can write this as $A_i = \partial \log f_i$ with $f_i \in C^\infty(U_i,\mathbb{C})$ satisfying $f_i = |z_i/z_j|^2 f_j$ and $\partial \log |z_i/z_j|^2 = dz_i/z_i - dz_j/z_j$.

Remark 2.6. Connections behave well with respect to the standard constructions with vector bundles: Let E and F be vector bundles over X, with connections ∇^E, ∇^F.

i) On the direct sum, we have the obvious connection

$$\nabla^{E\oplus F} = \begin{pmatrix} \nabla^E & 0 \\ 0 & \nabla^F \end{pmatrix},$$

ii) On the tensor product we have the connection $\nabla^{E\otimes F} = \nabla^E \otimes 1 + 1 \otimes \nabla^F$,

iii) On the dual E^*, we have the connection defined by the following equation

$$d\langle \alpha, s\rangle = \langle \nabla_{E^*}(\alpha), s\rangle + \langle \alpha, \nabla_E(s)\rangle, \quad \alpha \in \Gamma^\infty(M;E^*),\ s \in \Gamma^\infty(M;E),$$

using the dual pairing $\langle\ ,\ \rangle : \Gamma^\infty(M;E^*)\times\Gamma^\infty(M;E) \to C^\infty(M)$ between sections of E and E^*

iv) As a special case of *iii*), we obtain a connection on $\mathrm{End}(E) = E\otimes E^*$, defined by

(3) $$\nabla^{\mathrm{End}(E)}(A)(s) := \nabla_E(A(s)) - A(\nabla_E(s)) \quad \text{for } A \in \Gamma^\infty(X,\mathrm{End}(E),\ s \in \Gamma(X;E).$$

v) On the pull-back bundle f^*E for a smooth map $f : N \to M$, there is a natural pull-back connection $f^*\nabla_E$.

The curvature of a connection. Using the Leibniz identity, we can extend a connection to an operator $\nabla : \Omega^k(M,E) \to \Omega^{k+1}(M;E)$ by

$$\nabla(s\otimes\alpha) = \nabla s \wedge \alpha + s \otimes d\alpha, \quad s \in \Gamma^\infty(M,E),\ \alpha \in \Omega^k(M).$$

The operator ∇ thus defined doesn't turn $\Omega^\bullet(M;E)$ into a complex: $\nabla^2 \neq 0$. However we do have

$$\nabla^2(fs) = f\nabla^2(s), \quad \text{for all } f \in C^\infty(M)$$

so we can define the curvature $F(\nabla) \in \Omega^2(M,\mathrm{End}(E))$ by

$$F(\nabla)(s) := \nabla^2(s) \in \Omega^2(M;\mathrm{End}(E)) \quad \text{for all } s \in \Gamma^\infty(M,E).$$

In a local trivialization $\nabla|_{U_\alpha} = d + A_\alpha$ we see that

$$F_\alpha := F(d + A_\alpha) = (d + A_\alpha)^2 = dA_\alpha + A_\alpha \wedge A_\alpha, \tag{4}$$

where $(A_\alpha \wedge A_\alpha)(X, Y) = \frac{1}{2}[A_\alpha(X), A_\alpha(Y)]$. (Recall that A_α is matrix-valued. This also explains why $A_\alpha \wedge A_\alpha$ is nonzero.) Indeed with this we see that under the local gauge transformation (1) we have

$$\begin{aligned} F_\alpha &= d(\varphi_{\alpha\beta} A_\beta \varphi_{\alpha\beta}^{-1} - (d\varphi_{\alpha\beta})\varphi_{\alpha\beta}^{-1}) + (\varphi_{\alpha\beta} A_\beta \varphi_{\alpha\beta}^{-1} - (d\varphi_{\alpha\beta})\varphi_{\alpha\beta}^{-1}) \wedge (\varphi_{\alpha\beta} A_\beta \varphi_{\alpha\beta}^{-1} - (d\varphi_{\alpha\beta})\varphi_{\alpha\beta}^{-1}) \\ &= \varphi_{\alpha\beta} dA_\beta \varphi_{\alpha\beta}^{-1} + \varphi_{\alpha\beta}(A_\beta \wedge A_\beta)\varphi_{\alpha\beta}^{-1} \\ &= \varphi_{\alpha\beta} F_\beta \varphi_{\alpha\beta}^{-1}. \end{aligned}$$

This is precisely the transformation property of a section of the bundle $\mathrm{End}(E) \to M$ in local trivializations. It is remarkable that, although a connection is not a section of any bundle associated to E, the curvature does have this property. From the local expression for the curvature above, together with Eq. (**??**), we can deduce the useful formula

$$F(\nabla)(X, Y) = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}, \quad X, Y \in \mathfrak{X}(M).$$

Example 2.7. For line bundles, $M_1(\mathbb{C}) = \mathbb{C}$ is abelian, and therefore the second term in (4) vanishes. Therefore, the curvature of the connection of Example 2.5 on the universal bundle over $\mathbb{P}^n$ is given by

$$\omega_i = d\partial \log\left(\sum_{\ell=0}^{n} \frac{|z_\ell|^2}{|z_i|^2}\right) = \bar{\partial}\partial \log\left(\sum_{\ell=0}^{n} \frac{|z_\ell|^2}{|z_i|^2}\right),$$

because $d = \partial + \bar{\partial}$ and $\bar{\partial} \circ \bar{\partial} = 0$. Indeed, for rank $r = 1$, $\mathfrak{gl}(1, \mathbb{C}) = \mathbb{C}$ is abelian, so the second term in (4) is automatically zero. The argument above implies that the $\omega_i \in \Omega^2(U_i, \mathbb{C})$ patch to a global 2-form, called the *Fubini–Study form*. One can also explicitly check this:

$$\begin{aligned} \omega_j|_{U_{ij}} &= \partial\bar{\partial} \log\left(\sum_{k=0}^{n} \frac{|z_k|^2}{|z_j|^2}\right) \\ &= \partial\bar{\partial} \log\left(\frac{|z_i|^2}{|z_j|^2}\right) + \partial\bar{\partial} \log\left(\sum_{k=0}^{n} \frac{|z_k|^2}{|z_i|^2}\right) \\ &= \partial\bar{\partial} \log\left(\sum_{k=0}^{n} \frac{|z_k|^2}{|z_i|^2}\right) \\ &= \omega_i|_{U_{ij}} \end{aligned}$$

Lemma 2.8 (Bianchi identity)**.** *The curvature of a connection satisfies:*

$$\nabla^{\mathrm{End}(E)}(F(\nabla^E)) = 0.$$

Proof. Write out:

$$
\begin{aligned}
\nabla^{\mathrm{End}(E)}(F(\nabla^E))(s) &= \nabla^E(F(\nabla)(s)) - F(\nabla^E)(\nabla^E(s)) \\
&= (\nabla^E)^3(s) - (\nabla^E)^3(s) \\
&= 0.
\end{aligned}
$$

Here we have used the definition (3) of the connection $\nabla^{\mathrm{End}(E)}$ on the bundle $\mathrm{End}(E)$ induced by ∇^E. □