Geometric quantization

Let (X,ω) be ~~an~~ symplectic manifold. The goal of geometric quantization is to produce a Hilbert space $\mathcal{H}_X$ together with an "quantization map" $C^\infty(X) \to Op(\mathcal{H}_X)$ relating the Poisson bracket $\{\,,\,\}$ to the commutator.

Geometric quantization consists of two parts:

(i) construction of ~~the~~ a prequantum line bundle

(ii) finding a polarization.

Both steps involve existence & uniqueness issues.

§1 Prequantum line bundles

We are going to consider triples (L,∇,h) where

- $L \to X$ is a complex line bundle
- $\nabla : \Gamma(X;L) \to \Gamma(X;L\otimes T^*M)$ is a connection on L
- h is a hermitian metric on L, $h_x : L_x \times L_x \to \mathbb{C}$, compatible with the connection:

$$(*) \qquad d\,h(s_1,s_2) = h(\nabla s_1, s_2) + h(s_1,\nabla s_2) \qquad s_1,s_2\in\Gamma(X;L).$$

Remark: there is an obvious notion ~~of~~ a morphism of such triples.
With this we get a category (in fact a groupoid)
There ~~are~~ is an equivalence of categories

$$\{(L,\nabla,s) \text{ as above}\} \cong \left\{\begin{matrix}\mathbb{T}\text{-principal bundles}\\ \text{with connection}\end{matrix}\right\}$$

The connection ∇ extends, via the Leibniz rule, to

$$0 \to \Omega^0(X,L) \xrightarrow{\nabla} \Omega^1(X,L) \xrightarrow{\nabla} \Omega^2(X,L) \to \dots$$

The <u>curvature</u> of ∇ is given by $\nabla^2 =: F \in \Omega^2(X; \mathbb{C})$

(remark that $\mathrm{End}(L) = \mathbb{C} \times X \to X$)

When ∇ is compatible with h we have in fact $F(\nabla) \in \Omega^2(X; i\mathbb{R})$

This is the same curvature as that of the associated principal $\mathbb{T}$-bundle. (Remark that the space of connections on L compatible with h is an affine space modeled on $i\Omega^1(M, \mathbb{R})$: this follows from (*)).

Locally $\nabla = d + \alpha$, $\alpha \in \Omega^1(X)$, so $F(\nabla) = d\alpha$. This shows that the <u>Bianchi identity</u> $dF(\nabla) = 0$ holds true.

<u>Def</u>: Let (L, ∇) be a line bundle with connection. Its <u>first Chern form</u> $c_1(L,\nabla) := \frac{i}{2\pi} F(\nabla) \in \Omega^2_{cl}(X)$.

The first Chern class is given by the de Rham cohomology class $[c_1(L,\nabla)] \in H^2_{dR}(M)$.

Let $\omega \in \Omega^2_{cl}(X)$. We can represent homology class (of degree 2) by <u>smooth</u> maps $\gamma : \Delta^2 \to X$. (We have

$$\underset{\text{smooth sing chains}}{S_k^\infty(X)} \hookrightarrow \underset{\text{cont. sing chains}}{S_k(X)},$$ and this is a quasi-iso $\Rightarrow H_k^\infty(M) \cong H_k(M)$)

de Rham's homomorphism: $\alpha \in \Omega^k_{cl}(X)$, $\sigma : \Delta^k \to M, \sigma \in S^\infty_k(X)$

$$\langle \alpha, \sigma \rangle := \int_{\Delta^k} \sigma^* \alpha$$

Defines a map ~~$H_k(X) \to \mathbb{R}$. and co~~

$$\Omega^k(X) \longrightarrow Hom(S^\infty_k(X); \mathbb{R})$$

Thm (de Rham) this is a morphism of complexes & induces

$$H^k_{dR}(X) \cong H^k_{sing}(X).$$

Given $\alpha \in \Omega^k_{cl}(X)$, its periods are given by the integrals

$$\int_{\Delta^k} \sigma^* \alpha \in \mathbb{R}, \quad \sigma \in H^{sing}_k(X).$$

Since we work with smooth chains, we can in fact consider, for $\omega \in \Omega^2_{cl}(X)$:

Def: The period group of $\omega \in \Omega^2_{cl}(X)$ is the subgroup of $\mathbb{R}$ generated by the integrals

$$\int_\gamma \omega := \int_{S^2} \gamma^* \omega, \quad \text{for } \gamma : S^2 \to X \text{ smooth.}$$

<u>Prop</u> The periods of the first Chern form $c_1(L,\nabla)$ lie in $\mathbb{Z}$.

<u>Proof 1</u>: (i) Every line bundle can be obtained (up to iso) via pull-back of the universal bundle over $\mathbb{P}^N$.
Using functoriality of the first Chern form, it suffices to prove the statement for $\mathbb{P}^N$

(ii) $\mathbb{P}^N$ has a cell decomposition

$$\mathbb{P}^0 \subset \mathbb{P}^1 \subset \ldots \subset \mathbb{P}^{N-1} \subset \mathbb{P}^N$$

with $\mathbb{P}^i \setminus \mathbb{P}^{i-1} \cong \mathbb{C}^i$

so $\underset{S^2}{\overset{\mathbb{P}^1}{\cong}} \subset \mathbb{P}^N$ is the generator of $H_2(\mathbb{P}^N)$.

(iii) recall $\mathbb{P}^N := \left(\mathbb{C}^{N+1} \setminus \{0\}\right)/\mathbb{C}^*$ is a complex manifold.

The universal line bundle $K_N \to \mathbb{P}^N$ has fiber

$$\left(K_N\right)_{(z_0:\ldots:z_N)} := \text{line spanned by } (z_0,\ldots,z_N) \in \mathbb{C}^{N+1}$$

It has a canonical metric + connection with curvature equal to the Fubini-Study form

$$\omega\big|_{U_i} := \frac{i}{2\pi}\,\partial\bar\partial \log\left(\sum_{k=0}^{n} \frac{|z_k|^2}{|z_i|^2}\right).$$

(check that the forms agree on overlaps!)

(iv) So we have to do 1 computation:

$$\int_{\mathbb{P}^1} \omega_{FS} = \frac{i}{2\pi}\int_{\mathbb{C}} \frac{dz\wedge d\bar z}{(1+|z|^2)^2} = 2\int_0^\infty \frac{r\,dr}{(1+r^2)} = 1$$

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<u>Proof 2</u> i) Choose the following data: $\mathcal{U} = \{U_i\}_{i\in I}$ a covering of X such that over U_i we have nonzero sections (= trivializations) of $L|_{U_i} \overset{s_i}{\leftrightarrows} U_i$

ii) On $U_i \cap U_j =: U_{ij}$ we have transition functions $\varphi_{ij} : U_{ij} \to \mathbb{T}$

• On $L|_{U_i} \underset{\varphi_i}{\cong} U_i \times \mathbb{C}$ we can write the connection ∇ as

$$\varphi_i \circ \nabla \circ \varphi_i^{-1} = d + \alpha_i, \quad \alpha_i \in \Omega^1(U_i) \qquad \varphi_{ij} = \varphi_i \circ \varphi_j^{-1}$$

on U_{ij} we have $\varphi_i^{-1} \circ (d+\alpha_i) \circ \varphi_i = \nabla = \varphi_j^{-1} \circ (d+\alpha_j) \circ \varphi_j$

$$\Longrightarrow (d+\alpha_j) = \varphi_{ij}^{-1} \circ (d+\alpha_i) \circ \varphi_{ij} = d + \varphi_{ij}^{-1} d\varphi_{ij} + \alpha_i$$

so $\boxed{\alpha_j = \alpha_i + \varphi_{ij}^{-1} d\varphi_{ij}.} = \alpha_i + d \log \varphi_{ij}.$

On triple overlaps $\varphi_{ij}\varphi_{jk} = \varphi_{ik}$, so

$$d\log\varphi_{ij} + d\log\varphi_{jk} - d\log\varphi_{ik} = 0 \Rightarrow \log\varphi_{ij} - \log\varphi_{ik} + \log\varphi_{jk} = n_{ijk}\, 2\pi i \in 2\pi i \mathbb{Z}$$

so we find a Čech cocycle. $\{2\pi i\, n_{ijk}\}_{i,j,k\in I}$

$$\begin{tikzcd}
F(\nabla) \in \Omega^2(X) \arrow[r] & \prod_i \Omega^2(U_i) \ni d\alpha_i \arrow[r] & \prod_{ij} \Omega^2(U_{ij}) & \\
\Omega^1(X) \arrow[u, "d"] \arrow[r, "\delta"] & \prod_i \Omega^1(U_i) \ni \alpha_i \arrow[u, "d"] \arrow[r, "\delta"] & \prod_{ij} \Omega^1(U_{ij}) \ni d\log\varphi_{ij} \arrow[u, "d"] & \\
\Omega^0(X) \arrow[u] \arrow[r, "\delta"] & \prod_i \Omega^0(U_i) \arrow[u, "d"] \arrow[r, "\delta"] & \prod_{ij} \Omega^0(U_{ij}) \ni \log\varphi_{ij} \arrow[u, "d"] \arrow[r] & \prod_{ijk} \Omega^0(U_{ijk}) \ni 2\pi i\, n_{ijk}
\end{tikzcd}$$

this gives a (Čech cohomology class) in $H^2(X, \mathbb{Z})$

Claim: this the Chern class! (now with coefficients in $\mathbb{Z}$.)

Rk: $0 \to \mathbb{Z} \to \mathbb{R} \xrightarrow{e^{2\pi i -}} \mathbb{T} \to 0$ s.e.s. of sheaves: $\delta : H^1(X, \mathbb{T}) \to H^2(X, \mathbb{Z})$, $\ni \varphi_{ij}$

Thm (Weil) Let (X, ω) be a manifold with a closed 2-form ω.

(i) when $\mathrm{Per}(\omega) \subset \mathbb{Z} \Rightarrow \exists$ line bundle + connection s.t.

$$c_1(L, \nabla) = \omega$$

(ii) The set of iso-classes of such line bundles is a torsor over $H^1(X; \mathbb{T})$ = group of iso classes of flat line bundles.

Proof: Assume X simply connected, $\pi_1(X) = 1$

$$P(X) = \{ \gamma : I \to X \text{ piece-wise smooth}, \gamma(0) = x_0 \}$$

$$L_\omega := \left(P(X) \times \mathbb{C} \right) / \sim \quad : (\gamma_1, z_1) \sim (\gamma_2, z_2)$$

$$\Longleftrightarrow \gamma_1 \sim_H \gamma_2 \ \& \ z_2 = \exp i \int_H \omega \, z_1$$

$H : [0,1] \times [0,1] \to X$ homotopy.

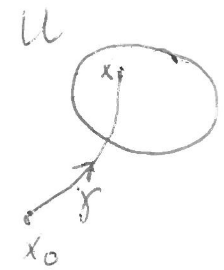
This $\sim$ does not depend on the choice of H because $\mathrm{Per}(\omega) \subset \mathbb{Z}$.

$\Rightarrow L_\omega \to X$ line bundle (proof that it is locally trivial)

$U \subset X$ s.t. $\omega = d\theta$ construct a section of L_ω by
U s.c.

$$s(x) = \left[\gamma, \exp\left(-2\pi i \int_\gamma \theta\right) \right].$$

γ a path from x_0 to x.



define a metric by $\|s\|^2 = 1$
(locally)

" a connection by $\nabla(s) = -2\pi i \theta s. \Rightarrow F(\nabla) = -2\pi i \omega.$

(X, ω) symplectic manifold.

Def A prequantum line bundle is a triple (L, ∇, h) s.t.

$$c_1(L, \nabla) = -\omega$$

By Weil's theorem, such a line bundle exists if $\mathrm{Per}(\omega) \subset \mathbb{Z}$, so this imposes a quantization condition on the symplectic form.

Example: For the sphere of radius r, $S_r \subset \mathbb{R}^3$ we have the symplectic form $\omega = r^2 \sin\theta \, d\theta \wedge d\varphi$. $(r=1 = \mathbb{P}^1)$
$\mathrm{Per}(\omega)$ is generated by $\mathrm{Area}(S_r) = \int_{S_r} r^2 \sin\theta \, d\theta \wedge d\varphi$

$$= 4\pi r^2$$

so we get the condition $r \in \mathbb{Z}$.

Thm: (Kostant - Souriau prequantization) Let $f \in C^\infty(X)$. Define

$$\rho(f)(s) = -i \nabla_{X_f}(s) + fs.$$

defines a morphism of Lie algebras $(C^\infty(X), \{,\}) \to (\mathrm{End}(\mathcal{L}), [,])$

Proof:
$$i[\rho(f), \rho(g)](s) = -i[\nabla_{X_f}, \nabla_{X_g}](s) - [\nabla_{X_f}, g](s) + [\nabla_{X_g}, f](s)$$
$$= -i[\nabla_{X_f}, \nabla_{X_g}](s) - dg(X_f)s + df(X_g)s$$
$$= -i[\nabla_{X_f}, \nabla_{X_g}](s) + iF(\nabla)(X_f, X_g) + \{f, g\}s$$
$$= -i \nabla_{[X_f, X_g]}(s) + \{f, g\}s.$$
$$= \rho(\{f, g\})(s)$$ ∎