Before we do QFT, let's focus on finite dimensional systems (Quantum mechanics) The classical theory (called "classical" or "Newtonian" mechanics) can be formulated in 2- ways:

Classical:	"Hamiltonian Mechanics" Symplectic Geometry T^*M "phase space"	$\xleftarrow{\text{Legendre transform}}$	Lagrangian Mechanics Variational principles TM
classical limit $\uparrow$ $\downarrow$ quantization	$\downarrow$ quantization of observables		$\downarrow \int Dq(t)\, e^{\frac{i}{\hbar}\int L\,dt}$.
Quantum	"Hamiltonian QM" Operator formalism Hilbert spaces $L^2(M)$ "wave functions"		Path integral formalism Measures on path spaces (Wiener measure in Euclidean times)

We will start in the upper right corner. Let's do the example of mechanics on a general manifold M. We shall (at some point) need a metric g on M, so let's focus on a Riemannian manifold (M, g).

<u>Def</u>: A "<u>Lagrangian function</u>" is a C^1-function $L: TM \to \mathbb{R}$.

Such a Lagrangian function leads to an <u>action functional</u>:

$$S(\gamma) := \int_I (\gamma^* L)\,dt = \int_{t_0}^{t_1} L(\gamma(t), \dot\gamma(t))\,dt$$

for $\gamma: I \to M$, $I = [t_0, t_1]$ (we allow $t_0 = -\infty$, $t_1 = \infty$ ...)

The <u>variational problem</u> for S now is: find the extrema of S.

Recall for a finite dimensional manifold X, the extrema of a C^1-function $f: X \to \mathbb{R}$, are the critical points $x_0 \in X$ where $df(x_0) = 0$. In other words

$$\frac{d}{d\varepsilon} f(x_0 + \varepsilon X) = 0 \quad \forall X \in T_{x_0} X.$$

The problem here now is the fact that we have to view „S' as a function

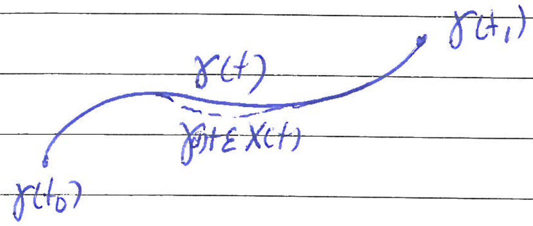
$$S' : C^2(I; M) \longrightarrow \mathbb{R},$$

and the space $C^2(I;M)$ of C^2-paths is – at best – an ∞-D manifold! In the notes so I have worked hard to make $C^2(I;M)$ into a Banach manifold. Then we can require $dS' = 0$ to find the extrema... Here we take a different route.

Def: A path $\gamma(t)$ is an extremum of S' when

$$\left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} S'(\gamma + \varepsilon X) = 0$$

for any compact perturbation of γ.



compact = compact support.

Question: what is a compact perturbation precisely?

Answer: if we really construct $C^2(I;M)$ as a Banach manifold, a compact perturbation would be an element of $\Gamma_c^\infty(I; \gamma^* TM)_0$ = "compactly supported vector fields along $\gamma(t)$".

For $X = \mathbb{R}^n$, this is easy; in this case $C^2(I;\mathbb{R}^n)$ is a Banach space with norm

$$\|\gamma\|_{C^2} := \sup_{x\in I} \|\gamma(t)\| + \sup_{x\in I} \|\gamma'(t)\| + \sup_{x\in I} \|\gamma''(t)\|.$$

and we can use the linear structure: a compact perturbation is just a $X \in C_c^2(I;\mathbb{R}^n)$.

extremum.

Def: $\gamma : I \to M$ is a critical point for S if $\forall$ C^1-families of paths $\gamma_s : [-\varepsilon,\varepsilon] \times I \to M$ with $\gamma(0) = \gamma$ we have

$$\frac{d}{ds} S(\gamma_s) = 0$$

and $\gamma_s(t_0) = \gamma(t_0)$, $\gamma_s(t_1) = \gamma(t_1)$

Prop: For $X = \mathbb{R}^n$, the extrema of the action are exactly the solutions of the Euler-Lagrange equations:

$$\frac{\partial L}{\partial x^i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}^i} = 0 \quad \text{where } \gamma(t) = (x^1(t), \dots, x^n(t)).$$

Pf: Let $\gamma_s(t) = (x_s^1(t), \dots, x_s^n(t))$ be a smooth variation of γ. Define $X^i(t) = \frac{d}{ds}\Big|_{s=0} x_s^i(t)$. Then we have

$$\frac{d}{ds}\Big|_{s=0} S(\gamma_s(t)) = \int_{t_0}^{t_1} \left(\frac{\partial L}{\partial x^i} X^i + \frac{\partial L}{\partial \dot{x}^i} \dot{X}^i \right) dt$$

$$= \int_{t_0}^{t_1} \left(\frac{\partial L}{\partial x^i} X^i - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}^i} \right) X^i \, dt + \underbrace{\left[\frac{\partial L}{\partial \dot{x}^i} X^i \right]_{t_0}^{t_1}}_{=0}.$$

We have: <u>Lemma</u>: if a continuous function $f(t)$ $t_0 \le t \le t_1$ satisfies $\int_{t_0}^{t_1} f(t) g(t)\, dt = 0$ for any g with $g(t_0) = g(t_1) = 0$ then $f \equiv 0$

$\square$

For general M, we have the following

Lemma: If $\gamma : [t_0, t_1] \to M$ is an extremum of S, for any sub interval $[t_0', t_1'] \subset [t_0, t_1]$, $\gamma|_{[t_0', t_1']}$ is an extremum for S,

Proof: let γ_s' be any C^1 variation of $\gamma|_{[t_0', t_1']}$. Then, extending γ_s' by zero, this is also a variation of γ. Therefore

$$\frac{d}{ds}\Big|_{s=0} S(\gamma_s') = 0 \text{ as required}$$

$\square$

<u>Prop</u>: For a general manifold, a path $\gamma : I \to M$ is an extremum of S if it satisfies the Euler-Lagrange equations in local charts on M

<u>Pf</u>: by the Lemma, we can divide the interval I into smaller intervals ~~contain~~ whose image under γ is contained in a local chart. $\square$

<u>Rk</u>: this is only if and only if, if $\gamma(I)$ lies in 1 coordinate chart! (this because the Lemma gives γ critical $\Rightarrow$ $\gamma|_{I'}$ critical.) so only necessary conditions.
Also, solving E-L in one coordinate system might be harder then solving it in another... Hamiltonian formalism!

<u>Examples</u>: (i) (zero order Lagrangians) suppose $L : TM \to \mathbb{R}$ is the pull-back of a function ~~f on~~ on M.

$$(E\text{-}L) : \quad \frac{\partial L}{\partial x^i}(\gamma(t)) = 0.$$

so γ should lie on the critical set of L on M.
generic $L \Rightarrow$ critical pts are isolated $\Rightarrow$ uninteresting

(ii) (first order Lagrangians) Suppose L is linear on the fibers of TM: $L(x,\xi) = L_0(x) + \alpha_x(\xi)$, $\alpha \in \Omega^1(M)$.

$$(E\text{-}L) : \quad \cancel{\frac{\partial L_0}{\partial x^i} \ldots \frac{\partial L_0}{\partial x^i}(\gamma(t)) + \frac{\partial \alpha}{\partial x^i} - \frac{d}{dt}}$$

$$\frac{\partial L_0}{\partial x^i}(\gamma(t)) + \frac{\partial \alpha}{\partial x^i}(\gamma(t), \dot\gamma(t)) - \frac{d}{dt}\alpha^i(\gamma(t), \cancel{\dot\gamma(t)}) \qquad \alpha = \alpha_i \xi^i$$

$$= \frac{\partial L_0}{\partial x^i}(\gamma(t)) + \left(\frac{\partial \alpha_j}{\partial x^i} \dot x^j \cancel{(\gamma(t), \dot\gamma(t))} - \frac{\partial \alpha^i}{\partial x^j}\frac{d x^j}{dt} \right)$$

coordinate free

$$= \cancel{\partial L_0} \; dL_0^{(\gamma(t))} + i_{\dot\gamma(t)} d\alpha = 0$$

Conditions for a minimum

In finite dimensions, $df(x_0)=0$ can mean that we have a (local) minimum, maximum or a saddle point. To distinguish these, we compute the Hessian $\frac{\partial^2 f}{\partial x^i \partial x^j}(x_0) = H_{ij}$. The cases are distinguished by $\det H_{ij} = \begin{cases} <0 & \text{max} \\ >0 & \text{min} \\ =0 & \text{saddle} \end{cases}$

Therefore, to investigate the behaviour ~~at~~ of our action, we consider its second variation:

$$\frac{d^2 S}{ds^2}(\gamma_s) = \sum_{i,j=1}^{n} \int_{t_0}^{t_1} \left\{ \frac{\partial^2 L}{\partial x^i \partial x^j} X^i X^j + 2 \frac{\partial^2 L}{\partial x^i \partial \xi^j} X^i \dot{X}^j + \frac{\partial^2 L}{\partial \xi^i \partial \xi^j} \dot{X}^i \dot{X}^j \right\} dt$$

$$\overset{\text{Integration by parts}}{=} \sum_{i,j} \int_{t_0}^{t_1} \left(Q_{ij} \dot{X}^i \dot{X}^j + P_{ij} X^i X^j \right) dt \qquad (*)$$

with $Q_{ij} = \frac{\partial^2 L}{\partial \xi^i \partial \xi^j}$, $P_{ij} = \frac{\partial^2 L}{\partial x^i \partial x^j} - \frac{d}{dt} \frac{\partial^2 L}{\partial x^i \partial \xi^j}$

where we assume $\frac{\partial^2 L}{\partial x^i \partial \xi^j} = \frac{\partial^2 L}{\partial \xi^i \partial x^j}$.

<u>Claim</u>: the dominant term in $(*)$ is given by Q.

this is because we can view $(*)$ itself as a quadratic functional on the space of paths $X^i(t)$, and call this I_γ. Clearly $I_\gamma(0) = 0$ and we are interested in the variation of I_γ for small X. Now we have $\dot{X}$ small $\Rightarrow$ X small, but not the other way around, we can easily construct a small X (in the C^0-norm) with large $\dot{X}$. Therefore we find Legendre's condition

$$\det\left(\frac{\partial^2 L}{\partial \xi^i \partial \xi^j}\right) \geq 0 \quad \text{along } \gamma(t)$$

Consider the variational problem associated to I_γ: its Euler-Lagrange equations are given by

$$\sum_{i,j=1}^{n} \left\{ \frac{\partial^2 L}{\partial x^i \partial x^j} X^j + \frac{\partial^2 L}{\partial x^i \partial \xi^j} \dot{X}^j - \frac{d}{dt}\left(\frac{\partial^2 L}{\partial x^i \partial \xi^j} X^j + \frac{\partial^2 L}{\partial \xi^i \partial \xi^j} \dot{X}^j \right) \right\} = 0$$

$\boxed{\text{Jacobi equations}}$

Prop if γ_s is a smooth family of solutions of E-L for $\mathcal{S}$, $X(t) := \frac{d}{ds}\Big|_{s=0} \gamma_s(t)$ is a solution of the Jacobi-equations

Proof differentiate E-L. $\square$

Thm: I_γ is positive definite if
(i) $\det\left(\frac{\partial^2 L}{\partial \xi^i \partial \xi^j}\right) > 0$ along $\gamma(t)$
(ii) the interval $(t_0, t_1]$ does not contain a point conjugate to t_0.

Rk: Legendre tried to prove the theorem without (ii).

Examples (continued) (iii) $\boxed{\text{quadratic Lagrangians}}$

Because of Legendre's condition, we are forced to take

$$L(x,\xi) = \frac{1}{2} g_x(\xi,\xi), \quad g \text{ a Riemannian metric.}$$
$$= \frac{1}{2}\|\xi\|^2 \implies \mathcal{S}(\gamma) = \frac{1}{2}\int_{t_0}^{t_1} \|\dot\gamma(t)\|^2 dt$$

Prop: The Euler-Lagrange equations for L are given by the geodesic equations $\nabla_{\dot\gamma} \dot\gamma = 0$.

Proof: let ∇ be the Levi-Civita connection associated to g, determined by the equation

$$d\, g(X,Y) = g(\nabla X, Y) + g(X, \nabla Y)$$

Let γ_s be a compact perturbation of γ_0. Then

$$
\begin{aligned}
\frac{d}{ds} S(\gamma_s(t)) &= \frac{1}{2} \frac{d}{ds} \int_{t_0}^{t_1} g\left(\frac{d\gamma_s}{dt}, \frac{d\gamma_s}{dt}\right) dt \\
&= \int_{t_0}^{t_1} g\left(\nabla_{\frac{d\gamma}{ds}}\left(\frac{d\gamma}{dt}\right), \frac{d\gamma}{dt}\right) dt \\
&= \int_{t_0}^{t_1} g\left(\nabla_{\frac{d\gamma}{dt}}\left(\frac{d\gamma}{ds}\right), \frac{d\gamma}{dt}\right) dt \\
&= \int_{t_0}^{t_1} \left\{ \frac{d}{dt} g\left(\frac{d\gamma}{ds}, \frac{d\gamma}{dt}\right) - g\left(\frac{d\gamma}{ds}, \nabla_{\frac{d\gamma}{dt}} \frac{d\gamma}{dt}\right) \right\} dt \\
&= - \int_{t_0}^{t_1} g\left(\frac{d\gamma}{ds}, \nabla_{\frac{d\gamma}{dt}} \frac{d\gamma}{dt}\right) dt
\end{aligned}
$$

$\square$

Rk: if we include a potential $V \in C^\infty(M)$, we get

$$
\nabla_{\frac{d\gamma}{dt}} \frac{d\gamma}{dt} = - \operatorname{grad}(V)
$$