The heat kernel

(M,g) Riemannian manifold, Δ_g Laplace operator on $C^2(M)$

Def: (M,g) is called complete if it is complete as a metric space for the metric

$$d_M(x,y) = \inf_{\substack{\gamma: I \to M \\ \gamma(0)=x \\ \gamma(1)=y}} \{ L(\gamma) \}$$

with $L(\gamma) = \int_0^1 |\dot\gamma(t)|\, dt$.

Thm (Hopf-Rinow) (M,g) is complete if and only if geodesics can be extended for all times.

Examples: (i) M cpt is complete
(ii) $\mathbb{R}$, $\mathbb{R}^n$ are complete
(iii) $(0,1)$, $[0,1]$ are not complete.

The heat equation on (M,g) is the parabolic PDE

$$\frac{\partial u}{\partial t} + \Delta_g u = 0 \qquad \text{(HE)}$$

for $u : \mathbb{R}^{>0} \times M \to \mathbb{R}$ an unknown function. Initial conditions are given by $\lim_{t \downarrow 0} u(t,x) = f(x)$ for some $f \in C^\infty(M)$.

Rk: if $\psi \in C^\infty(M)$ is an eigenfunction of Δ_g with eigenvalue λ, the function $\psi(x) e^{-\lambda t}$ solves (HE).

Thm: On a complete Riemannian manifold (M,g) there exists a unique C^∞ function $K : M \times M \times \mathbb{R}_{\geq 0} \to \mathbb{R}$ with the properties

- $(\partial_t + \Delta_{g,x}) K_t(x,y) = 0$
- $\lim_{t \downarrow 0} K_t f = f \quad \forall f \in C^\infty(M)$

where $(K_t f)(x) = \int_M K_t(x,y) f(y) \underbrace{dy}_{d\mathrm{vol}_g(y)}$

It follows that the (unique) solution to the heat equation ~~w.r.t~~ with intial data $f \in C^\infty(M)$ is given by $u_t = K_t f$.

Example: $M = \mathbb{R}^n$ with the standard Euclidian metric

$$K_t(x,y) = \frac{1}{\sqrt{4\pi t}} e^{-\|x-y\|^2/4t}$$

Proof of uniqueness: Since Δ_g is symmetric, we have

$$\langle K_t f, g\rangle_{L^2} = \langle f, K_t g\rangle_{L^2} \quad (*)$$

Namely: consider

$$f(\theta) = \oint \langle K_\theta f, K_{t-\theta} g\rangle \qquad 0 < \theta < t.$$

$$\Rightarrow \frac{d}{d\theta} f = -\langle \Delta_g K_t f, K_{t-\theta} g\rangle + \langle K_t f, \Delta_g K_{t-\theta} g\rangle = 0$$

so f is constant: $f(0) = f(\theta) \Rightarrow (*)$

Now suppose K_t^1 & K_t^2 are heat kernels. Following the same argument,

$$f(\theta) = \langle K^1_\theta f, K^2_{t-\theta} g\rangle \text{ is constant} \Rightarrow \langle K_t^1 f, g\rangle = \langle f, K_t^2 g\rangle$$

$$\langle f, K_t^1 g\rangle$$

so $K^1 = K^2$. $\square$

Prop: The operators $K_t(x,y)$ form a semigroup: $K_{t_1} K_{t_2} = K_{t_1+t_2}$.

Proof: The idea is the same. For $f \in C^\infty(M)$, the functions

$K_{t_1} K_{t_2} f$ & $K_{t_1+t_2} f$ solve the same initial value problem. The argument in Prop 4.12 of the lecture notes prove that they are the same $\square$

So it remains to prove Existence

Two approaches to existence 1) Spectral approach
2) By "construction".

Ad 1). We have to use some heavy analysis:

Thm 1 On a complete Riemannian manifold (M,g), The Laplacian Δ_g, acting on the dense domain $C^\infty(M)$, is essentially selfadjoint.

So it is symmetric, $\langle \Delta_g f, g\rangle = \langle f, \Delta_g g\rangle$ $\forall f,g \in C^\infty(M)$ and its closure $\bar{\Delta}_g$ ~~(defined by the closure of the graph)~~ is selfadjoint, i.e.; $D(\bar{\Delta}_g) = D(\bar{\Delta}_g^*)$)
(The domain of the closure is the second Sobolev space).

Thm 2 (Spectral theorem) Let $A : D(A) \subset \mathcal{H} \to \mathcal{H}$ be a selfadjoint operator. There exists a unique map $\hat{\phi}$ from the bounded Borel functions on $\mathbb{R}$ to $B(\mathcal{H})$ s.t.
(i) $\hat{\phi}$ is a $*$-hom.
(ii) $\|\hat{\phi}(f)\|_{B(\mathcal{H})} \leq \|f\|_\infty$
(iii) if $A\psi = \lambda\psi \Rightarrow f(A)\psi = f(\lambda)\psi$.

By this theorem we can define $e^{-t\bar{\Delta}_g}$, a bounded operator on $L^2(M)$.

Thm 3 (Schwartz kernel theorem) M cpt, ~~the map $C^\infty(M\times M)$~~ There is an isomorphism

$$C^\infty(M\times M) \longrightarrow \{\text{Bnd maps } C_c^{-\infty}(M) \to C^\infty(M)\}.$$

So then we have to prove that $e^{-t\bar{\Delta}_g}$ has this property!

Example: M cpt, the spectrum of Δ_g is discrete.

$$K_t(x,y) = \sum_{k=0}^{\infty} e^{-t\lambda_k} u_k(x) \otimes \overline{u_k(y)}$$

Can show that this sum converges!

<u>Ad 2</u>: On a Riemannian manifold (M,g) there exist so-called <u>normal coordinates</u> around a point $x_0 \in M$ given by the (inverse) of the exponential map $\mathrm{Exp}_{x_0} : T_{x_0}M \to M$ in a ball $B_r(x_0)$ around x_0.
In these coordinates we have the following Taylor expansion of the metric: $g_{ij}(x) \sim \delta_{ij} - \frac{1}{3}\sum_{kl} R_{ikjl} x^k x^l + \dots$
So locally, we can pretend we're in a vectorspace ...

<u>Thm</u>: For $x,y \in B_r(x_0)$, there exists a <u>formal solution</u> of the heat equation:

$$K_t^{formal}(x,y) = K_t^{T_{x_0}M}(x,y) \sum_{i=0}^{\infty} t^i \Phi_i(x,y) \quad \text{with}$$

$$\Phi_0(y,y) = 1$$

<u>Rk</u>: there is an explicit iterative formula for $\Phi_i(x,y)$ in terms of $\Phi_{i-1}(x,y)$

This is only <u>local</u> to get something on M, we multiply with a cut-off function on $M \times M$ around the diagonal, and stop after finite number of terms:

$$K_t^N(x,y) := \chi(x,y)\, K_t^{Eucl.}(x,y) \sum_{i=0}^{N} t^i \Phi_i(x,y)$$

Then $(\partial_t + \Delta_{g,x}) K_t^N = O(t^{N-\frac{n}{2}})$ $= R_t^N$ because of the cut-off function!

With this <u>approximate solution</u> we get

$$K_t(x,y) = \sum_{k=0}^{\infty} (-1)^k \int_{t\Delta_k} \int_{M^k} K^N_{t-t_k}(x,z_k)\, R^N_{t_k - t_{k-1}}(z_k, z_{k-1}) \cdots R^N_{t_1}(z_1,y)$$

um converges.