Chern - Simons theory

§ 1: Connections & curvature. G Lie group, $P \xrightarrow{\pi} M$ principal G-bundle

at each $p \in P$ we have an exact sequence of vector spaces

$$0 \to \mathfrak{g} \xrightarrow{i} T_pP \xrightarrow{d\pi} T_{\pi(p)}M \to 0 .$$

The following are equivalent:

(i) a map $\sigma_p : T_{\pi(p)}M \to T_pP$ s.t. $d\pi \circ \sigma = 1$

(ii) a map $\tau_p : T_pP \to \mathfrak{g}$ s.t. $\tau \circ i = 1$

(iii) a subspace $H_p \subset T_pP$ s.t. $T_pP = \mathfrak{g} \oplus H_p$.

} is called a splitting.

Varying $p \in P$ we get a short exact sequence of vector bundles:

$$0 \to \mathfrak{g} \times P \xrightarrow{i} TP \xrightarrow{d\pi} \pi^*TM \to 0 \qquad (*)$$

G acts on these vector bundles:
- trivially on π^*TM
- $TR_g : TP \circlearrowleft \ \forall g \in G$
- $Ad_g \times R_g$ on $\mathfrak{g} \times P$, since $\mathfrak{g}$ = left invariant v.f. on G.

<u>Def</u>: A connection on P is a G-equivariant splitting of $(*)$

By (ii) above, this is given by $A \in \Omega^1(P, \mathfrak{g})$ s.t.

- $i_{\xi_P} A = \xi \quad \forall \xi \in \mathfrak{g}$.
- $R_g^* A = Ad_g(A) \quad \forall g \in G$

<u>Example</u>: For the trivial bundle $P = M \times G$, we have $TP = TM \times \mathfrak{g}$, so it has a canonical connection.

<u>Lemma</u>: Every G-bundle has a connection. The space of all connections is an affine space modeled on $\Omega^1(\mathrm{ad}(P))$.

<u>Proof</u>: (i) use local triviality + partition of unity.

(ii) let A_1, A_2 be connections. Recall that

$$\mathrm{ad}(P) := (P \times \mathfrak{g})/G .$$

Then $A_1 - A_2$ is $\in \Omega^1(P;\mathfrak{g})$
- G-invariant
- G-basic: $\iota_{\xi_P}(A_1 - A_2) = 0$

so it descends to $(P \times \mathfrak{g})/G =: \mathrm{ad}(P)$. $\square$

The <u>curvature</u> of a connection is defined as follows: for X, Y vector fields on M, define $\tilde{X}, \tilde{Y}$ to be the <u>unique</u> lifts to P in the subbundle H defined by the connection.

$$R(X,Y) := [\tilde{X}, \tilde{Y}] - \widetilde{[X,Y]} ,$$

defines a form $R \in \Omega^2(\mathrm{ad}\, P)$

A connection is called <u>flat</u> if $R = 0$.

§2 Chern-Weil theory & the Chern Simons action

<u>Def</u>: An <u>invariant polynomial</u> is a polynomial map $P : \mathfrak{g} \to \mathbb{C}$ satisfying $P(\mathrm{Ad}_g(X)) = P(X)$ $\forall X \in \mathfrak{g}$, $g \in G$.

$\mathbb{C}[\mathfrak{g}]^G = I_{inv}(\mathfrak{g}) = \bigoplus_{k \geq 0} I^k_{inv}(\mathfrak{g})$ ring of invariant polynomials.

<u>Remark</u>: G reductive $\Rightarrow$ $\mathbb{C}[\mathfrak{g}]^G$ is a polynomial algebra with $\mathrm{rank}(G)$ generators. (Chevalley)

<u>Thm</u> (Chern-Weil). For $P \in I^k_{inv}(\mathfrak{g})$, the differential form $P(R) \in \Omega^{2k}_M$ is closed; $dP(R) = 0$ and its cohomology class is independent of the choice of connection.

<u>Proof</u> (Sketch). We can locally trivialize $P|_{\mathcal{U}}$ by choosing a local section $s : \mathcal{U} \to P$. Then consider $s^*A \in \Omega^1(\mathcal{U}, \mathfrak{g})$.

<u>Claim</u>: $R|_{\mathcal{U}} = d(s^*A) + \frac{1}{2}[s^*A, s^*A] \in \Omega^2(\mathcal{U}, \mathfrak{g})$. ?

Then $dR + [s^*A, R] = [ds^*A, s^*A] - [s^*A, ds^*A]$

$+ [s^*A, ds^*A] + [s^*A, [s^*A, s^*A]]$

$= 0$ (Bianchi)

So locally

$$d P(R, \dots, R) = \sum_{i=1}^{k} P(R, \dots, dR, \dots R)$$

$$= \sum_{i=1} P(R, \dots, (d + [s^*A, -])R, \dots R) \overset{\text{Bianchi}}{=} 0.$$

Now, let A_0, A_1 be two connections and consider the affine combination $A^{aff} := tA_0 + (1-t)A_1$ on the G-bundle

$$P \times [0,1] \longrightarrow M \times [0,1].$$

It has curvature

$$R_t^{aff} = dt \wedge (A_0 - A_1) + \cancel{t(d(A_0 - A_1))} + t\,dA_0 + (1-t)\,dA_1 + t^2 [A_0, A_0]$$
$$+ (1-t)^2 [A_1, A_1] + 2t(1-t)[A_0, A_1]$$

$$P(R_t^{aff}) \in \Omega^{2k}_{cl}(M \times [0,1])$$

Define

$$L(A_0, A_1) := \int_0^1 P(R_t^{aff})\, dt \in \Omega^{2k-1}(M)$$

Stokes for the fiber integral: $\int_0^1 dt : \Omega^k(M \times I) \longrightarrow \Omega^k(M)$

$$d \int_0^1 \alpha = \int_0^1 d\alpha - \alpha_{t=1} + \alpha_{t=0}.$$

In our case $dL(A_0, A_1) = P(R_1) - P(R_0)$

<u>Prop</u>: For 3 connections A_0, A_1, A_2, we have

$$L(A_0, A_1) + L(A_1, A_2) = L(A_0, A_2)$$

<u>Proof</u>: Put the three connections on $P \times \Delta^2 \longrightarrow M \times \Delta^2$ and use Stokes $\square$.

Recall that for a closed k-form ω, its <u>periods</u> are defined by the map

$$\begin{aligned} Per(\omega) : H_k(M, \mathbb{Z}) &\longrightarrow \mathbb{C} \\ \sigma &\longmapsto \int_{\Delta^k} \sigma^* \omega \end{aligned}$$

If the periods are in $\mathbb{Z}$, we call ω <u>integral</u>. They only depend on the class in $H^k_{dR}(M)$!

<u>Def</u>: A polynomial $P \in \mathbb{C}[\mathfrak{g}]^G$ is said to be integral if the form $P(R)$ is integral for one (and hence all) connections A with curvature $R(A)$.

<u>Example</u>: $G = GL(N, \mathbb{C})$. P_k = k-th symmetric polynomial in the eigenvalues

$$\mathbb{C}[gl(n, \mathbb{C})]^{GL(n,\mathbb{C})} \cong \mathbb{C}[P_1, \dots, P_n].$$

$$\left[\frac{1}{(2\pi\sqrt{-1})^k} P_k(A) \right] = c_k(P) \in H^{2k}_{dR}(M) \quad \text{"k-th Chern class"}.$$

Let P be an integral invariant polynomial. Suppose that $P \to M$ is a <u>trivial</u> principal G-bundle: then it admits a global section $s: M \to P$ which induces a trivialization $P \cong M \times G$, and this in turn defines a <u>flat</u> connection A_s.

For any other connection A with curvature R, we have:

$$A_s - A = \alpha \in \Omega^1(M, ad(P)) \Longrightarrow A^{aff} = A_s + t\alpha \quad 0 \le t \le 1.$$

$$(*) \quad \tilde{R}^{aff} = dt\wedge\alpha + t\,d\alpha + 2t[A_s,\alpha] + t^2[\alpha,\alpha]$$

$$L(A, A_s) := \int_0^1 P(R^{aff}, \dots, R^{aff}) =$$

$$\overset{\text{by}(*)}{=} \int_0^1 d\, P(A^{aff}, R^{aff}, \dots, R^{aff}) \overset{\text{Stokes}}{=} P(\alpha, R, \dots, R)$$

Now let $\dim M = 2k-1$.

<u>Lemma</u> $\int_M L(A, A_s) \in \mathbb{R}/\mathbb{Z}$ does not depend on s.

<u>Proof</u>: Let s, s' be 2 sections. By the prop. above we have

$$L(A, A_s) = L(A, A_{s'}) + L(A_{s'}, A_s)$$

Now $\displaystyle\int_M L(A_s, A_{s'}) = \int_M \int_0^1 P(R^{aff}(A_s, A_{s'}))$

$$= \int_M \int_{S^1} P(R^{aff}(A_s, A_{s'})) \in \mathbb{Z}$$

because $R^{aff}\big|_{t=1} = 0 = R^{aff}_{t=0}$

$\square$.

So $e^{2\pi\sqrt{-1}\int_M L(A,A_s)}$ is well-defined!

<u>Chern-Simons theory</u> : G compact Lie group.

$$B : \mathfrak{g} \times \mathfrak{g} \longrightarrow \mathbb{R} \qquad \text{integral multiple of the Killing form}$$

$$B(x,y) \sim \operatorname{tr}(\operatorname{ad}_x \operatorname{ad}_y)$$

B is G-invariant $\rightsquigarrow$ $B \in I^2_{inv}(G)$

$B(R,R) \in \Omega^4_{cl}(M)$ the associated Chern-Weil form.

$$CS(A) = B(A,R) = B(A,dA) + \frac{2}{3} B(A,[A,A])$$

<u>Topological fact</u>: ~~Ohra~~ G compact, simply connected, with simple Lie algebra (e.g. $SU(N)$)
Then any principal G-bundle over a manifold of dim ≤ 3 is trivial.

$$\Rightarrow \quad e^{2\pi\sqrt{-1}\,CS(A)} \quad \text{is well defined!}$$

<u>Lemma</u> : The critical points of CS are flat connections.

<u>Proof</u> : let $A + t\alpha$ be a first order deformation of A

$$\left.\frac{d}{dt}\right|_{t=0} CS(A_t) = B(\alpha, dA) + B(A, d\alpha) + 2B(\alpha,[A,A])$$

$$\cancel{= dB(\alpha, dA) + B(\alpha}$$

$$\cancel{dB(\alpha,A)} +$$

$$= -dB(A,\alpha) + 2B(\alpha, dA) + 2B(\alpha,[A,A])$$

$=0$ $\forall \alpha$ means $R(A) = 0$ $\square$.