Perturbation theory: a Toy model

In QM we have to evaluate integrals of the form

$$\langle q_{j_1}(t_1) \ldots q_{j_n}(t_n) \rangle := \int q_{j_1}(t_1) \ldots q_{j_n}(t_n)\, e^{iS(q)/\hbar}\, \mathcal{D}q .$$

or, in Euclidean signature

$$\langle q_{j_1}(t_1) \ldots, q_{j_n}(t_n) \rangle := \int q_{j_1}(t_1) \ldots q_{j_n}(t_j)\, e^{-S(q)/\hbar}\, \mathcal{D}q$$

where $S(q)$ = quadratic + h.o.t.

We want to evaluate such integrals in ~~the classical~~ perturbation theory.

In the finite dimensional situation, we can make this precise:

Theorem (Steepest descent). Suppose that $f, g \in C^\infty(B)$, where B is a closed box in a vector space V. Assume that f has a global ~~maximum~~ minimum at an interior point $c \in B$, such that the Hessian ~~$H(f)$~~ ($B := \mathrm{Hess}_c f$) is positive definite.

Then:

$$\int_B g(x)\, e^{-f(x)/\hbar}\, dx = \hbar^{\frac{d}{2}}\, e^{-f(c)/\hbar}\, I(\hbar)$$

where $I(\hbar)$ extends to a smooth function on $[0, \infty)$ such that $I(0) = (2\pi)^{\frac{d}{2}} \dfrac{g(c)}{\sqrt{\det B}}$.

Proof of steepest descent in dim 1

Thm: f, g smooth on $[a,b]$, f has a global minimum at $c \in (a,b)$ s.t. $f''(c) > 0$

"Laplace's method". Then
$$* \quad \int_a^b g(x) e^{-f(x)/\hbar} dx = \sqrt{\hbar}\, e^{-f(c)/\hbar} I(\hbar)$$

where $I(\hbar)$ extends to a smooth function on $[0,\infty)$ s.t. $I(0) = \sqrt{2\pi}\dfrac{g(c)}{\sqrt{f''(c)}}$

Proof: We replace $I(\hbar)$ by several functions: $\quad 0 < \varepsilon < \frac{1}{2}$

- $I_1(\hbar)$ defined by $*$, but with integration over $[c - \hbar^{\frac{1}{2}-\varepsilon}, c + \hbar^{\frac{1}{2}-\varepsilon}]$
  ~~since f has a global minimum,~~ $\cancel{\forall f(x) \ldots}$ $|f(x)| \geq f(c \pm \hbar^{\frac{1}{2}-\varepsilon})$
  Then $(I - I_1)(\hbar) = O(\hbar^N)$ $\forall N \geq 0$, $\hbar \downarrow 0$. So it suffices to prove ~~ev~~ the statement for $I_1(\hbar)$
  $y = (x-c)/\sqrt{\hbar} \Rightarrow I_1(\hbar) = \int_{-\hbar^{-\varepsilon}}^{\hbar^{-\varepsilon}} g(c + y\sqrt{\hbar}) e^{(f(c) - f(c+y\sqrt{\hbar}))/\hbar} dy$

- $I_2^N(\hbar)$: replace the integrand by its Taylor expansion in $\sqrt{\hbar}$ up to order $\hbar^N$: $|I_1(\hbar) - I_2^N(\hbar)| \leq C \hbar^{N-\varepsilon}$

- $I_3^N(\hbar)$: extend the integration to $(-\infty,\infty)$: $|I_2^N(\hbar) - I_3^N(\hbar)| = O(\hbar^N)$ $\forall N$, $\hbar \downarrow 0$.
  "rapidly decreasing".

So prove that $I_3^N(\hbar)$ has a Taylor expansion mod $\hbar^N$.
odd powers in $\sqrt{\hbar}$: odd functions of y. So integrals are zero. So Taylor series exist.

$$I_3(\hbar) = g(c)\int_{\mathbb{R}} e^{-\frac{f''(c) y^2}{2}} dy = \frac{\sqrt{2\pi}\, g(c)}{\sqrt{f''(c)}}. \qquad \square$$

For the n-dimensional case, we use the integral

$$\int_V e^{-B(x,x)/2} dx = (2\pi)^{d/2} (\det B)^{-1/2}$$

B a positive definite symmetric bilinear form.

Example : $\int_{-\infty}^{\infty} e^{-(x^2+x^4)/2\hbar}\, dx = \sqrt{2\pi\hbar}\; I(\hbar) \Longleftrightarrow I(\hbar) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-(y^2+\hbar y^4)/2}\, dy$

by Taylor: $I(\hbar) = \sum_{n\geq 0} a_n \hbar^n \qquad a_n := \frac{(-1)^n}{\sqrt{2\pi}} \int_{-\infty} e^{-y^2/2} \frac{y^{4n}}{2^n n!}\, dy$

$$\left(\Gamma(s+1) := \int_0^{\infty} t^s e^{-t}\, dt\,,\ s>0 \right) \qquad = \frac{(-1)^n}{\sqrt{2\pi}}\, 2^{n+\frac{1}{2}} \frac{\Gamma(2n+\frac{1}{2})}{n!}$$

divergent! $I(\hbar)$ is not analytic at $x=0$.
the Taylor series is only an asymptotic expansion:

Def $f(x) \sim \sum_{n=0}^{\infty} a_n x^n$, $x \to 0$ if, for each $N \geq 0$ we have

$\boxed{f: \mathbb{R}_{\geq 0} \to \mathbb{R}}$

$$f(x) - \sum_{n=0}^{N} a_n x^n = o(x^N) \qquad x \to 0.$$

$f'(0) = \lim_{x\to 0} \frac{f(x)-f(0)}{x}$

So $I(\hbar)$ is not determined by its asymptotic series. It is true that for any formal power series $\sum a_n x^n$, there exists a function which has this series as a Taylor series.

Suppose that $g(\hbar) = \sum_{n\geq 0} \frac{a_n \hbar^n}{n!}$ converges to an analytic function on $[0,\infty)$ at most exp growth at ∞.

(Borel summable). Define $I(\hbar) := \int_0^{\infty} g(\hbar u) e^{-u}\, du$

Then the Taylor series of $I(\hbar)$ is given by $\sum_n a_n x^n$, because

$$I(\hbar) = \sum_{n\geq 0} \frac{a_n \hbar^n}{n!} \underbrace{\int_0^{\infty} u^n e^{-u}\, du}_{n!} = \sum_{n\geq 0} a_n \hbar^n.$$

Ex: $\sum (-1)^n n!\, \hbar^n \Rightarrow I(\hbar) = \int_0^{\infty} \frac{e^{-u}}{1+\hbar u}\, du.$

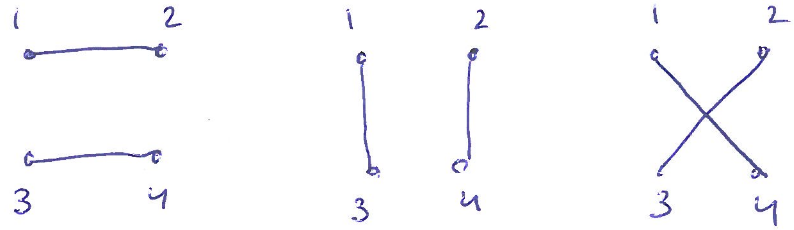
<u>Wick's theorem</u>, $V = \mathbb{R}^n$. We want to calculate integrals of the form $\int_V P(x) e^{-B(x,x)/2} dx$, $P(x)$ a polynomial on V.

<u>Thm</u> (Wick) Let $\alpha_1, \dots, \alpha_k \in V^*$. For k even, we have

$$\int_V \alpha_1(x) \dots \alpha_k(x) e^{-B(x,x)/2} dx = \frac{(2\pi)^{d/2}}{\sqrt{\det B}} \sum_{\substack{\sigma \in S_{2k} \\ \sigma^2 = 1 \\ \sigma(x) \neq x}} \prod_{i \in \{1,\dots,k\}/\sigma} B^{-1}(\alpha_i, \alpha_{\sigma(i)})$$

with B^{-1} = inverse form on V^*. When k is odd the integral is zero.

<u>Rk</u> σ is called a <u>pairing</u>: $\{1, 2, 3, 4\}$ we visualise σ as



terms: $(k-1)!!$

<u>Proof</u>: k odd: the ~~f~~integrand is odd, so obvious.
k even: l.h.s & r.h.s. are polynomials on $V^* \ni \alpha_i$, symmetric!
so it suffices to consider $\alpha = \alpha_1 = \dots = \alpha_k$.
Choose a coordinate system $x_1, \dots, x_n$ s.t. $B(x,x) = x_1^2 + \dots + x_n^2$
$\alpha(x) = x_1$

So we can assume $n = 1$, so we have

$$\int_{-\infty}^{\infty} x^{2l} e^{-x^2/2} dx = \sqrt{2\pi} \frac{(2l)!}{2^l l!} = \sqrt{2\pi}(2l-1)!! \qquad \square$$

$$2^l l! = \frac{(2l)!}{(2l)(2l-2)\dots} = (2l-1)!!$$

Computing the Taylor coefficients: Feynman diagrams.

We have seen that the function $I(\hbar) := \hbar^{-d/2} e^{S(c)/\hbar} \int_B g(x) e^{-S(x)/\hbar} dx$

has a Taylor expansion at $\hbar = 0$. The computation of its Taylor coefficients is a combinatorial problem: we have already seen that for this we have to consider the Taylor expansion of g & S around the critical point $c \overset{=0}{}$: Let us therefore assume that g is already a polynomial.

Expand $S(x) = \frac{B(x,x)}{2} + \sum_{r \geq 3} \frac{B_r(x,..,x)}{r!}$, $B_r(x) = d^r S(0)$

$\alpha_i \in V^*$, $\langle \alpha_1, \dots, \alpha_N \rangle := \frac{e^{S(c)/\hbar}}{\hbar^{-d/2}} \int_B \alpha_1(x) \dots \alpha_N(x) e^{-S(x)/\hbar} dx$.

"correlation functions".

By the proof of the theorem, we know that for Taylor coefficients, we change variables $x \mapsto x/\sqrt{\hbar}$ and extend the integral to all of V; (assume $S(0) = 0$)

$$\langle \alpha_1, \dots, \alpha_N \rangle \sim \hbar^{N/2} \int_V \alpha_1(x) \dots \alpha_N(x) e^{-\frac{B(x,x)}{2} - \sum_r \hbar^{r/2 - 1} \frac{B_r(x,..,x)}{r!}} dx$$

(So, the l.h.s. & the r.h.s. have the same asymptotic expansion in $\hbar$)

Thm: $\langle \alpha_1, \dots, \alpha_N \rangle = \frac{(2\pi)^{d/2}}{\sqrt{\det B}} \sum_{\Gamma \in G_{\geq 3}(N)} \frac{\hbar^{b(\Gamma)}}{|Aut(\Gamma)|} F_\Gamma(\alpha_1, \dots, \alpha_N)$,

where:

- $G_{\geq 3}(N)$ is the set of isomorphisms of Feynman diagrams: graphs with N "external" edges labeled $1,..,N$ and finite number of internal vertices of valency ≥ 3 (multiple edges, loops allowed)

N=2 | N=1

$|Aut| = 2$

- $Aut(\Gamma)$ = group of automorphisms = permutation of edges & vertices which preserves the graph structure
- $b(\Gamma)$ = # edges − # internal vertices

$F_\Gamma(\alpha_1,\dots,\alpha_N) \in \mathbb{C}$ is obtained as follows:

(i) put $\alpha_i \in V^*$ at the i'th external vertex

(ii) put $-B_r$ at each r-valent internal vertex.

(iii) contract everything using B^{-1}. (= "propagator")

The partition function: Write $S(x) = \frac{1}{2}B(x,x) + \sum_{k\geq 0} g_k \frac{B_k(x,\dots x)}{k!}$

g_i, formal parameters.

partition function: $Z := \hbar^{-d/2} \int_V e^{-S(x)/\hbar}\, dx \in \mathbb{C}[[\hbar,\hbar^{-1},g_0,g_1,\dots]]$

Thm: $(*)$ $$Z = \frac{(2\pi)^{d/2}}{\sqrt{\det B}} \sum_{\vec{n}} \left(\prod_i g_i^{n_i}\right) \sum_{\Gamma\in G(\vec{n})} \frac{\hbar^{b(\Gamma)}}{|\mathrm{Aut}(\Gamma)|} F_\Gamma,$$ where

- $\vec{n} = (n_0, n_1, \dots)$ sequence of integers $n_i \geq 0$, almost all zero
- $G(\vec{n})$ = iso class of graphs with n_0 0-valent vertices
  n_1 1-valent "
  n_2 2 - . . . "

The previous thm is a special case of this one: again $\langle\alpha_1,\dots,\alpha_N\rangle$ is symmetric, so it suffices to consider $\alpha_1 = \dots = \alpha_N = \alpha$.
To compute $\langle\alpha^N\rangle$ for all N, we consider the generating series

$$\langle e^{t\alpha}\rangle = \sum_N \langle\alpha^N\rangle \frac{t^N}{N!}$$

For this, we put $g_0 = g_2 = 0$, $g_i = 1$ $i\geq 3$, $g_1 = -\hbar t$, $B_0 = 0$, $B_2 = 0$, $B_1 = \alpha$
and apply the latter theorem. $(*)$

Proof of theorem (*)

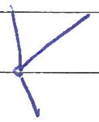
put $y = x/\sqrt{\hbar}$, $Z := \sum_{\vec{n}} Z(\vec{n})$, by expanding the exponential in a Taylor series

$$Z(\vec{n}) = \int_V e^{-B(y,y)/2} \prod_i \frac{g_i^{n_i}}{(i!)^{n_i} n_i!} \left(-\hbar^{\frac{i}{2}-1} B_i(y,\dots,y)\right)^{n_i} dy$$

B_i is a polynomial of degree i, so we can compute these integrals using Wick's theorem:

① assign to each B_r, a graph wit 1 vertex & r attached edges.

n_r of the graphs for B_r

$B_3 \longleftarrow$ 

② By Wick's theorem we have to contract these graphs using pairings σ of the external edges $\Rightarrow$ Feynman graphs, $F(\sigma) \in \mathbb{C}$.

③ Wick: $$Z(\vec{n}) = \frac{(2\pi)^{d/2}}{\sqrt{\det B}} \prod_i \frac{g_i^{n_i}}{(i!)^{n_i} n_i!} \hbar^{n_i(\frac{i}{2}-1)} \sum_\sigma F(\sigma)$$

Claim: $$\boxed{\sum_{\sigma} F(\sigma) = \sum_\Gamma \frac{\prod_i (i!)^{n_i} n_i!}{|Aut(\Gamma)|} F_\Gamma}$$ (check...)

Remark: $\sum_i n_i(\frac{i}{2}-1) = \#\text{edges} - \#\text{vertices}$.

The case of an imaginary exponent: stationary phase

Thm ("Stationary phase") Assume that f has a unique critical point ~~box~~ $c \in B \subset V$ such that $\det f''(c) \neq 0$, and that g vanishes with all derivatives on the boundary of the box. Then

$$\int_B g(x)\, e^{if(x)/\hbar}\, dx = \hbar^{d/2} e^{if(c)/\hbar}\, I(\hbar)$$

where $I(\hbar)$ extends to a smooth function on $[0,\infty)$ s.t.

$$I(0) = (2\pi)^{d/2} e^{\pi i \sigma/4} \frac{g(c)}{\sqrt{|\det f''(c)|}},$$

where σ is the signature of the Hessian of f at c.

Proof is more subtle than that of steepest descent. Have to use the conditionally convergent integral

$$\int_V e^{iB(x,x)/2}\, dx = \frac{(2\pi)^{d/2} e^{\pi i \sigma(B)/4}}{|\det(B)|^{-1/2}}$$