Lie's theorem (= Trotter for matrices) V = f.d. vector space $/\mathbb{R}, \mathbb{C}$.

Thm: Let $A, B \in \mathrm{End}(V)$ then

$$\exp(A+B) = \lim_{n\to\infty} \left[\exp(A/n)\exp(B/n)\right]^n$$

Proof: $S_n = \exp((A+B)/n)$, $T_n = \exp(A/n).\exp(B/n)$;

$$\& \quad S_n^n - T_n^n = \sum_{k=0}^{n-1} S_n^k (S_n - T_n) T_n^{n-k-1}.$$

$$\Rightarrow \|S_n^n - T_n^n\| \leq n\left(\max\{\|S_n\|, \|T_n\|\}\right)^{n-1} \|S_n - T_n\|$$

$$\leq n\, e^{\|A\|+\|B\|} \|S_n - T_n\|$$

also $$\|S_n - T_n\| = \left\|\sum_{k=0}^{\infty} \frac{1}{k!}\left(\frac{A+B}{n}\right)^k - \left(\sum_{\ell=0}^{\infty}\frac{1}{\ell!}\left(\frac{A}{n}\right)^\ell\right)\left(\sum_{m=0}^{\infty}\frac{1}{m!}\left(\frac{B}{n}\right)^m\right)\right\|$$

$$= \left\|(1-1) + \left(\frac{A+B}{n} - \frac{A}{n} - \frac{B}{n}\right) + O\left(\tfrac{1}{n^2}\right)\right\|$$

$$\leq \frac{C}{n^2}$$

So $\lim_{n\to\infty} \|S_n^n - T_n^n\| = 0$ $\quad\blacksquare$

The Wiener measure

(M, g) complete Riemannian manifold. Last week we have constructed the heat kernel $e^{-t\Delta_g}(x,y) = K_t(x,y) \in C^\infty(M\times M)$.
Let us assume that

$$\int_M K_t(x,y)\, dVol_g(y) = 1 \qquad \forall x, \forall t>0.$$

examples: $\mathbb{R}^n$, M cpt.

The Wiener space of M is defined as $(I = (a,T))$.

$$\mathcal{F}_M(I, x_0) = \{ \gamma : I \to M,\ \gamma(a) = x_0,\ \gamma \text{ cont.} \}$$

Def: $P = \{ s_0 = 0 \leq s_1 < .. < s_n = T \}$ is called a partition of length $n = |P|$
$(\Delta_i := |s_i - s_{i-1}|)$

For any partition P we have a map

$$\mathcal{F}_M(I, x_0) \xrightarrow{ev_P} M^{\times |P|}$$

Say that $P < P'$ if $|P| < |P'|$ & P' consists of P with a finite number of points added. Then we have

$$\begin{tikzcd} \mathcal{F}_M(I,x_0) \arrow[r, "ev_P"] \arrow[dr, "ev_{P'}"'] & M^{\times |P|} \arrow[d, phantom, "\circlearrowleft" near start, shift right=3ex] \\ & M^{\times |P'|} \arrow[u, "q"'] \end{tikzcd}$$

$P = \{ s_0 = 0 \leq s_1 < .. s_n = T \}$

$P' = = s_0' = 0 < s_1 .. < s_i < s_i' < .. \}.$

Thm: There exists a unique probability measure on $\mathcal{F}_M(I, x_0)$ such that for any partition P we have

$$\int_{\mathcal{F}(I,x_0)} ev_P^* f(\gamma)\, d\nu(\gamma) = \int_{M^{\times|P|}} \not{f}(x_1, \dots, x_{|P|}) \prod_{i=1}^{n} K_{\Delta_i}(x_{i-1}, x_i)\, dx_1 \dots dx_n$$

for all functions $\not{f} : M^{\times|P|} \to \mathbb{R}$.

Rk: For $P < P'$ we have and $\not{f} \in M^{\times|P'|}$. (Suppose P' has one s_i' more) $s_i < s_i' < s_{i+1}$.

$$\int_{\mathcal{F}(I,x_0)} (ev_{p*} f)(\gamma)\, d\nu(\gamma) = \int_{M^{\times|P'|}} f(\gamma(s_1), \dots \gamma(s_i), \cancel{\gamma(s_i')}, \gamma(s_{i+1}) \dots \gamma(s_n)) \prod_{i=1}^{n} K_{\Delta_i}(\gamma(s_i), \gamma(s_{i+1})).$$

K_{Δ_i}

$$= \int_{M^{\times|P'|}} f(\gamma(s_1), \dots, \gamma(s_n)) \prod_{i=1}^{n} K_{\Delta_i}(\gamma(s_i), y)\, dy.\, dx_1 \dots dx_n$$

So the definition is consistent. (by the semigroup property).

As $\Delta_i \to 0$, we have (see last week)

$$K_t(x,y) \sim \frac{1}{(2\pi t)^{n/2}} e^{-d_M^2(x,y)/2t}$$

$$\Rightarrow \prod_{i=1}^{n} K_{\Delta_i}(\gamma(s_i), \gamma(s_{i+1})) \sim \prod_{i=1}^{n} \frac{e^{-d_M^2(\gamma(s_i), \gamma(s_{i+1}))/2\Delta_i}}{(2\pi\Delta_i)^{n/2}}$$

$$= \frac{1}{Z_P} e^{-\frac{1}{2}\int_0^1 (\gamma'(t))^2 dt}$$

for γ a geodesic path of shortest length between $\gamma(s_i)$

$\Rightarrow$ Heuristically: $d\nu(\gamma) = \frac{1}{Z} e^{-\frac{1}{2}E(\gamma)} d\gamma$, with

$E(\gamma) = \int_0^1 |\gamma'(t)|^2 dt$ the energy of the path.

To get more insight we try to approximate $\mathcal{F}_M(I, x_0)$ by finite dimensional manifolds;

$$\mathcal{F}_M(P, x_0) = \{ \gamma : I \to M \text{ continuous}, \gamma \in C^2(I \setminus P), \nabla_{\dot\gamma} \dot\gamma(t) = 0 \ \forall t \in I \setminus P \}$$

"piecewise geodesic paths"

Claim: $\mathcal{F}_M(P, x_0)$ is a finite dimensional manifolds isomorphic to $\mathbb{R}^{|P|n}$

This is because $T_\gamma \mathcal{F}_M(P, x_0) = \{$ vector fields along γ, satisfying the Jacobi equations for $t \in I \setminus P$ $\}$.

Jacobi equations: $\nabla_{\dot\gamma}^2 X(t) = \underbrace{R_{\dot\gamma, X}}_{\in \Omega^2(\mathrm{End}(TM))}(\dot\gamma)$

~~1st~~ 2nd order ODE, so for this is determined by its initial conditions in $T_{\gamma(0)}M$: For any pair $v, w \in T_{\gamma(0)}M$ $\exists$ a Jacobi field X

with $X(0) = v$
$\dot X(0) = w$.

Jacobi fields

Let (M, g) be a Riemannian manifold, and let $\gamma_s : [0,1] \to M$, $s \in (-\varepsilon, \varepsilon)$ be ~~a geodesic in M~~, a family of geodesics in M (smooth)

Lemma: $X(t) := \frac{d\gamma_s}{ds}\Big|_{s=0}(t)$ satisfies the Jacobi equation

Proof:

$$\nabla_{\dot\gamma} \nabla_{\dot\gamma} X \underset{\substack{\nabla \text{ torsion} \\ \text{free}}}{=} \nabla_{\frac{\partial\gamma}{\partial t}} \nabla_{\frac{\partial\gamma}{\partial s}} \frac{\partial\gamma}{\partial t} \underset{\substack{\text{def of} \\ \text{curvature}}}{=} \nabla_{\frac{\partial\gamma}{\partial s}} \underbrace{\nabla_{\frac{\partial\gamma}{\partial t}} \frac{\partial\gamma}{\partial t}}_{=0 \text{ (geodesic equ)}} + R\left(\frac{\partial\gamma}{\partial t}, \frac{\partial\gamma}{\partial s}\right)\left(\frac{\partial\gamma}{\partial t}\right)$$

$$= R(\dot\gamma, X)(\dot\gamma)$$

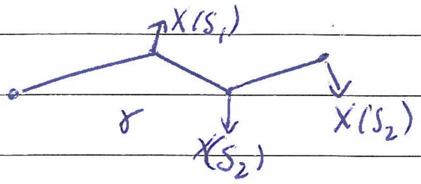
We have used (twice) that $\left[\frac{\partial\gamma}{\partial s}, \frac{\partial\gamma}{\partial t}\right] = 0$

$\square$

Def: For $X, Y \in T_\gamma F_M(P, x_0)$, define

$$g_P(X,Y) = \sum_{i=1}^{n} \left\langle \frac{\nabla X(s_{i-1}+)}{ds}, \frac{\nabla Y(s_{i-1}+)}{ds} \right\rangle \Delta_i s$$

with $\dfrac{\nabla X(s_{i-1}+)}{ds} = \lim_{s \downarrow s_{i-1}} \dfrac{\nabla X(s)}{ds}$



$\Rightarrow$ $(F_M(P, x_0), g_P)$ is a Riemannian manifold.

Claim: We now have

$P < P'$

$$\begin{tikzcd} F_M(P, x_0) \arrow[r] \arrow[d] & F_M(I, x_0) \\ F_M(P', x_0) \arrow[ur] & \end{tikzcd}$$

Claim: $f : F_M(I, x_0) \to \mathbb{R}$ bounded & continuous

$$\int_{F_M(I, x_0)} f(\gamma)\, d\nu(\gamma) = \lim_{|P| \to 0} \int_{F_M(P, x_0)} f(\gamma) e^{-\frac{1}{2}E(\gamma)}\, d\mathrm{Vol}_{g_P}(\gamma)$$