LECTURE 7: CHARACTERISTIC CLASSES

Consider the following integrals:

- In Maxwell theory, let consider an embedded sphere in space and integrate the field strength over it:
$$\int_{S^2} F = q_m$$
As explained in the previous lectures, this measure the magnetic charge. Consistency with Quantum Mechanics leads, as Dirac realized, to the condition that this integral is always an integer (in appropriate units). We will explain in §**??** that this observation leads to the fiber bundle formulation of Maxwell theory where the gauge field is a connection on a line bundle.
- In Yang–Mills theory over a 4-dimensional space-time M, the integral
$$\frac{1}{4\pi^2}\int_M \mathrm{Tr}(F\wedge F)$$
measures the instanton number. In the $SU(2)$-example over S^4 of the lecture, the integral measure the *winding number* of the transition map $S^3 \to SU(2) \cong S^3$ and is always an integer.

It is truly remarkable that the values of these integrals is always an integer, and this is by no means a coincidence: mathematically, such integrals are called *Chern numbers*. The integrands are given by certain polynomials of field strengths of vector bundles, and are called *characteristic classes*. Characteristic classes are certain cohomology classes associated to vector bundles that measure how "non-trivial" vector bundles are. We can give a mathematical definition:

Definition 0.1. A *characteristic class* is an assignment $E \mapsto c(E) \in H^\bullet(M)$ of a cohomology class to a vector bundle $E \to M$ that

i) depends only on the isomorphism class of E,
ii) is natural in the following sense: for any smooth map $f : N \to M$, we have
$$c(f^*E) = f^*c(E).$$

1. The Chern–Weil homomorphism

Let $E \to M$ be a complex vector bundle of rank r. Denote by $\mathrm{Mat}_r(\mathbb{C})$ the Lie algebra of $GL(r, \mathbb{C})$, given by $r \times r$ matrices with complex coefficients.

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Definition 1.1. An *invariant homogeneous polynomial* of degree k on $\mathrm{Mat}_r(\mathbb{C})$ is given by a symmetric multilinear map

$$P : \underbrace{\mathrm{Mat}_r(\mathbb{C}) \times \ldots \times \mathrm{Mat}_r(\mathbb{C})}_{k \text{ times}} \to \mathbb{C}$$

which is invariant under the action of $GL(r, \mathbb{C})$:

$$P(gA_1g^{-1}, \ldots, gA_kg^{-1}) = P(A_1, \ldots, A_k), \quad \text{for } A_i \in \mathrm{Mat}_r(\mathbb{C}) \text{ and } g \in GL(r, \mathbb{C}).$$

We denote the graded algebra of invariant polynomials of arbitrary degree by $I_{\mathrm{inv}} = \bigoplus_{k \geq 0} I^k_{\mathrm{inv}}$.

Given a vector bundle $E \to M$ and $P \in I^k_{\mathrm{inv}}$, we pick a connection ∇ on E and consider the differential form

$$P(F(\nabla), \ldots, F(\nabla)) \in \Omega^{2k}(M). \tag{1}$$

To make sense of this expression, choose, for a point $x \in M$ an isomorphism $E_x \cong \mathbb{C}^r$ of the fiber of E. This induces an isomorphism $\mathrm{End}(E)_x \cong \mathrm{Mat}_r(\mathbb{C})$, so that we can apply P to $F(\nabla) \in \Omega^2(M, \mathrm{End}(E))$ at that point. Since P is invariant, its value is in fact independent of the chosen isomorphism, and combining with the wedge product, this yields a smooth differential form of degree $2k$.

Proposition 1.2.

i) *The form $P(F(\nabla), \ldots, F(\nabla))$ is closed.*
ii) *The induced cohomology class in $H^{2k}_{\mathrm{dR}}(X)$ is independent of the chosen connection.*

Proof. First remark that by invariance of P we have

$$\sum_{i=1}^k P(A_1, \ldots, [A, A_i], \ldots, A_k) = 0, \quad A, A_1, \ldots, A_k \in \mathrm{Mat}_r(\mathbb{C}).$$

This identity can be obtained by using invariance with respect to conjugation with $g = \exp(tA)$ and differentiation. Therefore, in a local trivialization where we write $\nabla^E = d + A$, we find

$$\begin{aligned} dP(F(\nabla^E), \ldots, F(\nabla^E)) &= \sum_{i=1}^k P(F(\nabla^E), \ldots, dF(\nabla^E), \ldots, F(\nabla^E)) \\ &= \sum_{i=1}^k P(F(\nabla^E), \ldots, \nabla^{\mathrm{End}(E)} F(\nabla^E) - [A, F(\nabla^E)], \ldots, F(\nabla^E)) \\ &= 0 \end{aligned}$$

by the Bianchi identity and the invariance of P. This proves the first claim.

For the second, let ∇' be another connection. By Lemma **??** we have $\nabla' = \nabla + \alpha$ for some $\alpha \in \Omega^1(M; \mathrm{End}(E))$. Therefore the convex combination $\nabla_t = t\nabla' + (1-t)\nabla =$

$\nabla + t\alpha,\ t \in [0,1]$ is a family of connections interpolating between ∇ and ∇'. We now consider the connection $\nabla^{\text{aff}} := d_t + \nabla_t$ on the vector bundle $E \times [0,1] \to M \times [0,1]$.

A small computation shows that

$$F(\nabla^{\text{aff}}) = F(\nabla) + dt \wedge \alpha + t\nabla\alpha + t^2\alpha \wedge \alpha \in \Omega^2(M \times [0,1], \text{End}(E)).$$

For an invariant polynomial of degree k, we now consider the fiber integral over t-parameter:

$$L(\nabla, \nabla') := \int_0^1 P(F(\nabla^{\text{aff}})) \in \Omega^{2k-1}(M).$$

(To evaluate this integral, we pick the terms in $P(F(\nabla^{\text{aff}}))$ which contain one factor dt and then perform the integral.) This L is called the *transgression form*. Stokes' theorem now gives:

$$\begin{aligned} dL(\nabla, \nabla') &= d\int_0^1 P(F(\nabla^{\text{aff}})) \\ &= \int_0^1 dP(F(\nabla^{\text{aff}})) - P(F(\nabla^{\text{aff}}))\Big|_{t=1} + P(F(\nabla^{\text{aff}}))\Big|_{t=0} \\ &= P(F(\nabla)) - P(F(\nabla')). \end{aligned}$$

This proves the second claim. ☐

Corollary 1.3 (Chern–Weil homomorphism). *Given a vector bundle $E \to M$, there is a canonical homomorphism of graded algebras*

$$I_{\text{inv}} \to H^{2\bullet}_{\text{dR}}(M).$$

2. Chern classes

Before we define the Chern classes, let us make the following remark: in the definition (1) of the characteristic class associated to an invariant polynomial P, we are restricting P to the diagonal: $\tilde{P}(A) := P(A, \ldots, A)$. The function $\tilde{P}(A)$ clearly is a conjugacy invariant polynomial in the entries of A. Conversely, given a conjugacy invariant polynomial $\tilde{P}_k$ of degree k defines a symmetric multilinear $P_k : \text{Mat}_r(\mathbb{C}) \times \ldots \times \text{Mat}_r(\mathbb{C}) \to \mathbb{C}$ called its *polarization*:

$$P(A_1, \ldots, A_k) := \frac{(-1)^k}{k!} \sum_{j=1}^{k} \sum_{i_1 < \ldots < i_j} \tilde{P}(A_{i_1} + \ldots + A_{i_j}).$$

For example, for $k = 2$ we have

$$P_k(A_1, A_2) := \frac{1}{2}(P(A_1 + A_2) - P(A_1) - P(A_2)).$$

In the following we will therefore refer to both P as well as $\tilde{P}$ interchangeably as an invariant polynomial.

Consider now the invariant polynomials $P_k \in I^k_{\mathrm{inv}}$ defined by

$$\det(I + tA) =: P_0(A) + tP_1(A) + t^2 P_k(A) + \ldots, \quad A \in \mathrm{Mat}_r(\mathbb{C}).$$

(Of course, $\det(1 + tgAg^{-1}) = \det(1 + tA)$, so the polynomials are indeed invariant.) The polynomial P_k defines the *k-th Chern class*[1]

$$c_k(E) := \left(\frac{\sqrt{-1}}{2\pi}\right)^k P_k(F(\nabla)) \in H^{2k}_{\mathrm{dR}}(X).$$

For example, using the well-known expansion

$$\det(I + tA) = r + t\mathrm{Tr}(A) + \frac{t^2}{2}\left(\mathrm{Tr}(A^2) - \mathrm{Tr}(A)^2\right) + \ldots + t^r \det(A),$$

we find in low degrees

$$\begin{aligned} c_0(E) &= \mathrm{rank}(E) \in H^0_{\mathrm{dR}}(M), \\ c_1(E) &= \frac{\sqrt{-1}}{2\pi}\mathrm{Tr}(F(\nabla)) \in H^2_{\mathrm{dR}}(M), \\ c_2(E) &= -\frac{1}{4\pi^2}\left(\mathrm{Tr}(F(\nabla) \wedge F(\nabla)) - \mathrm{Tr}(F(\nabla)) \wedge \mathrm{Tr}(F(\nabla))\right) \in H^4_{\mathrm{dR}}(M). \end{aligned}$$

The *total Chern class* is defined as

$$c(E) := \sum_{k \geq 0} c_k(E).$$

Proposition 2.1. *The total chern class $c(E) \in H^\bullet_{\mathrm{dR}}(M)$ satisfies the following properties:*

$i)$ (Naturality) for $f : N \to M$ a smooth map, we have

$$c(f^*E) = f^*c(E) \in H^\bullet_{\mathrm{dR}}(N),$$

$ii)$ (Product formula) For a direct sum $E \oplus F$, we have

$$c(E \oplus F) = c(E)c(F)$$

The first property follows from the fact that the pull-back connection $f^*\nabla$ on f^*E has curvature equal to $F(f^*\nabla) = f^*F(\nabla)$. The second property follows from the fact that the direct sum connection $\nabla^E \oplus \nabla^F$ on $E \oplus F$ has curvature that can be written in matrix form as

$$\begin{pmatrix} F(\nabla^E) & 0 \\ 0 & F(\nabla^F) \end{pmatrix}.$$

In general, Chern classes measure how "nontrivial" a vector bundle is. To witness this point, we have:

Lemma 2.2. *For a trivializable vector bundle $E \to M$, all Chern classes $c_k(E)$, $k \geq 1$ are zero.*

[1]The reason for the normalization factor $\frac{\sqrt{-1}}{2\pi}$ is the fact that with precisely this factor the Chern classes are *integtral*, c.f. Theorem 3.2

Proof. Let us first remark that on the trivial vector bundle $M \times \mathbb{C}^r$ we can choose the trivial connection given by the exterior derivative d applied to vector-valued functions. This connection has curvature zero since $d \circ d = 0$ and therefore the trivial bundle has vanishing Chern classes. For a trivializable vector bundles, assume that $\varphi : E \xrightarrow{\cong} M \times \mathbb{C}^r$ is a trivialization. Then E carries a connection given by

$$\nabla = \varphi^{-1} \circ d \circ \varphi = d + \varphi^{-1} d\varphi.$$

We have already seen that under such "gauge transformations" φ, the curvature transforms neatly:

$$F(\varphi^{-1} \circ d \circ \varphi) = \varphi F(d) \varphi^{-1} = 0,$$

so again the Chern classes are zero. Notice that the theory implies that any other connection ∇ on E, its *Chern forms* $c_k(E, \nabla) \in \Omega^{2k}(M)$ are exact. $\square$

Finally, we come a crucial property of Chern classes, namely that they are *integral* cohomology classes. The definition of an integral cohomology class defined by a differential form relies on de Rham's theorem, but for now we record the following consequence:

Theorem 2.3. *Let $E \to M$ be a complex vector bundle. For any closed compact $2k$-dimensional oriented submanifold $S \subset M$ the integral*

$$\int_S c_k(E)$$

is an integer.

These numbers are called *Chern numbers*. Notice that the fact that the differential form $c_k(E) \in \Omega^{2k}(M)$ is closed explains that the value of the integral does not depend on the precise embedding of S into M, in fact by de Rham's theorem only the underling *homology class* in $H_{2k}^{\mathrm{sing}}(M, \mathbb{R})$ (obtained by taking the fundamental class) matters. However, the fact that the value of these integrals are always integers is truly remarkable, and a proof of this is sketched in the next section.

3. Integrality of Chern numbers[2]

To prove Theorem 2.3 we first need to give a proper definition of an *integral cohomology class* defined by a closed differential form. Observe that there is a natural inclusion $S_\infty^k(M, \mathbb{Z}) \subset S_\infty^k(M, \mathbb{R})$ which leads to a map $H_{\mathrm{sing}}^k(M, \mathbb{Z}) \to H_{\mathrm{sing}}^k(M, \mathbb{R})$.

Definition 3.1. A closed differential form $\omega \in \Omega_{\mathrm{cl}}^k(M)$ is called *integral* if its cohomology class $[\omega] \in H_{\mathrm{dR}}^k(M)$ corresponds, under the de Rham isomorphism to a class in $H_{\mathrm{sing}}^k(M, \mathbb{R})$ that lies in the image of the natural map $H_{\mathrm{sing}}^k(M, \mathbb{Z}) \to H_{\mathrm{sing}}^k(M, \mathbb{R})$.

[2]We did not really cover this section in the lecture. You can read it to get a general idea of how the proof of Theorem 2.3 works, as this involves some very nice ideas from topology such as classifying spaces and splitting principles.

Unraveling the definitions, this means that for any smooth singular chain $\sigma := \sum_i n_i \sigma_i \in S_k^\infty(M, \mathbb{Z})$ with $n_i \in \mathbb{Z}$ and $\sigma_i : \Delta^k \to M$ such that $\partial\sigma = 0$, we have

$$\sum_i \int_{\Delta^k} \sigma_i^* \omega \in \mathbb{Z}.$$

We can now properly state:

Theorem 3.2. *Let $E \to M$ be a complex vector bundle over a smooth manifold M. Then the Chern classes of E are integral: $c_k(E) \in H^{2k}(M, \mathbb{Z})$.*

Remark that this implies Theorem 2.3: this can be seen by taking the fundamental class of S, c.f. Remark **??**.

The integrality of the Chern classes is a very surprising fact, and at first sight it is not clear how to prove this fact. Certainly, we do not want to evaluate all the integrals by hand! We therefore consider a bit more sophisticated argument, which proceeds in several steps:

Step I: classifying line bundles. The idea is to first proof the Theorem for line bundles. For a line bundle $L \to M$, we have $\mathrm{End}(L) \cong M \times \mathbb{C}$, so that for a connection ∇ its curvature $F(\nabla) \in \Omega^2(M)$ is a closed 2-form. For $r = 1$, we have only one invariant polynomial, and therefore $c(L) = 1 + c_1(L)$, and we only have to evaluate the integrals

$$\frac{1}{2\pi\sqrt{-1}} \int_{\Delta^2} \sigma^* F(\nabla), \tag{2}$$

for all closed smooth singular 2-chains $\sigma : \Delta^2 \to M$. Once again, we do not want to evaluate all these integrals, but rather do one, universal computation. Let us therefore first try to classify al line bundles over a manifold.

Lemma 3.3. *For any line bundle L over M there exists a smooth map $\Phi : M \to \mathbb{P}^N$ for some $N > 0$ such that $L \cong \Phi^* T$.*

Proof. Consider the dual line bundle L^* to L. Choose N sections $\xi_1, \ldots, \xi_N$ of L^* such that for each $x \in M$ at least one ξ_i, $i = 1, \ldots, N$ is nonzero. Define V^* to be the complex vector space with basis $\xi_1, \ldots, \xi_N$. For each $x \in M$, we have evaluation maps

$$\mathrm{ev}_x : V^* \to L^*,$$

and these are surjective because at least one of the ξ_i is nonzero. Dually, this leads to an inclusion $\Phi_x : L \hookrightarrow V$, and therefore to a map $\Phi : M \to \mathbb{P}(V)$. Quite obviously, this leads to an isomorphism $\Phi^* T \cong L$. $\square$

Remark 3.4. There is a chain of inclusions of projective spaces

$$\mathbb{P}^0 \subset \mathbb{P}^1 \subset \ldots \subset \mathbb{P}^N \subset \ldots \subset \mathbb{P}^\infty,$$

with $\mathbb{P}^i \backslash \mathbb{P}^{i-1} \cong \mathbb{C}^i$. Here $\mathbb{P}^\infty$ can be taken to be the projective space of $L^2(S^1)$: considering the standard Fourier basis $e_n(\theta) = e^{2\pi i n\theta}$, $\theta \in S^1$, we can construct a chain of inclusions

$$\{0\} \subset \mathbb{C} \subset \mathbb{C}^2 \subset \ldots \subset \mathbb{C}^N \subset \ldots \subset L^2(S^1).$$

Taking the projective space, this leads to the chain above. Then the lemma can be rephrased as follows: *For any line bundle there is a map $\Phi : M \to \mathbb{P}^\infty$ such that $L \cong \Phi^* T$.* Furthermore, one can show that this map *is unique up to homotopy*. Mathematically this says that $\mathbb{P}^\infty$ is a model for *the classifying space for line bundles*: any line bundles arises by pull-back of the universal line bundle. Mathematicians (notably topologists) write $BU(1)$ for this space.

Step II: Computation for $\mathbb{P}^N$. To finish the proof of the integrality of the first Chern class of a line bundle it suffices by Lemma 3.3 and naturality of Chern classes Proposition 2.1 i) to prove that the first Chern class $c_1(T)$ of the tautological line bundle over $\mathbb{P}^N$ is integral. In this case, in degree 2 there is in fact only one nontrivial singular cycle over which to perform the integral:

Fact 3.5. *The image of the fundamental class of $\mathbb{P}^1 \hookrightarrow \mathbb{P}^N$ generates $H_2^{\text{sing}}(\mathbb{P}^N, \mathbb{Z})$.*

This follows from a computation similar to the cohomology computation of Example **??**. Using the homotopy invariance of the period map, we see that it suffices to compute the intergal of the Fubini–Study form over $\mathbb{P}^1$:

$$\begin{aligned}
\int_{\mathbb{P}^1} \omega_{FS} &= \frac{i}{2\pi} \int_{\mathbb{C}} \frac{1}{(1+|z|^2)^2} dz \wedge d\bar{z} \\
&= 2 \int_0^\infty \frac{r dr}{(1+r^2)^2} \\
&= 1.
\end{aligned}$$

Here we have covered $\mathbb{P}^1$ with one chart U_0 around zero by deleting the point at infinity. Applying the chart map φ_0 to $\omega_{FS}|_{U_0}$, one finds the 2-form appearing in the integral. In the second line we have used polar coordinates $z = re^{i\varphi}$.

Step III: The splitting principle. Now that we know that Theorem 3.2 holds true for line bundles, we want to reduce the statement for vector bundles to that of line bundles. This is achieved by the so-called *splitting principle*. This principle can be stated as follows:

Proposition 3.6 ("The splitting principle"). [3] *Let $E \to M$ be a vector bundle of rank r over a manifold. Then there exists a manifold N together with a smooth map $f : N \to M$ such that:*

[3] Informally, the splitting principle reads: to show an equality between certain characteristic classes, you may assume that the vector bundle is a direct sum of line bundles.

i) *The induced map on cohomology* $f^*: H^\bullet_{\mathrm{sing}}(M,\mathbb{K}) \to H^\bullet_{\mathrm{sing}}(N,\mathbb{K})$ *is injective, for* $\mathbb{K} = \mathbb{R}, \mathbb{Z}$.

ii) *There exist line bundles* $L_1, \ldots, L_r$ *over* N *together with an isomorphism of vector bundles*

$$f^*E \cong L_1 \oplus \ldots \oplus L_r.$$

Let us first explain how to find this space N. Consider the space $\mathbb{P}(E) = \coprod_{x\in M}\mathbb{P}(E_x)$ of lines in the fibers of E. Using the local triviality of the vector bundle $E \to M$, together with the manifold structure on projective space in Example **??**, one can show that $\mathbb{P}(E)$ is a smooth manifold, and that the projection $p : \mathbb{P}(E) \to M$ makes it into a smooth fiber bundle with fiber $\mathbb{P}^r$, where r is the rank of E. Over $\mathbb{P}^r(E)$ we have the tautological line bundle

$$T := \{(v, L) \in E \times \mathbb{P}(E),\ v \in L\} \subset p^*E$$

We now use the fact that for an inclusion of vector bundles, there is always a complement: we can find a vector bundle $E_1 \to \mathbb{P}(E)$ such that $p^*E \cong T \oplus E_1$. We then proceed by induction, applying the previous construction to E_1, until we have exhausted the rank of E. This shows the second property. To explain *i*), let us consider the cohomology of $\mathbb{P}(E)$. Denote by $u := c_1(T)$, the first Chern class of the tautological line bundle. Since the restriction of T to each fiber $\mathbb{P}(E_x)$ is the tautological line bundle over projective space, we see that the restriction of the differential forms $1, u, u^2, \ldots, u^r$ generate the cohomology of the fiber $\mathbb{P}(E_x)$ by Example **??**, the so-called *Leray–Hirsch Theorem* states that there is an isomomorphism:[4]

$$H^\bullet(\mathbb{P}(E)) \cong H^\bullet(M) \otimes \mathbb{C}\{1, u, \ldots, u^r\}.$$

This proves that the pull-back $p^* : H^\bullet(M) \to H^\bullet(\mathbb{P}(E))$ is injective.

To find the space N we now proceed by induction, splitting of a line bundle of the vector bundle E_1 over $\mathbb{P}(E)$, and continuing until the rank of the vector bundle is exhausted.

[4]This statment is only additive. If one takes into account the multiplicative structure induced by the wedge product, the statement is:

$$H^\bullet(\mathbb{P}(E)) \cong H^\bullet(M)[u]/(u^r + u^{r-1}c_1(E) + \ldots + c_r(E)).$$