MATHEMATICAL APPROACHES TO QFT

1. Introduction

Quantum field theory is an infinite dimensional generalization of quantum mechanics, which originally has been developed to combine the special theory of relativity with quantum mechanics. Quantum mechanics typically deal with with finite dimensional systems. Therefore we first discuss the mathematical content of finite dimensional systems, both classical and quantum mechanical. We will focus on the two main approaches to classical mechanics: the Lagrangian and the Hamiltonian approach. Their quantum mechanical counterparts are given by the operator approach and the path integral approach.

In the course of this chapter we will also help us to introduce some terminology for later use.

2. The Lagrangian approach to Classical Mechanics

Let us first recall the classical Newton equations for a particle moving in $\mathbb{R}^3$ under the influence of a potential $V \in C^1(\mathbb{R}^3, \mathbb{R})$. We use Euclidean coordinates $\vec{x}(t) = (x^1(t), x^2(t), x^3(t))$ to describe the position of the the particle. Then the Newton equation describing the motion of this particle is given by

$$\frac{d^2\vec{x}}{dt^2} = -\vec{\nabla}(V), \tag{1}$$

where the right hand side is simply minus the gradient of V. These equations are a system of second order ODE's, and in general we have local existence and uniqueness of solutions with prescribed initial position $\vec{x}(0) = \vec{x}_0 \in \mathbb{R}^3$ and velocity $\dot{\vec{x}}(0) = v_0 \in \mathbb{R}^3$ by the Picard–Lindelöf theorem. In the following we will consider the generalization of this classical mechanical system to an arbitrary Riemannian manifold (M, g).

The key point is that these equations of motion can be described by means of a variational principle as follows: consider the tangent bundle TM to M, called the state space, and fix a C^1-function $L : TM \to \mathbb{R}$, called a *Lagrangian function*.[1] For a C^2 map $\gamma : [t_0, t_1] \to M$, this defines an action

$$S(\gamma) := \int_{t_0}^{t_1} \dot{\gamma}^* L dt = \int_{t_0}^{t_1} L(\gamma(t), \dot{\gamma}(t)) dt. \tag{2}$$

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[1]We are ignoring the possibility for a time-dependent Lagangian here.

The aim of the associated variational problem is to look for extrema of this functional, keeping the endpoint fixed. This means that we fix $x_0,\ x_1 \in M$ and consider the curves $\gamma(t)$ in M that start in x_0 and end in x_1.

Principle of Least action
The classical trajectories are given by the extremal paths for the action S

For a differentiable function on a finite dimensional manifold, the extrema are easily found by equating the derivative to zero. In this case, since the Lagrangian is a function in infinite dimensions, we have to work a bit more in this case.

2.1. **The Geometry of mapping spaces.** The action S is a function from the mapping space $\mathrm{Map}(I, M)$ to $\mathbb{R}$ and so we are lead to consider the extrema of a function on an infinite dimensional space. To put this on solid grounds, we have to discuss the geometry of infinite dimensional manifolds. The right context for us is that of so-called *Banach manifolds*. In fact, the theory of Banach manifolds can be set up in almost the same manner as the usual theory of finite dimensional manifolds, and some books (e.g. [L] "at no extra costs") do this from the very beginning.

First when $M = \mathbb{R}^n$, we shall consider the Banach space $B = C^k(I, \mathbb{R}^n)$ equipped with the C^k-norm

$$\|f\|_{C^k} := \sum_{i=0}^{k} \sup_{t\in I} \left\| f^{(i)}(t) \right\|. \tag{3}$$

Remark 2.1. In the end we shall specify $k = 2$, a choice dictated by the fact that the classical equations of motion are second order.

Recall that a continuous map $F : U \to B_2$ from an open subset $U \subset B_1$ of a Banach space between to another Banach space B_2 is said to *Fréchèt differentiable* at $b \in U$ if there exists a bounded linear operator $A_b : B_1 \to B_2$ such that

$$\lim_{h\to 0} \frac{\|F(b+h) - F(b) - A_b(h)\|_{B_2}}{\|h\|_{B_1}} = 0.$$

In this case A_b is called the *Fréchèt derivative* of F at $b \in U$.

Next we consider the space $C^k(I, M)$ of maps from I to M that are k-times continuously differentiable. Since M, as a smooth manifold, is locally modeled on $\mathbb{R}^n$, we expect/hope $C^k(I, M)$ to be locally modeled on $B = C^k(I, \mathbb{R}^n)$. Let us first recall the definition of a Banach manifold:

Definition 2.2. Let B be a Banach space. A *Banach manifold modeled on B* is given by a Hausdorff topological space X equipped with a collection $\{U_i, \varphi_i, i \in I\}$ of local charts

$$\varphi_i : U_i \to \varphi_i(U_i) \subset B,$$

satsifying

i) The charts cover X: $\cup_{i\in I} U_i = X$,

ii) The transition maps

$$\varphi_j \circ \varphi_i^{-1} : \varphi_i(U_i \cap U_j) \to \varphi_j(U_i \cap U_j)$$

are smooth.

Remark 2.3. A few remarks about the definition:

i) For a finite dimensional manifold the model space is uniquely fixed by the dimension: $\mathbb{R}^n$ for an n-dimensional manifold. In infinite dimensions this is no longer the case, e.g. we could take $B = C^k(I, M)$ for different k's.

ii) Purists might object to the confusion between an atlas and a smooth structure, but this shouldn't lead to any problems.

For our purposes, we need the following space

$$\mathcal{F}_k(M; x_0, x_1) := \{f : I \to M \text{ of class } C^k,\ f(t_0) = x_0,\ f(t_1) = x_1.\}$$

Proposition 2.4. *Let M be a complete n-dimensional Riemannian manifold. Then the mapping space $\mathcal{F}_k(M; x_0, x_1)$ is a smooth Banach manifold modeled on the Banach space $\mathcal{F}_k(\mathbb{R}^n, 0, 0)$.*

Proof. Let us first equip $C^k(I, M)$ with the compact-open topology in which $f_n \to f$ if it converges uniformly. This topology is known to be Hausdorff. Since M is a metric space, this topology coincides with the one induced by the metric on $\mathcal{F}_k(M; x_0, x_1)$ given by

$$d(f, g) := \sup_{t\in I} d_M(f(t), g(t)).$$

Next, given $f \in C^k(I, M)$, we consider the Banach space

$$B_f := \{X \in \Gamma^k(I, f^*TM),\ X(t_0) = 0 = X(t_1)\},$$

equipped with the norm

$$\|X\|_{C^k} := \sum_{i=0}^{k} \sup_{t\in I} \left\| \nabla^i X \right\|,$$

where ∇ denotes the Levi–Civita connection. Alternatively, choose a trivialization $f^*TM \cong I \times \mathbb{R}^n$ and use the standard C^k-norm (3). This defines an equivalent norm on B_f. (These definitions assume f to be at least C^1, which may not be the case for $k = 0$. In that case, we make the definition only for C^1 maps in $\mathcal{F}_0(M; x_0, x_1)$ and use a denseness argument to prove that we obtain an atlas as described below.)

Since $f(I) \subset M$ is compact, there exists an $\epsilon_f > 0$ such that for each $x \in M$ in an ϵ_f-tubular neighborhood of $f(I)$, there is a unique geodesic from x to $f(I)$ of length less than ϵ_f. In other words, the map defined by

$$\varphi_{f,\epsilon_f}(X)(t) := \exp_{f(t)}(X(t)),$$

is a bijection from $B_{f,\epsilon_f} := \{X \in B_f,\ \|B_f\|_{C^k} < \epsilon_f\}$ to

$$U_{f,\epsilon_f} := \{g \in \mathcal{F}_k(M; x_0, x_1),\ d(f,g) < \epsilon_f\}.$$

Choosing a fixed $\epsilon_f > 0$ satisfying the property above, we obtain in this way an atlas $\{U_{f,\epsilon_f}\}_{f \in \Omega_k(M;x_0,x_1)}$ which clearly covers $\mathcal{F}_k(M; x_0, x_1)$. It remains to show that the transition maps between these charts are smooth. We omit the details of this verification, it uses the fact that the exponential map on a Riemannian manifold is smooth. □

Remark 2.5. We see from the proof that the tangent space to $\gamma \in \Omega(M, x_0, x_1)$ is given by

$$T_\gamma \mathcal{F}_k(M, x_0, x_1) = \{X \in \Gamma^k(I, \gamma^* TM),\ X(t_0) = 0 = X(t_1)\}.$$

This corresponds of course exactly to our intuition where a deformation of a C^k-curve $\gamma(t)$ is given by C^k-map $\gamma_\epsilon(s,t) : (-\epsilon, \epsilon) \times [t_0, t_1] \to M$, $\epsilon > 0$ with fixed endpoints: $\gamma_\epsilon(t_0) = \gamma(t_0) = x_0$ and $\gamma_\epsilon(t_1) = \gamma(t_1) = x_1$. Corresponding to such a deformation is the vector field along γ given by

$$\frac{\partial \gamma_\epsilon}{\partial s}(0,t) \in T_\gamma \mathcal{F}_k(I; M, x_0, x_1).$$

Conversely, given $X \in T_\gamma \mathcal{F}_k(I; M, x_0, x_1)$, we have the deformation

$$\gamma_s(t) := \exp_{\gamma(t)}(sX(t)),$$

where $-\epsilon < s < \epsilon$ for $\epsilon > 0$ small enough. Remark also that we can even think of $\Omega(M; x_0, x_1)$ as a Riemannian manifold with metric

$$\langle X, Y \rangle_\gamma := \int_{t_0}^{t_1} g_{\gamma(t)}(X(t), Y(t)) dt. \tag{4}$$

However, this is just formal: of course the tangent space, consisting of C^k-functions, is not complete for this inner product. Completion would lead to the theory of *Hilbert manifolds*.

2.2. **The variational problem in $\mathbb{R}^n$.** Let us consider for a moment the case $M = \mathbb{R}^n$, with euclidean coordinates denoted by x^i, $i = 1, \ldots, n$. We consider the action functional $S : C^2(I, \mathbb{R}^n) \to \mathbb{R}$ given by (2). For a curve $\gamma \in C^2(I, \mathbb{R}^n)$, we write $\gamma(t) = (x_1(t), \ldots, x_n(t))$. Then we have:

Proposition 2.6. *The functional* (2) *is Fréchèt differentiable with derivative*

$$D_\gamma S(X) = \int_{t_0}^{t_1} \left(\frac{\partial L}{\partial x^i} - \frac{d}{dt} \frac{\partial L}{\partial \dot{x}^i} \right) X_i dt + \left[\frac{\partial L}{\partial \dot{x}^i} X_i \right]_{t_0}^{t_1}.$$

Proof. $C^2(I, \mathbb{R}^n)$ is simply a Banach space so we apply the definition of the Fréchèt derivative. First we compute

$$\begin{aligned} |S(\gamma + X) - S(\gamma)| &= \left| \int_{t_0}^{t_1} \left(L(\gamma + X, \dot{\gamma} + \dot{X}, t) - L(\gamma, \dot{\gamma}, t) \right) dt \right| \\ &\leq \left| \int_{t_0}^{t_1} \left(\frac{\partial L}{\partial x^i} X_i + \frac{\partial L}{\partial \dot{x}^i} \dot{X}_i \right) dt \right| + o(\|X\|^2_{C^2}) \\ &= \left| \int_{t_0}^{t_1} \left(\frac{\partial L}{\partial x^i} - \frac{d}{dt} \frac{\partial L}{\partial \dot{x}^i} \right) X_i dt + \left[\frac{\partial L}{\partial \dot{x}^i} X_i \right]_{t_0}^{t_1} \right| + o(\|X\|^2_{C^2}). \end{aligned}$$

Here, to go to the third line, we have used integration by parts. In the second line, we have used Taylor's formula for the Lagrangian:

$$L(x^i_0 + x^i, \dot{x}^i_0 + \dot{x}^i, t) = L(x^i_0, \dot{x}^i_0, t) + \frac{\partial L}{\partial x^i} x^i + \frac{\partial L}{\partial \dot{x}^i} \dot{x}^i + R^1_{ij} x^i x^j + R^2_{ij} x^i \dot{x}^j + R^3_{ij} \dot{x}^i \dot{x}^j,$$

where the remainders $R^l_{ij}(x^i_0 + x^i, \dot{x}^i_0 + \dot{x}^i, t)$ go to zero as $x^i \to 0$, $\dot{x}^i \to 0$, for $l = 1, 2, 3$. When we put these remainders into the integral over t, we see that the overall remainder can be estimated in norm as $\leq C(X)||X||^2_{C^2}$ with $C(X) \to 0$ as $X \to 0$. In other words, the remainder is $o(\|X\|^2_{C^2})$ as claimed in the second line. Comparing with the definition of the Fréchèt derivative, this proves the proposition. $\square$

Definition 2.7. An *extremum* of the action S is a curve $q : I \to M$ such that

$$\frac{d}{d\epsilon} S(q + \epsilon h) = 0.$$

Remark 2.8. By the chain rule,

$$\left. \frac{d}{d\epsilon} S(q + \epsilon h) \right|_{\epsilon = 0} = D_q S(h)$$

The *variational derivative* of S at $q(t)$ is the vector field

$$\frac{\delta S}{\delta q(t)} \in T_q \mathcal{F}(M; x_0, x_1)$$

defined by the equation

$$D_q S(X) = \left\langle \frac{\delta S}{\delta q(t)}, X(t) \right\rangle_q,$$

using the pairing (4).

Theorem 2.9. *The extrema of the action* (2) *on curves with fixed initial point* $\gamma(t_0) = x_0$ *and endpoint* $\gamma(t_1) = x_1$ *in* $\mathbb{R}^n$ *satisfy the Euler–Lagrange equations:*

$$\frac{\delta S}{\delta \gamma(t)} = \frac{\partial L}{\partial x^i} - \frac{d}{dt} \frac{\partial L}{\partial \dot{x}^i} = 0. \tag{5}$$

Proof. This now clear from the previous proposition once we notice the following

Lemma 2.10 (The fundamental Lemma of the calculus of variations)**.** *If for all* $X \in \Gamma(I, \gamma^* TM)$ *with* $X(t_0) = 0 = X(t_1)$,

$$\left\langle \frac{\delta S}{\delta \gamma(t)}, X \right\rangle = 0, \quad \textit{then } \frac{\delta S}{\delta \gamma(t)} = 0.$$

we leave the proof of this Lemma to the reader. (cf. [Ar, §3.12]). □

From now on, we should be more precise about the type of Lagrangian we are considering. We are interested in Lagrangians of the form

$$L(q) = \frac{1}{2}||\dot{q}||^2 - V(q). \tag{6}$$

A straightforward application of the previous Theorem gives exactly Newton's law (1). This justifies the principle of least action.

2.3. **The variational problem on a general manifold.** On a general manifold, we choose a Lagrangian $L : TM \to \mathbb{R}$, and consider the variational problem given by the action

$$S : \mathcal{F}(I; M, x_0, x_1) \to \mathbb{R},$$

defined as in (2). Again, the equations of motion are given by the extrema of the action. To detect the extrema, we can choose a Riemannian metric on M to construct the Banach manifold structure on $\mathcal{F}(I; M, x_0, x_1)$ and look for zeros of the derivative of S. We will choose a slightly different path: The following Lemma allows us to work in local coordinates:

Lemma 2.11. *Suppose that* $\gamma : I \to M$ *is a curve minimizing the action* (2) *on* $\mathcal{F}(I; M, x_0, x_1)$ *and let* $t_0 \leq t_0' < t_1' \leq t_1$. *Then* $\gamma|_{[t_0', t_1']}$ *also minimizes the action on* $\mathcal{F}([t_0', t_1',]; M, \gamma(t_0'), \gamma(t_1'))$.

Proof. Suppose the statement is not true, so there is a curve $\eta \in \mathcal{F}([t_0', t_1,]; M, \gamma(t_0'), \gamma(t_1'))$ with $S(\eta) < S(\gamma)$. Construct the "broken curve" by replacing the piece $[t_0', t_1']$ in the curve γ with η. This yields a C^0-curve $\tilde{\gamma}$ for which, by additivity of the integral in (2) we have $S(\tilde{\gamma}) < S(\gamma)$. Here we have used the observation that the action S extends to the larger space of piecewise C^2-curves in M. Next, consider a "smoothing" of $\tilde{\gamma}$, i.e., a family $\tilde{\gamma}_\epsilon \in \mathcal{F}(I; M)$ for $\epsilon > 0$ such that $\lim_{\epsilon \to 0} \tilde{\gamma}_\epsilon = \tilde{\gamma}$ in the C^0-topology. Then $L(\tilde{\gamma}_\epsilon) \to L(\tilde{\gamma})$ point wise, where the latter is a bounded, piecewise smooth function on I. From this it follows that the extension of S is continuous in the sense that

$$\lim_{\epsilon \to 0} S(\tilde{\gamma}_\epsilon) = S(\tilde{\gamma}).$$

Choosing ϵ small enough, we now have a contradiction with the assumption that γ minimizes S. □

To detect the minima, we now use the smooth Banach manifold structure on $\mathcal{F}_k(I; M, x_0, x_1)$ and look for the critical points $\gamma \in \mathcal{F}_k(I; M, x_0, x_1)$ for which

$$\frac{d}{d\epsilon} S(\gamma_\epsilon) = 0,$$

where γ_ϵ is any smooth deformation of γ as in remark 2.5. Restricting a minimizing curve to a subinterval of I so that its image lies in a coordinate chart $(x^1, \ldots, x^n) : U \to \mathbb{R}^n$, we can now conclude:

Proposition 2.12. *A curve $\gamma : I \to M$ minimizing the action* (2) *satisfies the Euler–Lagrange equations* (5) *in a local chart on M.*

Let us now discuss some special cases:

i) (Zeroth order Lagrangian) Suppose that $L : TM \to \mathbb{R}$ is the pullback of a function on M ,i.e., does not depend on the variables $\dot{x}^i$. The Euler–Lagrange equations (5) then give
$$\frac{\partial L}{\partial x^i}(\gamma(t)) = 0.$$
In other words, $\gamma(t)$ should lie in the critical set of L on M. For a generic L, the critical sets, i.e., zeros of the derivative, are isolated in M, so $\gamma(t)$ must be a constant path.

ii) (First order Lagrangian) Suppose that L is *affine* in the fiber direction:
$$L(x, \xi) = L_0(x) + \alpha_x(\xi),$$
with $L_0 \in C^\infty(M)$ and $\alpha \in \Omega^1(M)$ a fixed one-form. This time the Euler–Lagrange equations write out in local coordinates as
$$\frac{\partial L_0}{\partial x^i}(\gamma(t)) = \sum_{j=1}^{n} \left(\frac{\partial \alpha_i}{\partial x^j} - \frac{\partial \alpha_j}{\partial x^i} \right) (\gamma(t)) \frac{d\gamma_j}{dt}.$$
If we assume the matrix whose entries are written out in the brackets is invertible, this is a system of first order ODE's which locally has a unique solution with initial point x_0. If x_1 is not on this curve, there is no γ minimizing the action.

So, to get something interesting, we should assume a nonlinear dependence in the fiber direction. Assume that
$$\det \left(\frac{\partial^2 L}{\partial \xi^i \partial \xi^j} \right) \neq 0, \tag{7}$$
and write (G_{ij}) for the inverse of this matrix. With this the Euler–Lagrange equations are given in local coordinates by
$$\frac{d^2 \gamma_i}{dt^2} = \sum_{j=1}^{n} G_{ij} \frac{\partial L}{\partial x^j}(\gamma(t), \frac{d\gamma}{dt}) - \sum_{j,k=1}^{n} G_{ij} \frac{\partial^2 L}{\partial \xi^j \partial x^k}(\gamma, \frac{d\gamma}{dt}) \frac{d\gamma_k}{dt}.$$
This is a system of second order ODE's that has a unique solution for initial conditions
$$\gamma(t_0) = x_0, \quad \frac{d\gamma}{dt}(t_0) = \xi_0,$$
with $\xi_0 \in T_{x_0}M$.

2.4. **The second order variation.** Recall that the Euler–Lagrange equations only give necessary conditions for a minimum.[2] To obtain sufficient conditions, we have to compute the second order derivative of the action. Again, we may assume that our curve $\gamma \in \mathcal{F}$ lies in the domain of a local chart $(x^1, \ldots, x_n)$. Let

$$\gamma_\epsilon(t) = \exp_{\gamma(t)}(\epsilon X)$$

be a one-parameter deformation defined by $X \in T_\gamma \mathcal{F}$. By a straightforward calculation, the second order derivative of the action (2) writes out as

$$\frac{d^2 S(\gamma_\epsilon)}{d\epsilon^2} = \sum_{i,j=1}^n \int_{t_0}^{t_1} \left(\frac{\partial^2 L}{\partial x^i \partial x^j} X^i X^j + 2 \frac{\partial^2 L}{\partial x^i \partial \xi^j} X^i \dot{X}^j + \frac{\partial^2 L}{\partial \xi^i \partial \xi^j} \dot{X}^i \dot{X}^j \right) dt.$$

This is called the *second variation* of the action. The quadratic nature of this second variation has some interesting consequences. First of all, we write the right hand side of the equation above as $I_\gamma(X, X)$ where the *index form* $I_\gamma(X, Y)$ is defined as

$$I_\gamma(X, Y) := \sum_{i,j=1}^n \int_{t_0}^{t_1} \left(\frac{\partial^2 L}{\partial x^i \partial x^j} X^i Y^j + 2 \frac{\partial^2 L}{\partial x^i \partial \xi^j} X^i \dot{Y}^j + \frac{\partial^2 L}{\partial \xi^i \partial \xi^j} \dot{X}^i \dot{Y}^j \right) dt. \tag{8}$$

Clearly, when $\gamma \in \mathcal{F}_k(I; M, x_0, x_1)$ is a minimum, we must have that $I_\gamma(X, X) \geq 0$. Next, choose $\eta \in \mathbb{R}^n$ and $t_2 \in (t_0, t_1)$ and consider the one parameter family $X_\delta \in \mathcal{F}_0(I; \mathbb{R}^n, 0, 0) \cong T_\gamma \mathcal{F}_0(I; M, x_0, x_1)$ given by

$$X_\delta(t) := \begin{cases} 0 & |t - t_2| > \delta \\ \delta\eta - |t - t_2|\eta & |t - t_2| \leq \delta. \end{cases}$$

By considering an appropriate "smoothening" of this family, keeping the support inside the interval $[t_2 - \delta, t_2 + \delta]$, we obtain a family of $X_\delta \in T_\gamma \mathcal{F}_k(I; M, x_0, x_1)$ with

$$\|X_\delta\|_{C^0} = O(\delta), \quad \|X_\delta\|_{C^1} = O(1), \quad \delta \to 0.$$

For such a family we therefor have

$$0 \leq I_\gamma(X_\delta, X_\delta) = \sum_{i,j=1}^n \int_{t_2-\delta}^{t_2+\delta} \left(\frac{\partial^2 L}{\partial \xi^i \partial \xi^j} \dot{\eta}^i \dot{\eta}^j \right) dt + O(\delta^2).$$

Taking the limit $\delta \to 0$, we obtain Legendre's necessary condition for a minimum:

$$\frac{\partial^2 L}{\partial \xi^i \partial \xi^j} \geq 0 \quad \text{along } \gamma(t).$$

Next, let us further analyse the case when $I_\gamma(X, X) = 0$ for a nonzero X. Since $I_\gamma(X, X) \geq 0$, we can view such an X as a minimum of the "action" $I_\gamma(X, X)$. By now, we know

[2]For physics, this seems to be enough as the Euler–Lagrange equations (i.e., Newton's equations) are all that matters. From that point of view, this section is only of mathematical interest

how to detect these: we write down the corresponding Euler–Lagrange equations! By the quadratic nature of the action, these are simply given by

$$\sum_{i,j=1}^{n}\left(\frac{\partial^2 L}{\partial x^i\partial x^j}X^j+\frac{\partial^2 L}{\partial x^i\partial\xi^j}\dot{X}^j-\frac{d}{dt}\left(\frac{\partial^2 L}{\partial x^i\partial\xi^j}X^j+\frac{\partial^2 L}{\partial\xi^i\partial\xi^j}\dot{X}^j\right)\right)=0, \tag{9}$$

a second order lineair homogeneous system of ODE's known as the *Jacobi equations*. Solutions are called Jacobi fields.

Proposition 2.13. *Let $\gamma_s(t)$ be a C^1-family of solutions to the Euler–Lagrange equations* (5)*. Then*

$$X(t) := \left.\frac{d}{ds}\gamma_s(t)\right|_{s=0}$$

is a Jacobi field.

Proof. Differentiate the Euler–Lagrange equations w.r.t s. □

Definition 2.14. Let $\gamma(t)$ be a solution to the Euler–Lagrange equations on $I=[t_0,t_1]$ for a Lagrangian L. we say that $\gamma(t_2)\in M$ and $\gamma(t_3)\in M$, for $t_0\leq t_2<t_3\leq t_2$ are *conjugate* to each other if there exists a not identically vanishing Jacobi field X along $\gamma|_{[t_2,t_3]}$ with $X(t_2)=0=X(t_3)$.

Theorem 2.15. *The index form $I_\gamma(X,Y)$ is positive definite if*

i) *(Legendre condition)*

$$\frac{\partial^2 L}{\partial\xi^i\xi^j}>0,$$

b) *The interval $(t_0,t_1]$ does not contain a point conjugate to t_0.*

So in this case, we really have a minimum. For the proof of this theorem, see [GF]. In fact, Legendre's condition is already enough to ensure a local minimum:

Proposition 2.16. *Let $\gamma(t)$, $t_0\leq t\leq t_1$ be a solution of the Euler–Lagrange equations for a Lagrangian satisfying Legendre's condition. Then for each sufficiently small subinterval $[t_0',t_1']\subset[t_0,t_1]$, $\gamma|_{[t_0',t_1']}$ is a local minimum of the action on $\mathcal{F}(I',M,\gamma(t_0'),\gamma(t_1'))$.*

Proof. The key is again the second variation $I_\gamma(X,X)$ given in (8), which consists of three terms: I, II and III. For the first two we have the estimates

$$|I|\leq C\,\|X\|^2_{L^2}\,,\quad |II|\leq D\,\|X\|_{L^2}\,\|\dot{X}\|_{L^2}\,.$$

For the last one we have, because of Legendre's condition:

$$III\leq E\,\|\dot{X}\|^2_{L^2}\,.$$

In these inequalities, $C,D,E>0$. Next, recall *Wirtinger's inequality*

$$\int_a^b\left|\frac{df}{dt}\right|^2dt\geq\frac{\pi^2}{(b-a)^2}\int_a^b|f|^2dt,$$

for any $f \in C^1(a,b)$ with $f(a) = 0 = f(b)$. Therefore, if $b - a$ is small enough, we find $I_\gamma(X,X) > 0$ for all X. $\square$

2.5. **The case of a Riemannian manifold.** On a general Riemannian manifold, things are slightly more complicated, but with our global set-up, we now have all the tools to derive the equations of motion. We consider the Lagrangian $L : TM \to \mathbb{R}$ given in (6), where $|| \, ||$ denotes the riemannian norm on T_xM defined by the metric g, and V is, as before, a potential function defined on M. Integrating over $t \in [t_0, t_1]$ we obtain the action as in (2): this is now a function

$$S : \mathcal{F}(I; M, x_0, x_1) \to \mathbb{R},$$

and we are lead to look for the extrema minimizing the action. We have seen in Theorem 2.4 that $\mathcal{F}(I; M, x_0, x_1)$ is a smooth Banach manifold, and it can be proved, analogous to Proposition 2.6 for $\mathbb{R}^n$ that S is differentiable. Therefore, we can detect the extrema by looking for the critical points of the action.

Proposition 2.17. *The Euler–Lagrange equations for the action given in equation* (6) *are given by*

$$\nabla_{\dot\gamma}\dot\gamma = -\mathrm{grad}(V).$$

In particular, when $V = 0$, particles move along geodesics along M.

Proof. Let us first assume our curve lands in a local coordinate chart $(x^1, \dots, x^n) : U \to \mathbb{R}^n$. The action is written in local coordinates as

$$L(x, \xi) = \frac{1}{2} \sum_{ij} g_{ij}(x) \xi^i \xi^j - V(x).$$

Then we can use our local formula for the Euler–Lagrange equations (5):

$$\begin{aligned} \frac{\partial L}{\partial x^i} - \frac{d}{dt}\frac{\partial L}{\partial \dot x^i} &= -\partial_i V + \partial_i g_{kl} \dot x^k \dot x^l - \partial_k g_{ij} \dot x^k \dot x^j - \partial_j g_{ki} \dot x^k \dot x^j - g_{ij} \ddot x^j - g_{ki} \ddot x^k \\ &= -\partial_i V - 2 g_{ij} \ddot x^j - (\partial_k g_{ji} + \partial_j g_{ki} - \partial_k g_{ji}) \dot x^k \dot x^j \\ &= 0. \end{aligned}$$

Applying the inverse of the metric g^{il} to this equation to raise one of the indices, we find:

$$\ddot x^i + \Gamma^i_{jk} \dot x^j \dot x^k = -g^{il} \partial_i V.$$

The left hand side is exactly the expression in local coordinates of $\nabla_{\dot\gamma(t)} \dot\gamma(t)$, whereas the right hand side is exactly $-\mathrm{grad}(V)$.

Alternatively, we can do a more global computation: let $\gamma_s(t)$ be a C^1-family with fixed endpoints such that $\gamma_0(t)$ is a solution to the Euler–Lagrange equations. Then we compute:

$$\begin{aligned}
\frac{dS}{ds}(\gamma_s) &= \frac{d}{ds}\int_{t_0}^{t_1}\left(\frac{1}{2}g(\frac{\partial\gamma}{\partial t},\frac{\partial\gamma}{\partial t}) - V(\gamma)\right)dt\\
&= \int_{t_0}^{t_1}\left(g(\nabla_{\partial\gamma/\partial s}\frac{\partial\gamma}{\partial t},\frac{\partial\gamma}{\partial t}) - dV(\frac{\partial\gamma}{\partial s})\right)dt \quad (\nabla \text{ is metric})\\
&= \int_{t_0}^{t_1}\left(g(\nabla_{\partial\gamma/\partial t}\frac{\partial\gamma}{\partial s},\frac{\partial\gamma}{\partial t}) - dV(\frac{\partial\gamma}{\partial s})\right)dt \quad (\nabla \text{ is torsionfree})\\
&= \int_{t_0}^{t_1}\left(\frac{d}{dt}\left(g(\frac{\partial\gamma}{\partial s},\frac{\partial\gamma}{\partial t})\right) - g(\frac{\partial\gamma}{\partial s},\nabla_{\partial\gamma/\partial t}\frac{\partial\gamma}{\partial t}) - dV(\frac{\partial\gamma}{\partial s})\right)dt\\
&= -\int_{t_0}^{t_1}\left(g(\frac{\partial\gamma}{\partial s},\nabla_{\partial\gamma/\partial t}\frac{\partial\gamma}{\partial t}) + dV(\frac{\partial\gamma}{\partial s})\right)dt
\end{aligned}$$

Applying g^{-1}, we get the desired Euler–Lagrange equations. □

Since for this Lagrangian, $\partial^2 L/\partial\xi^i\partial\xi^j = g_{ij}$ which is clearly positive definite by definition of a Riemannian metric, L satisfies Legendre's condition. Therefore, by Proposition 8, we find:

Corollary 2.18. *On a Riemannian manifold, geodesics are locally distance minimizing.*

The second variation also has a very interesting geometric interpretation. We put $V = 0$.

Proposition 2.19. *Let $\gamma(t)$ be a solution of the Euler–Lagrange equations in Proposition 2.17. The corresponding index form is given by*

$$I_\gamma(X,Y) = \int_{t_0}^{t_1}\left(g(\nabla_{\dot\gamma}X,\nabla_{\dot\gamma}Y) - g(R(\dot\gamma,X)(Y),\dot\gamma(t))\right)dt$$

Proof. We work globally: again, let $\gamma_s(t)$ be a C^1-family with fixed endpoints such that $\gamma_0(t)$ is a solution to the Euler–Lagrange equations. Then we compute

$$\begin{aligned}
\frac{d^2S}{ds^2}(\gamma_s) &= \frac{d}{ds}\int_{t_0}^{t_1}\left(g(\nabla_{\partial\gamma/\partial t}\frac{\partial\gamma}{\partial s},\frac{\partial\gamma}{\partial t})\right)dt\\
&= \int_{t_0}^{t_1}\left(g(\nabla_{\partial\gamma/\partial s}\nabla_{\partial\gamma/\partial t}\frac{\partial\gamma}{\partial s},\frac{\partial\gamma}{\partial t}) + g(\nabla_{\partial\gamma/\partial t}\frac{\partial\gamma}{\partial s},\nabla_{\partial\gamma/\partial t}\frac{\partial\gamma}{\partial s})\right)dt
\end{aligned}$$

(Again, we have used the fact that ∇ preserves the metric and is torsion free.) Using the definition of the curvature, the first term can now be written as

$$\begin{aligned}
&\int_{t_0}^{t_1}\left(g(\nabla_{\partial\gamma/\partial s}\nabla_{\partial\gamma/\partial t}\frac{\partial\gamma}{\partial s},\frac{\partial\gamma}{\partial t}\right)dt\\
&\quad= \int_{t_0}^{t_1}\left(g(\nabla_{\partial\gamma/\partial t}\nabla_{\partial\gamma/\partial s}\frac{\partial\gamma}{\partial s},\frac{\partial\gamma}{\partial t}) + g\left(R(\frac{\partial\gamma}{\partial s},\frac{\partial\gamma}{\partial t})(\frac{\partial\gamma}{\partial s}),\frac{\partial\gamma}{\partial t}\right)\right)dt
\end{aligned}$$

Using the metric property again, the first term is proportional to $\nabla_{\partial\gamma/\partial t}\frac{\partial\gamma}{\partial t}$, which cancels the last term in the previous equation by virtue of the Euler–Lagrange equations of Proposition 2.17. We now have the desired expression for the index form. □

The fact that the curvature appears is the most interesting aspect of this Proposition. For example, we immediately have:

Corollary 2.20. *On a manifold with nonpositive sectional curvature, geodesics γ with fixed endpoints are always locally minimizing in the sense that there exists some $\delta > 0$ such that for any other γ' with the same endpoint and $d_M(\gamma(t), \gamma'(t)) < \delta$ for all t, we have $E(\gamma') \geq E(\gamma)$.*

3. The Hamiltonian approach to Classical Mechanics

3.1. The Legendre transform.

Let $V = \mathbb{R}^n$ with Euclidean coordinates v^i and $F : V \to \mathbb{R}$ a C^2 function. Its *Hessian* at $p \in V$ is the quadratic function

$$D^2F(p)(u) := \frac{d^2}{dt^2}F(p+tu) = \frac{\partial^2 F}{\partial v^i \partial v^j}(p)u^i u^j$$

with $u = (u^1, \ldots, u^n) \in V$. F is called *strictly convex* if $D^2F(p) > 0$ for all $p \in V$.

Lemma 3.1. *Let F be strictly convex. Then the following are equivalent:*

a) *F has a critical point,*
b) *F has a global minimum at some point,*
c) *$F(p) \to \infty$ for $\|p\| \to \infty$.*

Now, given a strictly convex F and an element $\alpha \in V^*$, consider the function

$$F_\alpha(v) := F(v) - \alpha(v).$$

Clearly F_α is strictly convex iff F is. We now define the open convex (check!) subset $S_F \subset V^*$ by

$$\alpha \in S_F \quad \Longleftrightarrow \quad F_\alpha \text{ satisfies the conditions of Lemma 3.1.}$$

The *Legendre transform* $T_F : V \to V^*$ defined by F is now defined as

$$T_F(p) = dF(p) \in T_p^*V \cong V^*.$$

Lemma 3.2. *The Legendre transform yields an isomorphism $T_F : V \xrightarrow{\cong} S_F$.*

Proof. Define $T_F^{-1} : S_F \to V$ as follows: for $\alpha \in S_F$,

$$T_F^{-1}(\alpha) := \min F_\alpha.$$

One easily checks that this is indeed an inverse, i.e., $T_F \circ T_F^{-1} = id_{S_F}$ and $T_F^{-1} \circ T_F = id_V$. □

We now define the *dual function* $F^* : S_F \to \mathbb{R}$ by

$$F^*(\alpha) := -\min_{p \in V} F_\alpha(p).$$

Exercise 3.3. Show that $T_F^{-1} = T_{F^*}$.

On a general manifold M, with Lagrangian $L : TM \to \mathbb{R}$ satisfying Legendre's condition in Theorem 2.15 $i)$, we can apply the Legendre transform for each $x \in M$ to get

$$T_{L_x} : T_x M \xrightarrow{\cong} T_x^* M, \tag{10}$$

assuming that $S_{L_x} = T_x^* M$. The dual function to L is denoted by $H : T^*M \to \mathbb{R}$, called the *Hamiltonian*. It is easily checked that for the Lagrangian (6) for a particle in a potential V, the Legendre transform $T_L : TM \to T^*M$ is the isomorphism induced by the Riemannian metric g. The Hamilonian in this case writes out as

$$H(\alpha) = \frac{1}{2}\|\alpha\|^2 + (\pi^* V)(\alpha), \quad \alpha \in T^*M. \tag{11}$$

3.2. **Symplectic manifolds.** We recall the definition of a symplectic manifold:

Definition 3.4. A symplectic manifold is a pair (X, ω) of a smooth manifold X equipped with a non degenerate closed 2-form ω.

Closed means that $d\omega = 0$, where d is the exterior differential and non degenerate that for each $x \in X$, the map

$$T_x X \to T_x^* X, \quad \xi \mapsto \omega_x(\xi, -),$$

is an isomorphism.

Exercise 3.5. Show that a symplectic manifold is always even-dimensional. Show that the non-degeneracy of ω is equivalent to the condition that $\omega^{\dim X/2}$ is non-vanishing.

Example 3.6. We give the two main examples of symplectic manifolds:

i) Let $X = \mathbb{R}^{2n}$ with Euclidean coordinates given by $(x_1, \ldots, x_n, y_1, \ldots y_n)$. Then the constant 2-form
$$\omega = \sum_{i=1}^{2n} dx^i \wedge dy^i$$
is easily seen to be non degenerate. It is clearly closed, so defines a symplectic structure. In fact, this is an example of a *symplectic vector space*: this is a vector space equipped with an antisymmetric, non degenerate bilinear form. On the other hand, suing some linear algebra it is not difficult to prove that any symplectic vector space is of this form. In Theorem 3.7 below, we shall see that locally, any symplectic manifold looks like such a symplectic vector space.

ii) Let $X = T^*M$, the total space of the cotangent bundle of a smooth manifold M. This manifold carries a canonical 1-form, called the *Liouville form* defined by
$$\theta_\alpha(v) := \alpha(T\pi(v)), \quad \alpha \in T^*M,\ v \in T_\alpha(T^*M),$$
where $\pi : T^*M \to M$ denotes the canonical projection. We now define $\omega := d\theta$, and claim that this is a symplectic form. Being exact, it is clearly closed, so we

only have to check that it is non degenerate. Since this is a local property, we can do this in local coordinates $(x^1, \ldots, X^n)$ on M. These induce local coordinates $(x^1, \ldots, x^n, \nu^1, \ldots, \nu^n)$ on T^*M by putting $\nu^i := dx^i$. In these coordinates the projection π is given by the projection onto the first n coordinates, so that the Liouville form is given by

$$\theta(x, \nu) = \sum_{i=1}^{n} \nu^i dx^i.$$

With this, we see that $\omega = \sum_{i=1}^{n} d\nu^i \wedge dx^i$, which is clearly non degenerate.

Theorem 3.7 (Darboux)**.** *Let (X, ω) be a symplectic manifold. Around each point $x \in M$ there exists a local coordinate chart $(x_1, \ldots, x_n, y_1, \ldots, y_n)$ such that*

$$\omega = \sum_{i=1}^{n} dx_i \wedge dy_i$$

Proof. The following proof utilizes a well-known strategy, known as the *Moser-trick*: pick a symplectic basis of T_xM and use the exponential map with respect to some arbitrary metric to introduce coordinates $(x_1, \ldots, x_n, y_1, \ldots, y_n)$ in a neighborhood of x. In this neighborhood U, we consider two symplectic forms: $\omega_0 := \omega|_U$, and $\omega_1 := \sum_{i=1}^{n} dx_i \wedge dy_i$. By construction, we have that $\omega_0(x) = \omega_1(x)$. Since U is contractible and $\omega_1 - \omega_0$ closed, there exists a one-form $\nu \in \Omega^1(U)$ such that $\omega_1 - \omega_0 = d\nu$, and $\nu_x = 0$. Then we consider the convex combination

$$\omega_t := (1-t)\omega_0 + t\omega_1 = \omega_0 + td\nu.$$

Since being non degenerate is clearly an open condition, and ω_t is symplectic at $x \in U$, we can shrink U so that ω_t is symplectic for all $0 \leq t \leq 1$. Then we consider the *Moser equation*:

$$\iota_{X_t}\omega_t + \nu = 0.$$

Since ω_t is non degenerate, this defines a t-dependent family of vector fields in a neighborhood around x. Let $\varphi : U \times [0,1] \to M$ be the flow generated by this time-dependent vector field:

$$\frac{d}{dt}\varphi_t(x) = X_t(\varphi_t(x)).$$

Since this is a first order (nonautonomous) ODE, we know that solutions locally exist and shrinking U again we can assume φ_t is defined on all of U. Then we compute

$$\begin{aligned}\frac{d}{dt}\varphi_t^*\omega_t &= \varphi_t^*(L_{X_t}\omega_t + \frac{d\omega_t}{dt}) \\ &= \varphi_t^*(d\iota_{X_t}\omega_t + d\nu) \\ &= 0.\end{aligned}$$

Since $\varphi_0 = id_U$, we therefore have that $\varphi_t^*\omega_t = \omega_0$ for all $0 \leq t \leq 1$. Therefore φ_1 is a local diffeomorphism having x as a fixed point that satisfies $\varphi_1^*\omega_1 = \omega_0$. Composing the coordinates $(x_1, \dots, x_n, y_1, \dots, y_n)$ with φ_1, we get the desired coordinates. □

Remark 3.8. This result is very surprising if you compare symplectic geometry with Riemannian geometry. We have seen, see Appendix A, that the local model for Riemannian geometry is given by (an open subset of) $\mathbb{R}^n$ together with a *smoothly varying* metric $g_{ij}(x)$. In symplectic geometry however, the local model is given by (an open subset of) $\mathbb{R}^{2n}$ together with a *constant* antisymmetric non degenerate bilinear form!

Given a function $f \in C^1(M)$, we can associate to it a vector field X_f, called the *Hamiltonian vector field*, defined by the equation

$$\iota_{X_f}\omega = df, \tag{12}$$

and associated to this vector field is its (local) flow $\varphi^f : [0, \epsilon) \times M \to M$ determined by the first order ODE

$$\frac{d}{dt}\varphi^f(t, x) = X_f(x), \quad x \in M.$$

We see that on a symplectic manifold any function gives rise to dynamics, and it is up to physics to determine which function (called the Hamiltonian and usually written as H) governs the equations of motions that describe reality. If we choose local Darboux coordinates $(x^1, \dots, x^n, y^1, \dots y^n)$ as in Theorem 3.7, these *Hamilton equations* (as they are called) take the form

$$\frac{dx^i}{dt} = \frac{\partial H}{\partial y^i}$$
$$\frac{dy^i}{dt} = -\frac{\partial H}{\partial x^i}$$

Another way to describe the dynamics is by introducing the *Poisson brackets* of two C^1-functions by

$$\{f, g\} := \omega(X_f, X_g).$$

Proposition 3.9. *For $f, g \in C^2(M, \mathbb{R})$, we have*

$$[X_f, X_g] = -X_{\{f,g\}}.$$

Proof. This is a direct computation, using the fact that the Lie derivative is a derivation with respect to the contraction between vector fields and differental forms:

$$\begin{aligned}
\iota_{[X_f, X_g]}\omega &= \iota_{L_{X_f}X_g}\omega \\
&= L_{X_f}(\iota_{X_g}\omega) - \iota_{X_g}L_{X_f}\omega \\
&= -d(\omega(X_f, X_g)).
\end{aligned}$$

In this computation we have used Cartan's magic formula $L_X = d\iota_X + \iota_X d$ together with the definition of a Hamiltonian vector field in (12). □

Corollary 3.10. *The Poisson bracket $\{-,-\}$ defines a Lie bracket on smooth functions.*

Definition 3.11. A *Poisson manifold* is a smooth manifold M whose ring of smooth functions $C^\infty(M)$ is equipped with a Lie bracket $(f,g) \mapsto \{f,g\}$ satisfying the *Leibniz rule*:

$$\{f,gh\} = \{f,g\}h + g\{f,h\}. \tag{13}$$

We see that a symplectic manifold is also a Poisson manifold, but there are Poisson manifolds that are not symplectic, e.g., they can be odd-dimensional. Given a function $f \in C^1(M,\mathbb{R})$ on a Poisson manifold, it follows from the Leibniz rule that $\{f,-\}$ defines a derivation of the ring of functions on M, i.e., a vector field: this is (again) called the Hamiltonian vector field X_f of f, so these are defined on any Poisson manifold, not just symplectic ones.

3.3. **The Hamilton equations.** This section is crucial in classical mechanics, since it links the Euler–Lagrange equations on M to the Hamilton equations on the symplectic manifold T^*M. The connection between the two is given by the Legendre transform, with Hamiltonian defined by the dual of the Lagrangian:

Theorem 3.12. *For a Lagrangian $L : TM \to \mathbb{R}$ satisfying Legendre's condition, the Euler–Lagrange equations are equivalent to the Hamilton equations on T^*M for the Legendre transform $H := L^*$ of the Lagrangian: For a C^2-curve $\gamma(t)$ in M we have*

$$\boxed{\gamma(t) \text{ satisfies Euler–Lagrange}} \iff \boxed{(\gamma(t), T_L(\dot\gamma(t))) \text{ satisfies the Hamilton equations}}$$

Proof. It suffices to check this in local coordinates $(x^1, \dots, x^n)$ on M. As usual, we write the associated local coordinates on TM by $\xi_i := \partial/\partial x^i$, and on T^M by $v^i = dx^i$. In these coordinates, the Legendre transform (10) is given by

$$T_L(x,\xi) = \sum_{i=1}^n \frac{\partial L}{\partial \xi^i} v^i.$$

The Hamiltonian, i.e., the dual of the Lagrangian, is given by

$$H(x,v) := L^*(x,v) = \sum_{i=1}^n v^i \xi_i - L(x,\xi), \quad \text{with } \xi = T_L^{-1}(v).$$

The local coordinates (x^i, v^i) on T^*M are in fact Darboux coordinates, so that the *Hamilton equations* for $H(x,v)$ are given by

$$\begin{aligned} \frac{dx^i}{dt} &= \frac{\partial H}{\partial v^i} \\ \frac{dv^i}{dt} &= -\frac{\partial H}{\partial x^i}. \end{aligned}$$

For the dual of the Lagrangian, the first equation amounts to

$$\frac{dx^i}{dt} = \frac{\partial H}{\partial v^i} = \frac{\partial L^*}{\partial v^i} = T_{L^*}(v_i) = T_L^{-1}(v_i),$$

so this is automatically satisfies if we define $v^i = T_L(dx^i/dt)$. For the second equation, we notice that

$$\frac{\partial L}{\partial x^i}(x,\xi) = -\frac{\partial H}{\partial x^i}(x,v), \quad \text{with } \xi = T_L^{-1}(v).$$

With this, the second equation gives

$$\frac{d}{dt}\frac{\partial L}{\partial \xi_i} = \frac{dv^i}{dt} = -\frac{\partial H}{\partial x^i} = \frac{\partial L}{\partial x^i},$$

exactly the Euler–Lagrange equations. □

3.4. **The appearance of symplectic geometry form the Lagrangian approach.** Let us assume that the Lagrangian $L : TM \to \mathbb{R}$ is such that the dynamics is complete, i.e., solutions of the Euler–Lagrange equations exist for all times. We now drop the boundary conditions and consider the space $\mathcal{F}(I;M)$ of "free" curves in M. Inside $\mathcal{F}$, the subset of solutions to the Euler–Lagrange equations is denoted by $\mathcal{M}$. Using the Picard–Lindelöf theorem for the existence and uniqueness of solutions of second order ODE's, and the fact that these depends smoothly on the initial conditions, we see that $\mathcal{M}$ carries a smooth structure and is isomorphic to TM.

From Proposition 2.13, we see that the Tangent space $T_\gamma\mathcal{M}$ to $\gamma \in \mathcal{M}$ is given by vector fields $X(t)$ along γ that satisfy the Jacobi equation (9). We now consider the boundary term in the derivation in Proposition 2.6 of the Euler–Lagrange equations to define the one-parameter family of forms $\alpha_L^t \in \Omega^1(\mathcal{M}), t \in \mathbb{R}$ by

$$\alpha_L^t(X) := \sum_{i=1}^n \frac{\partial L}{\partial \xi^i}(\gamma(t),\dot\gamma(t))X^i(t).$$

Restricting the action S to $\mathcal{M}$, we find by Proposition 2.6

$$dS = \alpha_L^{t_1} - \alpha_L^{t_0},$$

so $d\alpha_L^t$ is independent of t.

Proposition 3.13. *Combining the isomorphism $\mathcal{M} \cong TM$ with the Legendre transform $TM \cong T^*M$, the form $d\alpha_L^t$ is mapped to the canonical symplectic form on T^*M.*

4. Quantum mechanics

Classical mechanics gives a macroscopic description of physical systems, whereas quantum mechanics gives a microscopic description. The fact that we work on small scales in quantum mechanics is indicated by the appearance of $\hbar$, *Planck's constant*. Of course, in physics this is a real number with fixed numerical value, but here we shall treat it mathematically as a deformation parameter: the limit $\hbar \to 0$, called the "classical limit" means that we go over to the macroscopic limit where we should recover classcal mechanics. The Hamiltonian approach to quantum mechanics insists that we work on the Hilbert space $\mathcal{H} = L^2(X, \mathrm{Vol}(g))$, where $\mathrm{Vol}(g)$ means the natural volume element on M given in local coordinates by

$$\mathrm{Vol}(g) := \sqrt{\det(g)(x)}dx^1 \wedge \ldots \wedge dx^n.$$

Elements of the Hilbert space $\mathcal{H}$ are called "states" of the system Additionally, we need to "quantize" classical observables, that is, functions $f \in C^\infty(T^*M)$, to operators $Q_\hbar(f)$ on $\mathcal{H}$. In general, this quantization procedure is still very mysterious but (from general principles) we insist that the correspondence $f \mapsto Q_\hbar(f)$ is linear and satisfies

$$Q_\hbar(f)Q_\hbar(g) = Q_\hbar(fg) + O(\hbar), \tag{I}$$

$$[Q_\hbar(f), Q_\hbar(g)] = i\hbar Q_\hbar(\{f,g\}) + O(\hbar^2), \tag{II}$$

where $f, g \in C^\infty(T^*M)$.

4.1. **Differential operators and quantization.** Any manifold M carries a natural non-commutative algebra, namely the algebra of differential operators $\mathcal{D}(M)$. Let us first recall its definition:

Definition 4.1. The *algebra of differential operators* $\mathcal{D}(M)$ is the algebra generated inside $\mathrm{End}(C^\infty(M))$ by $f \in C^\infty(M)$, acting by multiplication, and $X \in \mathfrak{X}(M)$ acting by the Lie derivative.

Recall that an *increasing filtration* on an algebra A is given by a chain of sub vector spaces

$$F_0(A) \subset F_1(A) \subset F_2(A) \subset \ldots, \quad A = \bigcup_{k\geq 0} F_k(A),$$

with the property that $F_i(A) \cdot F_j(A) \subset F_{i+j}(A)$. For a filtered algebra, the associated *graded algebra* $\mathrm{Gr}(A)$ is defined as

$$\mathrm{Gr}(A) := \bigoplus_{k\geq 0} F_k(A)/F_{k-1}(A),$$

with the product induced by that of A.

The algebra of differential operators $\mathcal{D}(M)$ is filtered by means of the inductive definition

$$D \in \mathcal{D}_k(M) \iff [D, f] \in \mathcal{D}_{k-1}(M), \quad \mathcal{D}_0(M) := C^\infty(M).$$

It is easy to see that this means that $D \in \mathcal{D}_k(M)$ in local coordinates $(x^1, \ldots x^n)$ looks like

$$D = \sum_{\substack{i_1,\ldots,i_p \\ p\leq k}} a_{i_1,\ldots,i_p}(x) \frac{\partial^p}{\partial x^{i_1} \cdots \partial x^{i_p}}, \tag{14}$$

where $a_{i_1,\ldots,i_k}(x) \not\equiv 0$.

Proposition 4.2. *The associated graded algebra of* $\mathcal{D}(M)$ *is commutative, and canonically isomorphic to*

$$\mathrm{Gr}(\mathcal{D}(M)) \cong \Gamma^\infty(M; \mathrm{Sym}(TM)),$$

with isomorphism induced by the principal symbol map

$$\sigma_k(D) : \mathcal{D}_k(M) \to \Gamma^\infty(M; \mathrm{Sym}^k(TM))$$

given in local coordinates (14) *by*

$$\sigma_k(D)(\nu) := \sum_{i_1,\ldots,i_k} a_{i_1,\ldots,i_k}(x)\nu^{i_1}\cdots\nu^{i_k},$$

where $\nu = \sum_i \nu_i dx^i \in T^*_x M$

Proof. First, check that the local expression for the principal symbol above is indeed coordinate invariant: for this it is essential that we only consider the top degree part of the differential operator D. (Why?) In local coordinates it is then easy to check that the principal symbol map fits into a short exact sequence

$$0 \longrightarrow \mathcal{D}_{k-1}(M) \longrightarrow \mathcal{D}_k(M) \xrightarrow{\sigma_k} \Gamma^\infty(M; \mathrm{Sym}^k(TM)) \longrightarrow 0.$$

This observation shows that σ induces an isomorphism $\mathrm{Gr}(\mathcal{D}(M)) \cong \Gamma^\infty(M; \mathrm{Sym}(TM))$ of vector spaces. To see that it is in fact an algebra isomorphism, one uses the property

$$\sigma_{k_1+k_2}(D_1 D_2) = \sigma_{k_1}(D_1)\sigma_{k_2}(D_2), \tag{15}$$

for $D_i \in \mathcal{D}_{k_i}(M)$, $i = 1, 2$. Again, this is immediate from the expression of the symbol in local coordinates, and shows that on the graded quotient, σ is an algebra homomorphism. ☐

Remark 4.3 (PBW for Lie algebras)**.**

The idea is now to construct a right inverse map to the symbol map to construct a quantization. By a right inverse we mean a map $Q : \Gamma^\infty(M; \mathrm{Sym}(TM)) \to \mathcal{D}(M)$ satisfying

$$\sigma \circ Q = 1.$$

Definition 4.4. Let A be a filtered algebra. The *Rees algebra* associated to the filtration is defined as

$$\mathrm{Rees}(A) := \bigoplus_{k\geq 0} F_k(A)\hbar^k \subset A \otimes \mathbb{C}[\hbar].$$

When $A = \mathcal{D}(M)$, we write $\mathcal{D}^\hbar(M)$ for the Rees algebra associated to the filtration by order of differentiation.

For the construction of the right inverse Q, we need the notion of an *iterated covariant derivative*: for a connection ∇ on TM, we get a natural connection on T^*M, the dual also denoted by ∇ and therefore also on $\mathrm{Sym}^k(T^*M)$, for all k. With this, define $\nabla^{(k)} : \Gamma(M; \mathrm{Sym}^k(T^*M)) \to \Gamma(M; \mathrm{Sym}^{k+1}(T^*M))$ by

$$(\nabla^{(k)}\gamma)(X_1, \ldots, X_{k+1}) := \frac{1}{k!} \sum_{\sigma \in S_{k+1}} (\nabla_{X_{\sigma(1)}}\gamma)(X_{\sigma(2)}, \ldots, X_{\sigma(k+1)}).$$

For $\alpha \in \Gamma(M; \mathrm{Sym}^k(TM))$, we can now define

$$Q_\hbar(\alpha)(f) := \hbar^k \iota_\alpha(\nabla^k(f)), \quad \text{for all } f \in C^\infty(M).$$

In this formula, $\nabla^k := \nabla^{(k)} \circ \nabla^{(k-1)} \circ \ldots \circ \nabla^{(0)}$, remark that $\nabla^{(0)} = d$, the exterior derivative.

Proposition 4.5. *The map*

$$Q_\hbar : \Gamma^\infty(M; \mathrm{Sym}(TM)) \to \mathcal{D}^\hbar(M)$$

defines a quantization of the sub Poisson algebra of functions on T^*M that are polynomial along the fibers of $\pi : TM \to M$.

Proof. Extend the principal symbol map $\mathbb{C}[\hbar]$-linearly to $\mathcal{D}^\hbar(M)$. First we shall show that

$$\sigma_k(Q_\hbar(\alpha)) = \hbar^k \alpha, \quad \text{for all } \alpha \in \Gamma^\infty(M; \mathrm{Sym}^k(TM)).$$

We know from the proof of Proposition 4.2 that it suffices to compute this locally, and that only the highest order (i.e., k) derivative contributes to the principal symbol. In local coordinates $(x^1, \ldots, x^n)$ on M the connection ∇ on T^*M can be written as

$$\nabla_{\partial/\partial x^i}\beta = \partial_i \beta + G_i(\beta),$$

with $G_i(x) \in \mathrm{End}(T^*M)$. Therefore we find

$$\sigma_k(Q_\hbar(\alpha)) = \sum_{i_1, \ldots, i_k} \alpha_{i_1, \ldots, i_k} v^{i_1} \cdots v^{i_k}, \quad \alpha = \sum_{i_1, \ldots, i_k} \alpha_{i_1, \ldots, i_k} dx^{i_1} \cdots dx^{i_k}.$$

This shows that $Q_\hbar$ is, up to a power of $\hbar$ a right inverse to σ. This, together with property (15) shows that

$$\sigma(Q_\hbar(\alpha_1\alpha_2) - Q_\hbar(\alpha_1)Q_\hbar(\alpha_2)) = 0,$$

and therefore

$$Q_\hbar(\alpha_1\alpha_2) - Q_\hbar(\alpha_1)Q_\hbar(\alpha_2) \in \mathcal{D}^\hbar_{k_1+k_2-1}(M),$$

where the degrees of α_1, α_2 are k_1 resp. k_2. This shows property (I) above.

By a similar argument, we see that $Q_\hbar$ and σ induce an antisymmetric bracket on $\Gamma^\infty(M; \mathrm{Sym}(TM))$:

$$\{\alpha_1, \alpha_2\} := \sigma_{k_1+k_2-1}([Q_\hbar(\alpha_1), Q_\hbar(\alpha_2)].$$

Using property I of $Q_\hbar$ again, we easily verify that the bracket satisfies the Leibniz rule as in (13). It is therefore determined by the following two Poisson brackets, that are easily computed:

$$\{f, g\} = 0, \quad \{X, Y\} := [X, Y], \quad \{X, f\} = X(f),$$

where $X, Y \in \mathfrak{X}(M)$ and $f, g \in C^\infty(M)$. These brackets are determined by the fact that $Q_\hbar(f) = f$ and $Q_\hbar(X) = L_X$. We now see that the bracket is nothing but the canonical Poisson bracket on T^*M, restricted to functions that are polynomial along the fibers of $\pi : T^*M \to M$. Altogether, this proves (II) $\square$

4.2. **The Laplacian on a manifold.** Let us again consider our favorite example: the free particle on a Riemannian manifold (M, g). Of course, in this case we use the Levi–Civita connection to define the quantization. As we have seen, the Hamiltonian is given by $H(v) := \|v\|^2/2$ so that its quantization gives, in local coordinates $(x^1, \ldots, x^n)$:

$$Q_\hbar(H) = -\frac{\hbar^2}{2}\Delta_g := \frac{\hbar^2}{2}\sum_{i,j} g^{ij}\nabla_i\partial_j$$

The second order differential operator Δ_g is called the *Laplace–Beltrami operator*. Because the Levi–Civita connection is completely determined by the metric, we can rewrite Δ_g as

$$\Delta_g = -\frac{1}{\sqrt{\det(g)}}\sum_{i,j=1}^n \frac{\partial}{\partial x^i}\left(\sqrt{\det(g)}g^{ij}\frac{\partial}{\partial x^j}\right). \tag{16}$$

We will now analyze the structure of the Laplace–Beltrami operator. For this we need the construction of *Hodge $*$-operator* on Riemannian manifolds.

Let us first consider the following construction in Linear Algebra: If $(V, \langle\ ,\ \rangle)$ is an oriented real vector space of dimension n with an inner product $\langle\ ,\ \rangle : V \times V \to \mathbb{R}$, the exterior powers $\Lambda^k V$ of V also inherit an inner product by setting

$$\langle w_1 \wedge \ldots \wedge w_k, v_1 \wedge \ldots \wedge v_k\rangle := \det\langle w_i, v_j\rangle.$$

In other words, if $e_1, \ldots, e_n$ is an orthonormal basis of V, an orthonormal basis for $\Lambda^k V$ is given by $e_{i_1} \wedge \ldots \wedge e_{i_k} \in \Lambda^k V$ with $i_1 < \ldots < i_k$. Recall that the choice of orientation of V is simply a choice of component of $\Lambda^n V \backslash \{0\}$, and the metric fixes a unique volume element in this component

$$\mathrm{Vol} := e_1 \wedge \ldots \wedge e_n,$$

of norm 1. The Hodge $*$-operator $* : \Lambda^k V \to \Lambda^{n-k} V$ is defined by the equation

$$\alpha \wedge *\beta = \langle \alpha, \beta\rangle\, \mathrm{Vol}, \quad \alpha, \beta \in \Lambda^k V.$$

In terms of the orthonormal basis:

$$*(e_{i_1} \wedge \ldots \wedge e_{i_k}) = (-1)^\sigma e_{j_1} \wedge \ldots \wedge e_{j_{n-k}},$$

where $\{i_1, \ldots, i_k, j_1, \ldots, j_{n-k}\} = \{1, \ldots, n\}$ and $\sigma = (i_1, \ldots, i_k, j_1, \ldots, j_{n-k})$. From this we easily derive that

$$*^2 = (-1)^{k(n-k)}.$$

Also remark that $*1 = \mathrm{Vol}$.

Let us now consider the de Rham complex on M:

$$0 \longrightarrow \Omega^0(M) \xrightarrow{d} \Omega^1(M) \xrightarrow{d} \ldots \xrightarrow{d} \Omega^n(M) \longrightarrow 0.$$

With the metric, we have the Hodge $*$-operator and we can introduce L^2-type inner products on $\Omega^k(M)$:

$$\langle \alpha, \beta \rangle := \int_M \alpha \wedge *\beta. \tag{17}$$

Lemma 4.6. *The formal adjoint of $d : \Omega^{k-1}(M) \to \Omega^k(M)$ is given by*

$$d^* = (-1)^{n(k+1)+1} * d*$$

Proof. This follows easily from Stokes' formula. Let $\alpha \in \Omega^{k-1}(M)$ and $\beta \in \Omega^k(M)$ and compute:

$$\begin{aligned}
0 &= \int_M d(\alpha \wedge *\beta) \\
&= \int_M d\alpha \wedge *\beta + (-1)^{k-1} \int_M \alpha \wedge d * \beta \\
&= \int_M d\alpha \wedge *\beta - (-1)^{n(k+1)+1} \int_M \alpha \wedge *(*d * \beta).
\end{aligned}$$

Interpreted in terms of the inner product (17), the statement now follows. □

Definition 4.7. The *Laplacian* acting on k-forms is the second order differential operator defined as

$$\Delta_g^k := dd^* + d^*d : \Omega^k(M) \to \Omega^k(M).$$

Proposition 4.8. *Restricted to* 0*-forms, Δ_g^0 agrees with* (16)

Corollary 4.9. *The Laplacian, acting on the dense domain given by $C_c^\infty(M)$, is a positive symmetric operator on $L^2(M)$.*

Proof. We now have $\Delta_g = d^*d$ it is clearly symmetric. Also for $f \in C_c^2(M)$, we see that

$$\langle f, \Delta_g f \rangle = \langle df, df \rangle = \|df\|_{L^2}^2 \geq 0,$$

so the operator is nonnegative. □

We therefore know that the Laplacian has a selfadjoint extension, namely Friedrichs' extension given in §B.2. However, on a complete Riemannian manifold, this is the *unique* selfadjoint extension, as the following theorem shows:

Theorem 4.10. *On a complete Riemannian manifold, the Laplace–Beltrami operator is essentially selfadjoint.*

See [St] for a proof, it amounts to showing that $H_{min} = H_{max}$.

4.3. **Adding a potential.** Recall that in general, the Hamiltonian (11) has a potential term $V \in C^\infty(X)$, which should be taken into account in the quantization. But this is easily done: since V is the lift of a function on the base manifold, its quantization is simply the multiplication operator on $\mathcal{H} = L^2(M)$. Recall that for $f \in C^\infty(M)$ and $g \in L^2(M)$ we have

$$\|fg\|_{L^2} \leq \|f\|_\infty \|g\|_{L^2},$$

and therefore the multiplication operator defined by f is bounded when f is a bounded function. When it is not, it defines an unbounded operator on the domain $D(f)$ of all $g \in L^2(M)$ for which $fg \in L^2(M)$.

In total, we now have the Hamiltonian

$$Q_\hbar(H) = -\frac{\hbar^2}{2}\Delta_g + V. \tag{18}$$

We have seen that for $V = 0$, this is an essentially selfadjoint operator. Treating V as a perturbation of this operator, there is a very general theorem, the Kato–Rellich theorem on perturbations, c.f. [RSII, §X.2], which guarantees us that (18) is still essentially selfadjoint.

4.4. **Dynamics.** Let us now assume that (M, g) is complete, and denote by $\bar{H}$ the selfadjoint extension of the Hamiltonian H acting on $D(\bar{H}) \subset \mathcal{H}$. In Quantum Mechanics, the dynamics of the system is described by the *Schrödinger equation*

$$\frac{du}{dt} = -\frac{\sqrt{-1}}{\hbar}\bar{H}u, \quad u \in \mathcal{H}. \tag{19}$$

The solution to this equation, subject to $u(0) = u_0$, describes the evolution of the system described by a time dependent vector $u(t)$ in $\mathcal{H}$.

Proposition 4.11. *For $u_0 \in D(\bar{H})$, the Schrödinger equation* (4.11) *has a unique solution given by*

$$u(t) = e^{-\sqrt{-1}t\bar{H}/\hbar}u_0.$$

Proof. Write $U(t) = e^{-\sqrt{-1}t\bar{H}/\hbar}$ for the one-parameter group generated by $\bar{H}$ as in Proposition B.5. Let $u_0 \in D(\bar{H})$, and check that indeed $u(t) := U(t)u_0$ is a solution:

$$\begin{aligned}
\frac{du}{dt} &= \lim_{h\to 0} \frac{(U(t+h) - U(t))u_0}{h} \\
&= \lim_{h\to 0} \frac{(U(h) - 1)U(t)u_0}{h} \\
&= -\frac{\sqrt{-1}}{\hbar}\bar{H}U(t)u_0 \\
&= -\frac{\sqrt{-1}}{\hbar}\bar{H}u.
\end{aligned}$$

We skip the proof of uniqueness. □

This is the so-called Schrödinger picture of Quantum Mechanics. In Quantum Mechanics, observables are given by selfadjoint operators on the Hilbert space $\mathcal{H}$. A vector $u \in \mathcal{H}$ defines a *state*

$$E_u(A) := \frac{\langle u, Au \rangle}{\langle u, u \rangle}, \tag{20}$$

where A is a selfadjoint operator on $\mathcal{H}$. It gives the expected (or average) value of the observable A in the state u. In the Schrödinger picture of Quantum Mechanics therefore the state evolves over time, whereas the observables are fixed.

Since the quantities (20) yield all the physical information one can deduce from a Quantum Mechanical system, there is a dual Heisenberg picture: observable evolves, states don't. Indeed with the solution of Proposition 4.11 to the Schrödinger equation, one easily see that we may as well define the time evolution of an observable as

$$A(t) := e^{\sqrt{-1}tH/\hbar} A e^{-\sqrt{-1}tH/\hbar}, \tag{21}$$

and keep u fixed: the evaluation (20) then gives the same result:

$$E_{u(t)}(A) = E_u(A(t)).$$

The Euclidean formalism. Let us (formally) substitute $\tau = -it$. The Schrödinger equation then transforms to the (abstract) *Heat equation*:

$$\begin{aligned} \frac{du}{d\tau} + \frac{\bar{H}}{\hbar} u &= 0 \\ u(0) &= u_0. \end{aligned}$$

Proposition 4.12. *When $\bar{H}$ is a positive selfadjoint operator,*

$$u(\tau) := e^{-\tau \bar{H}} u_0$$

is the unique solution to the heat equation with $u_0 \in D(\bar{H})$.

Proof. By Proposition B.9, the heat semigroup $e^{-\tau \bar{H}}$ is well-defined as a semigroup of contraction operators. The verification that $u(\tau)$ solves the heat equation is done in the same way as in the proof of Proposition 4.11. To verify uniqueness, suppose that $u^1(\tau)$ and $u^2(\tau)$ are two solutions with the same initial value u_0. Then $v(\tau) = u^1(\tau) - u^2(\tau)$ is a solution with $v(0) = 0$. Then we compute

$$\frac{d}{d\tau} \|v(\tau)\|^2 = -2 \langle v(\tau), Av(\tau) \rangle \leq 0.$$

But since clearly $\|v(\tau)\|^2 \geq 0$, we find that $\|v(\tau)\|^2 = 0$, i.e., $v(\tau) = 0$. This proves uniqueness. □

4.5. **Heat kernels and Propagators.** Let (M, g) be a complete Riemannian manifold. We take another look at the semigroup generated by the (selfadjoint extension of) the Laplacian Δ_g. The following theorem gives another description as a semigroup of kernels:

Theorem 4.13. *Let (M, g) be a complete Riemannian manifold. There exists a unique real-valued C^∞ function $K_\tau(x, y)$ on $\mathbb{R}_+ \times M \times M$, symmetric in x and y such that*

$$(e^{-\tau\bar{\Delta}_g}u)(x) = \int_M K_\tau(x, y)u(y)\mathrm{Vol}(g)(x), \quad \textit{for all } u \in L^2(M, g).$$

This function $K_\tau(x, y)$ is called the heat kernel *of (M, g).*

Remark 4.14. Because of Proposition 4.12 we see that the heat kernel satisfies

$$\frac{\partial}{\partial\tau}K_\tau(x, y) + \Delta_x k_\tau(x, y) = 0 \tag{22}$$

$$\lim_{\tau\downarrow 0}\int_M K_\tau(x, y)s(y)\mathrm{Vol}(g)(y) = s(x), \quad \text{for all } s \in C_c^\infty(M).$$

The semigroup property amounts to

$$K_{t_1+t_2}(x, y) = \int_M K_{t_1}(x, z)K_{t_2}(z, y)\mathrm{Vol}_g(z)$$

There are two approaches to prove the existence of the heat kernel: first there is a direct approach trying to solve (22) and then using uniqueness to show the equality with $e^{-\tau\bar{\Delta}_g}$. Second, one can use the spectral theory of Δ_g in good cases to construct the heat kernel. Below we give an outline of this argument in the case that M is *compact*.

For Schrödinger equation (4.11), there also exists a kernel $K_t(x, y)$, called the *propagator* but its properties are not as nice as that of the heat kernel, showing that the formal substitution $t \mapsto -it$ isn't that innocent. Most importantly, it may not be a smooth function anymore, but rather a distribution. (This happens in for example in the compact case mentioned above.) The analysis of this propagator is really very subtle, and we shall no more about it in general.

4.6. **Examples.**

Free particle on $\mathbb{R}^n$. On $\mathbb{R}^n$, equipped with the euclidean metric, the Laplacian takes the usual form

$$\Delta = -\sum_{i=1}^n \frac{\partial^2}{\partial x_i^2},$$

where $(x_1, \ldots, x_n)$ are the standard euclidean coordinates. In this case, one easily verifies that

$$K_\tau(x, y) = (4\pi t)^{-n/2}\exp\left(-\frac{||x-y||^2}{4t}\right)$$

solves the heat equation, i.e., the first equation in (22). By using Fourier transformation, one shows that the second equation holds true, i.e., $K_\tau(x, y)$ is indeed the heat kernel.

Compact Riemannian manifolds. The case of a *compact* Riemannian manifold differs from the general (noncompact) case since the spectrum of the Laplace–Beltrami operator is discrete. Recall that $u_n \in L^2(M)$ is said to be an *eigenfunction* of Δ_g with *eigenvalue* $\lambda_n \in \mathbb{R}$ when it satisfies

$$\Delta_g u_n = \lambda_n u_n.$$

The following theorem summarizes the whole spectral theory of the Laplace–Beltrami operator on a compact Riemannian manifold:

Theorem 4.15. *Let (M, g) be a compact Riemannian manifold. Then the following holds true:*

i) The spectrum of the Laplace–Beltrami operator consists of

$$0 \leq \lambda_0 < \lambda_1 < \dots, \quad \lambda_n \to \infty,\ n \to \infty.$$

ii) Each eigenspace $\mathcal{H}_k \subset L^2(M)$ is finite dimensional and consists of eigenfunctions that are smooth*: $u_n \in C^\infty(M)$.*

iii) The set of eigenvectors is complete*: there is a Hilbert space decomposition*

$$L^2(M, \mathrm{Vol}(g)) \cong \bigoplus_{n \geq 0} \mathcal{H}_n.$$

We will not prove this theorem, it heavily relies on the fact that Δ_g is *elliptic*: this is a property of differential operators that is very easy to check, namely it means that the symbol $\sigma(\Delta_g) \in C^\infty(T^*M)$ defined in Proposition 4.2 is nonzero off the zero section $M \subset T^*M$. However, this innocently looking property has profound consequences, some of which are stated in the theorem: the spectrum is discrete and consists of smooth functions (elliptic regularity). The fact that eigenvalues are real and positive of course follows immediately from the fact that Δ_g is symmetric and positive.

The basic question of *spectral geometry* is: how much of the geometry can we extract from the spectrum of Δ_g? ("Can one hear the shape of a drum?") The following classical result shows that at least the volume of M can be recovered:

Theorem 4.16 (Weyl)**.** *Let (M, g) be a compact Riemannian manifold. For $\lambda_k \to \infty$,*

$$\lambda_k \sim 4\pi \left(\frac{\Gamma(n/2+1)}{\mathrm{Vol}(M)} \right)^{n/2} k^{2/n}.$$

Let us now outline how the heat kernel of Δ_g can be constructed from the two theorems above, c.f. [LM]. Consider the following expression:

$$K_\tau(x, y) := \sum_{k=0}^{\infty} e^{-\tau \lambda_k} u_k(x) \otimes u_k(y). \tag{23}$$

Using a fundamental result for elliptic differential operators, the so-called *elliptic estimates*, one shows that

$$\|u_k\|_{C^r} \leq c(1 + \lambda_k^s),$$

for some $c > 0$ and $s > (n/4) + (r/2)$. Using this estimate, we can show that $K_\tau(x,y)$, as defined above, converges in the C^r norm for any $r > 0$, using the integral test for the integral

$$\int_0^\infty e^{-tx} x^s dx < \infty.$$

We therefore conclude that $K_\tau(x,y)$ is smooth in both variables. Second, it is easily seen that it satisfies the heat equation:

$$\begin{aligned}\frac{\partial}{\partial\tau} K_\tau(x,y) &= -\sum_{k=0}^\infty \lambda_k e^{-\tau\lambda_k} u_k(x) \otimes u_k(y)\\ &= -\sum_{k=0}^\infty e^{-\tau\lambda_k} (\Delta_x^g u_k(x)) \otimes u_k(y)\\ &= -\Delta_x^g K_\tau(x,y).\end{aligned}$$

Finally, using the completeness of the set of eigenfunctions, we immediately get that

$$\lim_{\tau\downarrow 0} \int_M K_\tau(x,y) s(y) \mathrm{Vol}_g(y) = s(x),$$

where the convergence is in the L^2-sense.

5. The Path integral approach to Quantum Mechanics

The previous section dealt with Quantum Mechanics from the Hamiltonian point of view. The analogue of the Lagrangian approach to classical mechanics is given by the path integral. Physically, this is a beautiful incorporation of the principle of least action in Quantum Mechanics. Mathematically however, there are many problems to give a rigorous construction of this integral.

5.1. **Derivation from the Hamiltonian approach.**

Theorem 5.1 (The Trotter product formula)**.** *Let A and B be selfadjoint operators on a Hilbert space $\mathcal{H}$ and suppose that $A+B$ is selfadjoint on $D(A) \cap D(B)$. Then*

$$\limsup_{n\to\infty} \left(e^{itA/n} e^{-itB/n}\right)^n = e^{it(A+B)}.$$

When A and B in addition are bounded from below, we have

$$\limsup_{n\to\infty} \left(e^{-tA/n} e^{tB/n}\right)^n = e^{-t(A+B)}$$

Proof. (c.f. [S]) We write $S(t) := e^{\sqrt{-1}t(A+B)}$, $V(t) = e^{\sqrt{-1}tA}$, $W(t) = e^{\sqrt{-1}tB}$ and $U(t) = V(t)W(t)$, and $u(t) = S(t)u_0$ for some $u_0 \in \mathcal{H}$. Then we have

$$\begin{aligned}\|(S(t) - U(t/n)^n)u_0\| &= \left\| \sum_{j=0}^{n-1} U(t/n)^j (S(t/n) - U(t/n)) S(t/n)^{n-j-1} u_0 \right\|\\ (*) \qquad &\le n \sup_{0\le s\le t} \|(S(t/n) - U(t/n))u(s)\|\end{aligned}$$

If $u_0 \in D(A) \cap D(B)$, we have

$$\lim_{s \to 0} \frac{(S(s)-1)u_0}{s} = \sqrt{-1}(A+B)u_0,$$

and

$$\begin{aligned} \frac{(U(s)-1)u_0}{s} &= V(s)(\sqrt{-1}Bu_0) + V(s)\left(\frac{(W(s)-1)}{s} - \sqrt{-1}B\right)u_0 + \frac{(V(s)-1)u_0}{s} \\ &\xrightarrow{s \to 0} \sqrt{-1}(A+B)u_0. \end{aligned}$$

Therefore we have

$$(**) \qquad \lim_{n \to \infty} \left(n \left\| S(t/n) - U(t/n)u_0 \right\|\right) = 0, \quad \text{for all } u_0 \in D(A) \cap D(B).$$

We can equip $D = D(A) \cap D(B)$ with the norm $\|u_0\|_D := \|u_0\| + \|(A+B)u_0\|$. By assumption, $A + B$ is selfadjoint on D, and therefore closed: it follows that D equipped with the norm above is a Banach space. We view $n(S(t/n) - U(t/n))$ as a family of bounded operators from D to $\mathcal{H}$, and we see from the calculation above that

$$\sup_n \{\|n(S(t/n) - U(t/n))u_0\|\} < \infty.$$

By the uniform boundedness principle, we have

$$\|n(S(t/n) - U(t/n))u_0\| \leq C\,\|u_0\|_D\,, \quad \text{for all } u_0 \in D(A) \cap D(B),$$

for some $C > 0$, independent of n. This inequality implies that the limit in $(**)$ is uniform over compact subsets of D. For $u_0 \in D$, the map $s \mapsto u(s)$ is continuous from $[0, t]$ to D, so the image is compact in D. We now see that in the limit as $n \to \infty$, $(*)$ goes to zero for $u_0 \in D$. Since the operator $S(t) - U(t/n)^n$ has operator norm bounded by 1, and D is dense, this shows that the limit is zero for all $u_0 \in \mathcal{H}$. This proves the first statement. $\square$

We now return to the vector space situation: $V = \mathbb{R}^n$, with Hamiltonian

$$\hat{H} := -\frac{1}{2}\sum_{i,j} \frac{\partial^2}{(\partial x^i)^2} + V(x),$$

which we write as $H = H_0 + V$. Then $e^{it\hat{H}_0}$ is the operator with kernel

$$K_0(x, y, t) = (2\pi i t)^{-n/2} \exp\left(i\frac{||x-y||^2}{2t}\right).$$

It follows that the operator $(e^{-itH_0/n}e^{-itV/n})^n$ has kernel

$$K^{(n)}(x_0, x_n, t) = \left(\frac{2\pi i t}{n}\right)^{3n/2} \int \exp\left(iS(x_0, x_1, \ldots, x_n; t)\right) dx_1 \cdots dx_{n-1},$$

where

$$S(x_0, x_1, \ldots, x_n; t) = \sum_{i=1}^{n} \left(\frac{n}{2t}\right) ||x_{i-1} - x_i||^2 - \sum_{i=1}^{n} V(x_i)\left(\frac{t}{n}\right)$$

This expression is an approximation to the action

$$S(\gamma) := \frac{1}{2}\int_0^t \left\|\frac{d\gamma}{ds}\right\|^2 ds - \int_0^t V(\gamma(s))ds,$$

with γ the "polygonal path" that passes through x_j at time jt/n. Therefore as $n \to \infty$ we formally obtain

$$K_t(x,y) = \int_{\substack{\gamma(0)=x\\ \gamma(t)=y}} \exp\left(iS(\gamma)\right) d\gamma, \tag{24}$$

where $d\gamma$ is some kind of "Lebesque-type" measure on the space of paths with fixed beginning and end-point. Unfortunately, such a translation-invariant measure simply doesn't exist, due to the infinite-dimensionality of the space of paths. In the euclidean case however, one can make sense of the combination $\exp\left(-S(\gamma)\right) d\gamma$ as a probability measure on the space of continuous paths in M: this is the Wiener measure.

5.2. **The Wiener measure.** Let (M,g) be a Riemannian manifold. As we have seen, when M is complete, the Laplace–Beltrami operator Δ_g is positive and selfadjoint, and therefore generates a unique semigroup $e^{-t\bar{\Delta}_g}$, called the Heat-Kernel, also written as the kernel function $K_\tau(x,y)$. This heat kernel is essential in the construction of the Wiener measure on the path space on M. Below we outline this construction following [T, §11]. Let us write $\mathcal{F}(I,x_0)$ for the space of continuous maps $\gamma : [0,t] \to M$ with $\gamma(0) = x_0$, with $x_0 \in M$ fixed.

We shall assume that the heat kernel satisfies

$$\int_M K_\tau(x,y)\mathrm{Vol}_g(y) = 1, \tag{25}$$

for all $x \in M$, $\tau > 0$. This for example the case for $M = \mathbb{R}^n$, and also compact M. With this property, the heat kernel $K_\tau(x,y)$ defines, for each x a probability distribution that we interpret as the probability for a random particle to be at place y after time τ.

The idea of the Wiener measure is that we can use the heat kernel to try to define a measure on $\mathcal{F}(I,x,y)$ in the following way: for $0 < t_1 < \ldots < t_n < T$ and Borel sets $E_1,\ldots,E_n \subset M$, the measure evaluated on the *cylinder subset* of paths γ with the property that $\gamma(t_i) \in E_i,\ i = 1,\ldots,n$ is given by

$$\begin{aligned} &W(\gamma \in \mathcal{F}(I;x_0,x_1),\ \gamma(t_i) \in E_i,\ i = 1,\ldots,n\}) \\ &\quad = \int_{M^{\times n}} \chi_{E_1\times\ldots\times E_n} K_{T-t_n}(y,x_n)\cdots K_{t_2-t_1}(x_2,x_1)K_{t_1}(x_1,x)\mathrm{Vol}_g(x_1)\cdots\mathrm{Vol}_g(x_n). \end{aligned} \tag{26}$$

Theorem 5.2. *Let (M,g) be a Riemannian manifold satisfying* (25). *Then the heat kernel induces unique probability measures W_{x_0} on $\mathcal{F}(I;M,x_0)$, called the* Wiener measure, *and $W_{x_0}^{y_0}$ on $\mathcal{F}(I;M,x_0,x_1)$ called the* conditional Wiener measure. *They are related by the equality*

$$\int_{\mathcal{F}(I;M,x_0)} f(\gamma)dW_{x_0}(\gamma) = \int_M \left(\int_{\mathcal{F}(I;M,x_0,y_0)} f(\gamma)dW_{x_0}^{y_0}(\gamma)\right) d\mathrm{Vol}_g(y_0),$$

for all integrable functions f on $\mathcal{F}(I;M,x_0)$

Proof. We give a sketch of the construction, c.f. [T, §11]. The idea is to approximate paths in M by their locations at rational τ. For this we define

$$\mathfrak{F} := \prod_{\tau \in [0,T] \cap \mathbb{Q}} \dot{M}$$

($\dot{M}$ is the one-point compactification of M.) Equipped with the product topology, this is a compact, metrizable space. By Riesz' theorem, a probability measure on $\mathfrak{F}$ corresponds to a positive linear functional $E : C(\mathfrak{F}) \to \mathbb{R}$ satisfying $E(1) = 1$. We first define E on the subspace $C^{\text{fin}}_{\tau_1,\ldots\tau_k}(\mathfrak{F})$ of functions depending on *finitely many* $\tau \in [0,T] \cap \mathbb{Q}$, i.e., functions that can we written as

$$\varphi(\gamma) = F(\gamma(\tau_1), \ldots, \gamma(\tau_k)), \quad \tau_1 < \tau_2 < \ldots < \tau_k,$$

with F continuous on $\prod_{i=1}^k \dot{M}$. On these functions we define

$$E(\varphi) := \int_{M^{\times k}} K_{\tau_1}(x_0, x_1) \cdots K_{\tau_k - \tau_{k-1}}(x_{k-1}, x_k) F(x_1, \ldots, x_k) d\mathrm{Vol}_g(x_1) \cdots d\mathrm{Vol}_g(x_k). \tag{27}$$

There is an obvious map $C^{\text{fin}}_I(\mathfrak{F}) \to C^{\text{fin}}_J(\mathfrak{F})$ for $I \subset J \subset [0,T] \cap \mathbb{Q}$, and the semigroup property of $K_\tau(x,y)$ ensures that E is compatible with this inclusion. Clearly, E is positive linear on $C^{\text{fin}}(\mathfrak{F})$ and satisfies $E(1) = 1$.

By the Stone Weierstrass theorem, $C^{\text{fin}}(\mathfrak{F}) \subset C(\mathfrak{F})$ is dense and E has a unique continuous extension to $C(\mathfrak{F})$ having the same properties. By Riesz'theorem, there exists a measure W_{x_0} such that

$$E(\varphi) := \int_{\mathfrak{F}} \varphi(\gamma) dW_{x_0}(\gamma).$$

Inside $\mathfrak{F}$, we restrict to the subset $\mathfrak{F}_0$ of paths from $I \subset \mathbb{Q}$ to M that are *uniformly continuous*. These are exactly the paths that extend uniquely to continuous paths from $[0,\tau]$ to M. The crucial point is then:

Claim[3]: $\mathfrak{F}_0$ is a Borel subset of measure 1.

This concludes the construction of the Wiener measure on the space of continuous paths in M, starting at x_0. The conditional Wiener measure is constructed in a similar way, we skip the details. □

Remark 5.3. Choose $0 < \tau_1 < \ldots < \tau_k < T$ and functions $f_1, \ldots, f_k \in L^\infty(M)$. With these we can define the cylindrical function $\varphi^{\tau_1,\ldots,\tau_k}_{f_1,\ldots,f_k} \in C^{\text{fin}}(\mathfrak{F})$ by

$$\varphi^{\tau_1,\ldots,\tau_k}_{f_1,\ldots,f_k}(\gamma) := f_1(\gamma(\tau_1)) \cdots f_k(\gamma(\tau_k)).$$

This function is integrable with respect to the Wiener measure and defines the *correlation functions*

$$\langle f_1(\tau_1) \cdots f_k(\tau_k) \rangle := \int_{\mathcal{F}(I;M)} \varphi^{\tau_1,\ldots,\tau_k}_{f_1,\ldots,f_k}(\gamma) dW(\gamma). \tag{28}$$

[3]The proof of this claim is highly nontrivial, c.f. [T]

5.3. **The Feynman–Kac formula.** The Wiener measure gives an integral formula for the kernel of the free Hamiltonian $H_0 = \Delta_g$ in euclidean quantum mechanics. The general case of a Hamiltonian $H = \Delta_g - V$ is given by the so-called Feynman–Kac formula:

Theorem 5.4 (Feynman–Kac formula)**.** *Let $H = \Delta_g - V$ with $V \in L^\infty(M, \mathbb{R})$ a potential. Then the semigroup generated by H is given by*

$$(e^{-tH}s)(x) = \int_{\mathcal{F}(I,M,x_0)} s(\gamma(\tau)) \exp\left(\int_0^\tau V(\gamma(s))ds\right) dW_x(\gamma).$$

Proof. We sketch the proof in the case that V is continuous and bounded. Applying the Trotter product formula we get

$$\begin{aligned}
&\left(e^{-t\Delta/n}\ e^{tV/n}\right)^n s(x_0) \\
&\qquad = \int_{M^{\times n}} \exp\left(\frac{t}{n}\sum_{j=1}^n V(x_j)\right) s(x_n) K_{t/n}(x_n, x_{n-1}) \cdots K_{t/n}(x_1, x_0) dx_n \cdots dx_1 \\
&\qquad = \int_{\mathcal{F}(I;M,x_0)} \exp\left(\frac{t}{n}\sum_{j=1}^n V(\gamma(jt/n))\right) s(\gamma(t)) dW_{x_0}(\gamma).
\end{aligned}$$

Since $V \circ \gamma : [0, t] \to \mathbb{R}$ is continuous, the Riemann sum will converge to an integral:

$$\frac{t}{n}\sum_{j=1}^n V(\gamma(jt/n)) \xrightarrow{n\to\infty} \int_0^t V(\gamma(s))ds.$$

To complete the proof, one needs to argue that one can bring the limit $n \to \infty$ inside the integral over $\mathcal{F}(I; M, x_0)$, for the details see [RSII, Thm X.68]. $\square$

With the Feynman–Kac formula, we can now give the rigorous statement for the Kernel, in accordance with the heuristic formula (24):

Corollary 5.5. *The integral kernel for the operator $e^{-t(\Delta_g - V)}$ is given by*

$$K_t^H(x, y) = \int_{\mathcal{F}(I;M,x,y)} \exp\left(\int_0^t V(\gamma(s))ds\right) dW_x^y(\gamma).$$

Let $\mathcal{H}$ be a separable Hilbert space with basis $\{e_i\}_{i\in\mathbb{N}}$. For a positive bounded operator S on $\mathcal{H}$, define its trace to be

$$\mathrm{Trace}(S) = \sum_{i=0}^\infty \langle Se_i, e_i\rangle_{\mathcal{H}} \in [0, \infty]. \tag{29}$$

One can prove (check!) that this trace is independent of the chosen basis. For $1 \leq p < \infty$, define the *p-th Schatten class* $B_p(\mathcal{H})$ to be the set of bounded operators for which

$$||T||_p := \sqrt[p]{\mathrm{Trace}(|T|^p)} < \infty,$$

where $|T| = \sqrt{T^*T}$ defined by means of the functional calculus. For $p = 2$, these operators are called *Hilbert–Schmidt*. One can prove that each $B_p(\mathcal{H})$ is an ideal inside the compact operators $K(\mathcal{H})$. For $T \in B_1(\mathcal{H})$, the sum $\sum_i \langle Te_i, e_i \rangle$ converges absolutely to a linear functional

$$\mathrm{Trace} : B_1(\mathcal{H}) \to \mathbb{C},$$

satisfying

$$\mathrm{Trace}(ST) = \mathrm{Trace}(TS), \quad \text{for all } S \in B(\mathcal{H}),\ T \in B_1(\mathcal{H}). \tag{30}$$

Remark 5.6 (The partition function)**.** When M is compact, the heat kernel $K_\tau(x,y)$ is trace class. One can show this fact using equation (23) together with asymptotics of the eigenvalues given by Weyl's Theorem 4.16. Since the heat kernel is a positive operator, It follows easily from the same equation (23) and the definition of the trace in equation (29) that

$$\mathrm{Tr}\left(e^{-\tau\Delta_g}\right) = \int_M K_\tau(x,x)\mathrm{Vol}_g(x).$$

This function of τ is called the *partition function* of the model, and it has a beautiful interpretation in statistical physics. More generally, from Corollary we see that

$$\mathrm{Tr}\left(e^{-\tau H}\right) = \int_M \int_{\mathcal{F}(I;M,x,x)} \exp\left(\int_0^t V(\gamma(s))ds\right) dW_x^y(\gamma)\mathrm{Vol}_g(x).$$

5.4. **The mathematical structure of Euclidean Quantum Mechanics.** Let us now assume that (M,g) is compact. Then the constant function 1 defines a vector $\Omega \in \mathcal{H} = L^2(M, \mathrm{Vol}_g)$ called the *vacuum vector*. Indeed it is the unique eigenvector of the Laplacian Δ_g with "energy" zero, which is therefore fixed by the heat semigroup:

$$e^{-t\Delta_g}\Omega = \Omega, \quad \text{for all } t > 0.$$

With this vector, we now see that the Hamiltonian formulation of euclidean quantum mechanics uses the following list of items:

1 the Hilbert space $\mathcal{H}$ together with the vacuum vector $\Omega \in \mathcal{H}$,
2 the strongly continuous one-parameter semigroup of operators $S(t) = e^{-tH}$,
3 the ring of operators $Q(f)$, $f \in L^\infty(M)$.

We have not really stressed the last item so far. Recall that for a function f on the base manifold M the quantization procedure described in §4.1 simply imposes that $Q(f)$ is the operator on $L^2(M)$ given by multiplication with f. (Notice that this operator is bounded by compactness of M.)

For the following, we recall the definition of the correlation functions in Remark 5.3 using the Wiener measure. The following Theorem explains how they are related to the ingredients of the Hamiltonian approach:

Theorem 5.7. *Let* $0 < \tau_1 < \ldots < \tau_k < T$ *and* $f_1, \ldots, f_k \in L^\infty(M)$. *For the free Hamiltonian* $H = \Delta_g$, *we have the equality*

$$\langle f_1(\tau_1)\cdots f_k(\tau_k)\rangle = \langle \Omega, S(T-\tau_k)Q(f_k)S(\tau_k - \tau_{k-1})Q(f_{k-1})\cdots S(\tau_2-\tau_1)Q(f_1)S(\tau_1)\Omega\rangle_{\mathcal{H}}$$

Proof. This follows by the definition (28) of the correlation function, using equation (27) for the value of the Wiener measure on cylindrical functions. ☐

6. SUPERSYMMETRIC QUANTUM MECHANICS

There is a supersymmetric extension of the model we have discussed so far –that of a free particle on a Riemannian manifold– that can be introduced without much further analytical complications, and has very interesting geometrical and topological consequences. For this, we go back to the Hamiltonian of the theory, namely the Laplace operator on a Riemannian manifold (M, g); the Laplace–Beltrami operator. As we have seen in Definition 4.7, there is a natural extension of this operator acting on differential forms. This suggests to look at the quantum mechanical model where the Hilbert space is given by

$$\mathcal{H} := \bigoplus_{p\geq 0} \Omega^p_{L^2}(M),$$

where $\Omega^p_{L^2}(M)$ denotes the L^2-completion of $\Omega^p(M)$ in the inner product (17).(Remark that the sum above is finite.) As Hamiltonian we now take the Laplace–Beltrami operator defined in Definition 4.7 acting on each degree.

With this, we see we have enlarged our Hilbert space, $L^2(M\mathrm{Vol}_g)$ is the piece in $\mathcal{H}$ in degree zero, and extended our original Hamiltonian. It turns out that this model has a richer structure than the original one, referred to as "supersymmetry" by physicists.

First of all, we can decompose $\Omega^\bullet(M) = \Omega^{\mathrm{ev}}(M) \oplus \Omega^{\mathrm{odd}}(M)$, which induces a $\mathbb{Z}/2$-grading of the Hilbert space

$$\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-.$$

($+$ corresponds to even-, and $-$ to odd-degree differential forms.) With respect to this grading we can view the de Rham operator as an unbounded operator

$$d : \mathcal{H}_\pm \to \mathcal{H}_\mp,$$

acting on the dense domain of smooth differential forms with compact support. We have already seen in Lemma 4.6 that its (formal) adjoint is given by $(-1)^{n(k+1)+1} * d*$, and also acts on the same domain of definition as an unbounded operator $d^* : \mathcal{H}_\pm \to \mathcal{H}_\mp$. With these operators, we now define

$$Q := \begin{pmatrix} 0 & d + d^* \\ d + d^* & 0 \end{pmatrix}.$$

Then we find, by virtue of $d^2 = 0 = (d^*)^2$, that

$$Q^2 = \begin{pmatrix} dd^* + d^*d & 0 \\ 0 & dd^* + d^*d \end{pmatrix} = \Delta_g.$$

Physicist say that the model now has *supersymmetry* and call Q the supersymmetry generator. Supersymmetric quantum mechanical models were introduced in mathematics in the paper [W]. Mathematically, the existence of this supersymmetry has a remarkable consequence if we look at the euclidean partition function

$$Z_M(\tau) := \mathrm{Tr}_{\mathcal{H}}\left(\gamma e^{-\tau\Delta_g}\right) = \mathrm{Tr}_{\mathcal{H}_+}\left(e^{-\tau\Delta_g}\right) - \mathrm{Tr}_{\mathcal{H}_-}\left(e^{-\tau\Delta_g}\right),$$

where

$$\gamma = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Theorem 6.1 (Mckean–Singer)**.** *Let (M, g) be a compact Riemannian manifold. $Z_M(\tau)$ does not depend on τ and equals the Euler number $\chi(M)$.*

Proof. The proof uses the formula (23) for the heat kernel, which continues to hold for the Laplacian acting on forms if we consider all eigenforms $\psi_\lambda \in \Omega^\bullet(M)$, $\lambda \in \mathbb{R}_+$. Again we have a complete eigenspace decomposition

$$\mathcal{H} = \bigoplus_{\lambda\in\sigma(\Delta_g)} \mathcal{H}_\lambda,$$

and we write $n_\lambda^\pm := \dim(\mathcal{H}_\lambda^\pm) < \infty$. With this notation, we have

$$\mathrm{Tr}(\gamma e^{-\tau\Delta_g}) = \sum_{\lambda\geq 0}(n_\lambda^+ - n_\lambda^-)e^{-t\lambda}.$$

One easily checks that $[Q, \Delta_g] = 0$[4], so we see that $Q_\pm : \mathcal{H}_\lambda^\pm \to \mathcal{H}_\lambda^\mp$. The composite

$$\mathcal{H}_\lambda^+ \xrightarrow{Q_+} \mathcal{H}_\lambda^- \xrightarrow{\lambda^{-1}Q_-} \mathcal{H}_\lambda^+$$

is the identity for $\lambda \neq 0$, so we see that $\mathcal{H}_\lambda^+ \cong \mathcal{H}_\lambda^-$. It follows that $n_\lambda^+ = n_\lambda^-$ and we find

$$\mathrm{Tr}(\gamma e^{-\tau\Delta_g}) = n_0^+ - n_0^- = \dim\ker\left(\Delta_g|_{\Omega^{\mathrm{ev}}(M)}\right) - \dim\ker\left(\Delta_g|_{\Omega^{\mathrm{odd}}(M)}\right).$$

By the Hodge Theorem, we have that

$$\ker\left(\Delta_g|_{\Omega^k(M)}\right) \cong H^k(M),$$

and therefore

$$\mathrm{Tr}(\gamma e^{-\tau\Delta_g}) = \sum_{i=0}^{k}(-1)^i \dim H^i(M) = \chi(M).$$

This proves the theorem. ☐

Remark 6.2. Observe that again we have been able to extract a geometric/topological invariant of the manifold from the spectrum of the Laplacian, namely the Euler number!

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[4]Working with commutation relations between unbounded operators can be very tricky, c.f. [RSI], but we only need this relation for the operators restricted to the eigenspaces which consists of smooth compactly supported forms, so no surprises appear

7. Topological Quantum Field Theory

We now move on to discuss the basic mathematical structures underlying Quantum field theory. As mentioned, Quantum field theory is an infinite dimensional generalization of Quantum mechanics, and the structure we want to discuss (partially) generalizes that discussed in §5.4.

7.1. **The cobordism category.**

The notion of *cobordism* between manifolds is classical in differential topology. Recall the notion of a *manifold with boundary*:

Definition 7.1. A *manifold with boundary* is a paracompact Hausdorff space M covered by charts taking values in $\mathbb{R}^n$ or the half space

$$\mathbb{R}^n_+ := \{(x_1,\ldots,x_n),\ x_n \geq 0\}.$$

Transition functions should be smooth, where $\varphi : \mathbb{R}^n_+ \to \mathbb{R}^n_+$ is said to be smooth if it admits a smooth extension to an open subset of $\mathbb{R}^n$. The boundary of M, written ∂M is given by the inverse image of the plane $x_n = 0$ under the chart maps.

Remark 7.2. Since we assume that all the transition functions of the charts that take value in the half space $\mathbb{R}^n_+$ can be extended to a small open neighborhood in $\mathbb{R}^n$, the usual definition of the tangent space $T_xM,\ x \in \partial M$ makes perfect sense. As a result, we have that $T_x\partial M \subset T_xM$ as a codimension one subspace.

Theorem 7.3 (c.f. [H, §8.2])**.** *Suppose that M and N are manifolds with boundary, and $f : \partial M \xrightarrow{\cong} \partial N$ a diffeomorphism of its boundaries. Then there exists a manifold structure on*

$$M \cup_f N := \left(M \coprod N\right)/\sim_f, \quad x \sim_f y \Longleftrightarrow f(x) = y,$$

inducing the given manifold structure on M and N. Furthermore, if α and β are two manifold structures on $M \cup_f N$ with the property above, there exists a diffeomorphism $\phi : (M \cup_f N, \alpha) \to (M \cup_f N, \beta)$ with $\phi|_{\partial M} = \mathrm{id}$.

Proof. Using the canonical projection $M \coprod N \to M \cup_f N$, we give $M \cup_f N$ the quotient topology: a set is open if its inverse image is open in M and in N. With this, we see that the gluing is well defined as topological spaces. The subtleties arise when one considers the smooth structure. To construct a smooth structure, we use the following

Lemma 7.4 (Existence of collars, c.f. [H, §6.2])**.** *Let M be a compact manifold with boundary ∂M. Then there exists a neighborhood of ∂M that is diffeomorphic to $\partial M \times [0, \epsilon)$.*

Proof. Choose a smooth function $f : M \to \mathbb{R}_+$ with $f^{-1}(0) = \partial M$ and $df \neq 0$ in a neighborhood U of ∂M. We also choose a metric g on M and consider the *gradient flow* of f: this is the solution $\varphi : [0, \epsilon) \to M$ of the initial value problem

$$\frac{d}{dt}\varphi(t) = \mathrm{grad}(f)(\varphi(t)), \quad \varphi(0) = x_0 \in \partial M.$$

As usual, the Picard–Lindelöf gives us the local existence and uniqueness of the flow. Next, observe that

$$\frac{d}{dt}(f\circ\varphi)(t) = df(\varphi(t))(\frac{d}{dt}\varphi(t)) = df(\varphi(t))(\mathrm{grad}(f)(\varphi(t))) = \|\mathrm{grad}(f)(\varphi(t)\|_g^2 \geq 0.$$

We see that f must be monotonically increasing along the flow line $\varphi(t)$. With this, we see that the map $[0,\epsilon)\times\partial M \to M$, given by $(t, x_0) \mapsto \varphi(t, x_0)$ is a diffeomorphism onto a neighborhood of ∂M. □

With the help of this Lemma, we fix a collaring of M and N and construct a smooth map $\psi : (-\epsilon,\epsilon)\times\partial M \to M\cup_f N$ with $\psi(0,x) = x$ for all $x\in\partial M$ which is a local diffeomorphism onto an open neighborhood of ∂M inside $M\cup_f N$. If we write M° and N° for the interior of M, resp. N, we see that we can cover $M\cup_f N$ by three open sets, namely M°, N° and the image of $(-\epsilon,\epsilon)\times\partial M$ under ψ. We can now introduce charts on these three open sets, using the manifold structures on M°, N° and ∂M, and these are, by construction, compatible with each other. □

Next, we add orientations. Suppose that M is an n-dimensional manifold with boundary, and $\Sigma\subset\partial M$ a connected component of the boundary of M. Assume M and Σ to be oriented. We say that a vector $v\in T_xM$, $x\in\Sigma$ is *positive normal* if $\{v_1,\ldots,v_{n-1},v\}$ is a positive basis for a choice of positive basis $\{v_i\}_{i=1}^{n-1}$ of $T_x\Sigma$. If such a vector points *inwards*, we call Σ an *incoming boundary*, if it points *outwards* we call it *outgoing*. (One should check that this is independent of the various choices made.) We can now define the notion of an *oriented cobordism*:

Definition 7.5. Let Σ_0 and Σ_1 be two closed oriented $(n-1)$-dimensional manifolds. An *oriented cobordism from* Σ_0 *to* Σ_1 is a triple (f_0, M, f_1) consisting of an oriented n-dimensional manifold with boundary, together with maps $f_i : \Sigma_i \to \partial M$, $i = 0, 1$ such that $\partial M = f_0(\Sigma_0)\coprod f_1(\Sigma_1)$, and which embed Σ_0 as an incoming boundary and Σ_1 as an outgoing boundary.

We shall write such a cobordism as

$$\Sigma_0 \overset{M}{\rightsquigarrow} \Sigma_1,$$

to emphasize the fact that this is *not* a function from Σ_0 to Σ_1. Next, if we have cobordisms

$$\Sigma_0 \overset{M}{\rightsquigarrow} \Sigma_1, \quad \Sigma_1 \overset{N}{\rightsquigarrow} \Sigma_2$$

we would like to define their composition as

$$\Sigma_0 \overset{M\cup_{\Sigma_1} N}{\rightsquigarrow} \Sigma_2,$$

where we glue M and N along that part of the boundaries diffeomorphic to Σ_1 using the composition of the two maps present in the two triples representing the two cobordisms. As we have seen above, such a gluing is only well defined up to diffeomorphism. Therefore, we make the following

Definition 7.6. The *cobordism category* Bord_n has closed[5] $(n-1)$-dimensional oriented manifolds as its objects. Morphisms are given by diffeomorphism classes of cobordisms with composition given by gluing along boundaries as indicated above.

To check that this is indeed a category, one needs to check that composition of morphisms is associative (do this yourself). Also, one needs to show the existence of unit morphisms $1 : \Sigma \rightsquigarrow \Sigma$ in Bord_n for any object Σ. Indeed, for any $(n-1)$-dimensional oriented manifold we always cobordisms of the form $\Sigma \times I$, from Σ to Σ (check that this is consistent with orientations) where I any closed interval in $\mathbb{R}$. One easily shows that these cobordisms are all diffeomorphic for different choices of interval I, so that we really have defined a unique morphism $\Sigma \rightsquigarrow \Sigma$. Furthermore, it is indeed not difficult to show that element is a unit in the sense that composition from the left or the right acts as a unit.

We can "twist" the unit cobordism by any diffeomorphism of Σ as follows: for any $f \in \mathrm{Diff}^+(\Sigma)$, instead of considering the unit cobordism $(i_0, \Sigma \times [0,1], i_1)$, we precompose i_0 by f to obtain the cobordism $(i_0 \circ f, \Sigma \times I, i_1)$. One easily shows that this defines a homomorphism

$$\mathrm{Diff}^+(\Sigma) \to \mathrm{Hom}_{\mathsf{Bord}_n}(\Sigma, \Sigma).$$

Proposition 7.7. *Two diffeomorphisms induce the same cobordism class if and only if they are smoothly homotopic.*

Proof. Exercise. □

Remark 7.8 (Monoidal structure)**.** The category $\mathsf{Bord}_n^{\mathrm{or}}$ has more structure: it is a symmetric monoidal category. Let us first recall what this exactly is:

Definition 7.9. A **tensor category** is a category $\mathcal{C}$ equipped with

i) a bifunctor $\circledast : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$,

ii) associativity isomorphisms

$$\alpha_{UVW} : (U \circledast V) \circledast W \xrightarrow{\cong} U \circledast (V \circledast W),\ U, V, W \in \mathrm{Ob}(\mathcal{C})$$

natural in U, V and W,

iii) a unit object $\mathbf{1} \in \mathrm{Ob}(\mathcal{C})$ with natural isomorphisms

$$\lambda_V : \mathbf{1} \circledast V \xrightarrow{\cong} V,\ \rho_V : V \circledast \mathbf{1} \xrightarrow{\cong} V,$$

[5] meaning compact and without boundary

such that the so called pentagon diagram

$$\begin{tikzcd}
((V_1 \circledast V_2) \circledast V_3) \circledast V_4 \arrow[rr, "\alpha_{123} \circledast id_4"] \arrow[d, "\alpha_{1\circledast 2,3,4}"'] & & (V_1 \circledast (V_2 \circledast V_3)) \circledast V_4 \arrow[d, "\alpha_{1,2\circledast 3,4}"] \\
(V_1 \circledast V_2) \circledast (V_3 \circledast V_4) \arrow[dr, "\alpha_{1,2,3\circledast 4}"'] & & V_1 \circledast ((V_2 \circledast V_3) \circledast V_4) \arrow[dl, "id_1 \circledast \alpha_{234}"] \\
& V_1 \circledast (V_2 \circledast (V_3 \circledast V_4)) &
\end{tikzcd}$$

commutes for all $V_1, V_2, V_3, V_4 \in \mathrm{Ob}(\mathcal{C})$, as well as the triangle axiom:

$$\begin{tikzcd}
(V_1 \circledast \mathbf{1}) \circledast V_2 \arrow[rr, "\alpha_{1,0,2}"] \arrow[dr, "\rho_1 \circledast id_2"'] & & V_1 \circledast (\mathbf{1} \circledast V_2) \arrow[dl, "id_1 \circledast \lambda_2"] \\
& V_1 \circledast V_2 &
\end{tikzcd}$$

The easiest example of a tensor category is the category of finite dimensional vector space $\mathrm{Vect}_{\mathbb{C}}$, where the monoidal structure is given by the tensor product of vector spaces. Another example is given by $\mathrm{Rep}(G)$, the category of finite dimensional representations of a compact Lie group G, where the monoidal structure is given by taking the tensor product of representations. Notice that in these examples, there is an isomorphism $V_1 \otimes V_2 \cong V_2 \otimes V_1$, for any pair of objects V_1 and V_2. This motivates the following

Definition 7.10. A tensor category is said to be **braided** if it comes equipped with a natural isomorphisms

$$\beta_{VW} : V \circledast W \xrightarrow{\cong} W \circledast V,$$

such that the hexagon diagram

$$\begin{tikzcd}
& V_1 \circledast (V_2 \circledast V_3) \arrow[r, "\beta_{1,2\circledast 3}"] \arrow[dl, "\alpha^{-1}_{1,2,3}"'] & (V_2 \circledast V_3) \circledast V_1 \arrow[dr, "\alpha_{2,3,1}"] & \\
(V_1 \circledast V_2) \circledast V_3 \arrow[dr, "\beta_{1,2} \circledast id_3"'] & & & V_2 \circledast (V_3 \circledast V_1) \\
& (V_2 \circledast V_1) \circledast V_3 \arrow[r, "\alpha_{2,1,3}"] & V_2 \circledast (V_1 \circledast V_3) \arrow[ur, "id_2 \circledast \beta_{1,3}"'] &
\end{tikzcd}$$

commutes for all $V_1, V_2, V_3 \in \mathrm{Ob}(\mathcal{C})$, and so does the same diagram with β replaced by its inverse.

Notice that a braiding is simply a natural isomorphism of functors $\beta : \circledast \to \circledast\tau$, where $\tau : \mathcal{C} \times \mathcal{C} \to \mathcal{C} \times \mathcal{C}$ is the flip, i.e., $\tau(V, W) = (W, V)$. A tensor category with a braiding is called a braided tensor category, or a monoidal category. It is called symmetric if $\beta_{VW} \circ \beta_{WV} = 1_{W \circledast V}$, $\forall V, W \in \mathrm{Ob}(\mathcal{C})$. The name braided category comes from the "universal" example, the category of braids, see e.g. [Ka].

The reader may now check that the operation of disjoint union of manifolds $(\Sigma_1, \Sigma_2) \mapsto \Sigma_1 \coprod \Sigma_2$ defines a monoidal structure on the cobordism category $\mathsf{Bord}_n^{\mathrm{or}}$. The unit is given by the empty set $\emptyset$. With this structure, it becomes a symmetric monoidal category.

7.2. **Axioms for TQFT.** We now have all the mathematical notions to give the definition of a Topological Quantum Field Theory, following Atiyah, who in turn was inspired by Segal's approach to conformal field theory.

Definition 7.11 (Atiyah–Segal)**.** An n-dimensional *Topological Quantum Field Theory* (TQFT) is given by a monodical functor

$$Z : \mathsf{Bord}_n^{\mathrm{or}} \longrightarrow \mathsf{Vect}_{\mathbb{C}}.$$

Let us spell out what this means. Given an n-dimensional TQFT, we get:

- a vector space $Z(\Sigma)$ for each oriented $(n-1)$-dimensional manifold,
- a linear map $Z(M) : Z(\Sigma_0) \to Z(\Sigma_1)$, for each oriented cobordism from Σ_0 to Σ_1.

These assignments should satisfy:

i) equivalent cobordisms $M \cong M'$ give the same map: $Z(M) = Z(M')$.

ii) composition of cobordisms is sent into the composition of maps:

$$Z(M_1 \cup_\Sigma M_2) = Z(M_1) \circ Z(M_2),$$

iii) the unit cobordism is sent to the identity map:

$$Z(\Sigma \times I) = \mathrm{id}_{Z(\Sigma)},$$

iv) disjoint union corresponds to taking tensor products of vector spaces and maps:

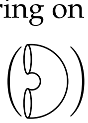
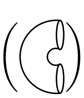
$$Z(\Sigma_1 \coprod \Sigma_2) = Z(\Sigma_1) \otimes Z(\Sigma_2), \quad Z(M_1 \coprod M_2) = Z(M_1) \otimes Z(M_2),$$

v) the empty manifold is sent to the ground field: $Z(\emptyset) = \mathbb{K}$.

Let us now analyse some simple consequences of these axioms.

Proposition 7.12. *For any oriented $(n-1)$-dimensional manifold, there is a canonical isomorphism $Z(\overline{\Sigma}) \cong Z(\Sigma)^\vee$, where $\overline{\Sigma}$ is the manifold Σ with the orientation reversed.*

Proof. We have already considered the identity cobordism $\Sigma \times [0,1]$ from Σ to Σ. If we now change the orientation on outgoing copy $\Sigma \times \{1\}$, we obtain a cobordism from $\Sigma \coprod \overline{\Sigma}$ to $\emptyset$. Similarly, if change the orientation on the ingoing copy we get a cobordism from $\emptyset$ to $\Sigma \coprod \overline{\Sigma}$. Under the functor of a given TQFT, Σ is mapped to a vector space $V := Z(\Sigma)$ and $\overline{\Sigma}$ to $W := Z(\overline{\Sigma})$. Furthermore, these cobordisms are mapped to a pairing and a copairing on V and W:

$\beta := Z\Bigg($  $\Bigg) : V \otimes W \to \mathbb{K}, \quad \gamma := Z\Bigg($  $\Bigg) : \mathbb{K} \to W \otimes V.$

The axioms of a TQFT now imply that the maps

$$V \xrightarrow{id_V \otimes \gamma} V \otimes W \otimes V \xrightarrow{\beta \otimes id_V} V,$$
$$W \xrightarrow{id_W \otimes \gamma} W \otimes V \otimes W \xrightarrow{\beta \otimes id_W} W,$$

are both the identity, since they are the image of the unit cylinder cobordism. This implies that β is in fact a nondegenrate pairing, and identifies $Z(\overline{\Sigma}) \cong Z(\Sigma)^\vee$. □

The proof of this proposition actually shows something more:

Corollary 7.13. *The image of closed oriented $(n-1)$-dimensional manifold under an n-dimensional TQFT is a finite dimensional vector space.*

Proof. The copairing γ sends 1 to an element $\sum_i v_i \otimes w_i$. Now use the previous argument to show that the $v_i \in V$ span V (and $w_i \in W$ span W). □

Proposition 7.14. *The vector space $Z(\Sigma)$ of a TQFT carries a representation of the mapping class group $\Gamma_\Sigma := \pi_0(\mathrm{Diff}^+(\Sigma))$.*

Remark 7.15 (TQFT versus classical algebraic topology)**.** Given any closed oriented n-dimensional manifold M, we can view it as a cobordism from $\emptyset$ to $\emptyset$. Therefore, evaluation on any n-dimensional TQFT Z yields a linear map from $\mathbb{K}$ to $\mathbb{K}$, in other words a number

$$Z(M) \in \mathbb{K},$$

which only depends on the diffeomorphism class of M. In other words, it is a differential topological invariant. Even more: a TQFT gives ways to compute this invariant. Suppose we cut M open along an $(n-1)$-dimensional closed sub manifold Σ, so that $M \cong M_1 \cup_\Sigma M_2$. Then the TQFT produces two vectors

$$Z(M_1) \in Z(\Sigma), \quad Z(M_2) \in Z(\Sigma)^\vee,$$

and the topological invariant of M is given by

$$Z(M) = \langle Z(M_1), Z(M_2) \rangle_{Z(\Sigma)}, \tag{31}$$

using the canonical pairing $\langle\ ,\ \rangle : Z(\Sigma) \times Z(\Sigma)^\vee \to \mathbb{K}$. Equation (31) expresses the *locality* of a quantum field theory, and one should think of this equation as the analogue of the Mayer–Vietoris property for cohomology theories: both are giving a means to compute the invariants of a space from those of smaller pieces. The fundamental difference is that the Mayer–Vietoris sequence is additive, whereas (31) is multiplicative.

7.3. **Examples.** Let us first look at some low dimensional examples:

$n = 1$. In this case, there is only one connected 0-dimensional manifold, namely the point. A TQFT therefore only gives us one vector space $Z(pt.) := V$. Furthermore, any connected one dimensional manifold with boundary is differeomorphic to an interval in $\mathbb{R}$, and all the maps given by the TQFT are just the natural one given by contracting tensors in V and $V^\vee$. In fancy language: *the category of 1-dimensional TQFT's is equivalent to the category of vector spaces.*

$n = 2$. The 2-dimensional case is a favorite example of a TQFT, and there are many sources for this, e.g., [Ko]. A first remark in this case is that up to diffeomorphism, there is only one connected closed 1-manifold, namely the circle S^1. Next we turn to the classification of 2-dimenionsional cobordisms. This is the subject of a classical theorem in differential topology, known as the *classification of surfaces*:

Theorem 7.16 (c.f. [H, §9.3])**.**
i) Two connected closed oriented surfaces are diffeomorphic if and only if they have the same genus, or, equivalently, Euler characteristic.
ii) Two oriented 2-dimensional cobordisms are diffeomorphic if and only if they have the same genus and the same number of incoming and outgoing boundaries.

Recall that the Euler characterstic of a surface of genus g and n boundary components is given by

$$\chi(M_{g,n}) = 2 - 2g - n.$$

The Euler characteristic is most elementary defined using triangulations, and it does not distinguish between incoming and outgoing boundaries. It enjoys the nice property (check using the formula above!) that it is additive under composition of cobordisms:

$$\chi(M_1 \cup_\Sigma M_2) = \chi(M_1) + \chi(M_2).$$

As for the $n = 1$ case, 2-dimensional TQFT's are classified by algebraic objects. In this case the main object is the following:

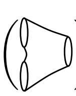
Definition 7.17. A *Frobenius algebra* is a finite dimensional commutative unital $\mathbb{K}$-algebra A equipped with a trace $\tau : A \to$ for which the pairing

$$\langle a_1, a_2 \rangle := \tau(a_1 a_2), \quad a_1, a_2 \in A,$$

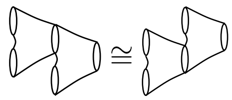
is nondegenerate.

Theorem 7.18. *There is a bijective correspondence between 2d-TQFT's and finite dimensional Frobenius algebras.*

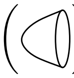
Proof. Let us first prove the implication from left to right. Given a 2-dimensional TQFT Z, define $A := Z(S^1)$. A pair of pants surface induces a linear map

($\star$) $Z\Big($  $\Big) : A \otimes A \to A.$

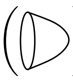
By Theorem 7.16 there exists a diffeomorphism



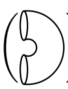
so we can view ($\star$) as an associate product on A. Furthermore, by symmetry on the functor Z, this product is commutative, and the unit is given by

$Z\Big($  $\Big) = 1 : \mathbb{K} \to A$

as one can check with pictures. Dually, the trace is given by

$Z\Big($  $\Big) = \tau : A \to \mathbb{K}.$

To check that the pairing of Definition 7.17 is non degenerate, one observes that, again by the axioms of a TQFT, is given by

$Z\Big($  $\Big) : A \otimes A \to \mathbb{K},$

which, as we have seen in the proof of Proposition 7.12, is non degenerate. this proves that $A := Z(S^1)$ is indeed a Frobenius algebra.

Conversely, given a Frobenius algebra A, let us construct a 2-dimensional TQFT. First notice that the non degenerate pairing in Definition 7.17 yields an isomorphism $A \cong A^\vee$ of left A-modules by $a \mapsto \langle a, - \rangle$. Now the multiplication on A induces a *comultiplication* on $A^\vee$, which we can pullback to A using this isomorphism. With this we get a comultiplication

$$\Delta : A \to A \otimes A,$$

for which the trace $\tau : A \to \mathbb{K}$ is a *counit*. We see that a Frobenius algebra has a canonical coalgebra structure. Let us now construct the *Euler element* in A by

$$\theta := m(\Delta(1)) \in A,$$

where $m : A \otimes A$ is the multiplication. With this element, the TQFT assigns to a cobordism $M_{g,p,q}$ of genus g with p incoming boundaries and q outgoing boundaries, the linear map $Z : A^{\otimes p} \to A^{\otimes q}$ given by

$$Z(M_{g,p,q}) = \Delta^{(q)} \circ \theta^g \circ m^{(p)},$$

with $m^{(p)} : A^{\otimes p} \to A$ the map given by multiplying all p elements, $\Delta^{(q)} : A \to A^{\otimes q}$ by applying q-times the comultiplication, and θ^g stand for multiplication with θ^g. We skip the proof that this indeed defines a TQFT. $\square$

n=3. In dimension 3, things get really interesting.

Finite group TQFT. This is an example that lives in arbitrary dimension, and it can also be viewed as a Toy model for dealing with gauge symmetries. Before we start, recall the definition of a principal bundle. Let X be a smooth manifold.

Definition 7.19. Let F be a manifold. A *fiber bundle* over X with fiber F is a surjective submersion $\pi : E \to X$ which is locally trivial in the following sense: each $x \in X$ has an open neighbourhood U for which there is a diffeomorphism $\phi : \pi^{-1}(U) \to U \times F$, which makes the following diagram commute

$$\begin{tikzcd} \pi^{-1}(U) \arrow[r, "\phi"] \arrow[d, "\pi"'] & U \times F \arrow[dl, "\mathrm{proj}_1"] \\ U & \end{tikzcd}$$

where proj_1 is the projection onto the first factor.

Notice that it follows that $\pi^{-1}(x) \cong F$ for each $x \in X$. Let G be a Lie group.

Definition 7.20. A principal G-bundle is a fiber bundle $p : P \to X$ with a free fiberwise right action of G on P such that $P/G \cong X$.

It follows that G can be taken to be the fiber of the bundle. A morphism of principal bundles is a smooth map $f : P_1 \to P_2$ which commutes with the right G-action. A section s of a principal G-bunde is a smooth map $s : X \to P$ satisfying $p \circ s =$identity.

There is a "cocycle view" on principal bundles over X as follows: by definition, we can find an open covering $\{U_\alpha\}_{\alpha \in I}$ of X such that P has local trivialisations $\phi_\alpha : p^{-1}(U_\alpha) \to U_\alpha \times G$. Two local trivializations (U_α, ϕ_α), and (U_β, ϕ_β) define a smooth map

$$g_{\alpha\beta} : U_\alpha \cap U_\beta \to G,$$

by

$$(\phi_\alpha \circ \phi_\beta^{-1})(x, g) = (x, g g_{\alpha\beta}(x)).$$

These functions are called *transistion functions*. One easily verifies the following conditions satisfied by the transition functions of a principal bundle:

i) for three local trivializations (U_α, h_α), (U_β, h_β) and (U_γ, h_γ),

$$g_{\alpha\beta} g_{\beta\gamma} g_{\gamma\alpha} = 1,$$

on $U_\alpha \cap U_\beta \cap U_\gamma$,

ii) for each local trivialization (U_α, h_α),

$$g_{\alpha\alpha} = 1.$$

We now fix a finite group Γ, and we associate to each manifold M the following category $\mathsf{Bun}_\Gamma(M)$: its objects are principal Γ-bundles P over M, and morphisms are given by morphisms $f : P_1 \to P_2$ of principal Γ-bundles that cover the identity map on M. One easily checks that this is a category with the special property that *every morphism is invertible*: such a category is called a *groupoid*. We shall now construct another groupoid, written $\mathrm{Hom}(\pi_1(M), \Gamma) \rtimes \Gamma$. Its objects are homomorphisms ϕ from $\pi_1(M)$ to Γ. A morphism in $\mathrm{Hom}(\pi_1(M), \Gamma) \rtimes \Gamma$ from ϕ_1 to ϕ_2 is given by an element $\gamma \in \Gamma$ such that

$$\phi_1(u) = \gamma \phi_2(u) \gamma^{-1}, \quad u \in \pi_1(M).$$

Again, one easily checks that this defines a groupoid. The crucial point is now:

Proposition 7.21. *There is a canonical equivalence of categories*

$$\mathsf{Bun}_\Gamma(M) \cong \mathrm{Hom}(\pi_1(M), \Gamma) \rtimes \Gamma.$$

Proof. We shall construct two functors in opposite directions and prove that they are inverses of each other up to natural isomorphisms. In one direction, the functor

$$F : \mathsf{Bun}_\Gamma(M) \to \mathrm{Hom}(\pi_1(M), \Gamma) \rtimes \Gamma,$$

is obtained by taking the holonomy of a connection. In the other direction, the functor

$$G : \mathrm{Hom}(\pi_1(M), \Gamma) \rtimes \Gamma \to \mathsf{Bun}_\Gamma(M),$$

associates to a homomorphism $\phi : \pi_1(M) \to \Gamma$ the principal Γ-bundle given by

$$G(\phi) := \tilde{M} \times \Gamma / \sim, \quad \text{where } (x, \gamma) \sim (\phi(\gamma')x, \gamma\gamma'),\ \gamma' \in \Gamma.$$

Here $\tilde{M}$ is the universal covering of M, which is a principal $\pi_1(M)$-bundle. □

Associated to any groupoid G is its *coarse moduli space* $\mathcal{M}(\mathsf{G})$, the space of objects modulo isomorphisms. The Proposition above then tells us that for the groupoid of principal Γ-bundles, we have

$$\mathcal{M}(M, \Gamma) \cong \mathrm{Hom}(\pi_1(M), \Gamma)/\Gamma.$$

We can now describe the TQFT associated to a finite group. The functor Z assigns to each $(n-1)$- dimensional closed oriented manifold Σ the vector space $Z(\Sigma)$ of functions on $\mathcal{M}(\Sigma, \Gamma)$. Since $\mathcal{M}(\Sigma, \Gamma)$ is just a finite set, we can write this vector space as

$$Z(\Sigma) := \mathbb{C}[\mathcal{M}(\Sigma, \Gamma)].$$

It has a linear basis given by $\delta_{[P]}$, the delta function associated to an isomorphism class of principal Γ-bundles $[P]$:

$$\delta_{[P]}([P']) = \begin{cases} 1 & [P] = [P'] \\ 0 & [P] \neq [P'] \end{cases}$$

Let us now describe the functor Z on morphisms. For any cobordism $M : \Sigma_0 \rightsquigarrow \Sigma_1$, we have a *correspondence of groupoids*

$$\begin{tikzcd} & \mathsf{Bun}_\Gamma(M) \arrow[dl, "r_0"'] \arrow[dr, "r_1"] & \\ \mathsf{Bun}_\Gamma(\Sigma_0) & & \mathsf{Bun}_\Gamma(\Sigma_1) \end{tikzcd}$$

In this diagram, r_0 and r_1 are just the maps given by restricting principal bundles to the boundary components Σ_0 and Σ_1. When we have two cobordisms $M_1 : \Sigma_0 \rightsquigarrow \Sigma_1$ and $M_2 : \Sigma_1 \rightsquigarrow \Sigma_2$, we have a diagram

$$\begin{tikzcd} & & \mathsf{Bun}_\Gamma(M_1 \cup_{\Sigma_1} M_2) \arrow[dl] \arrow[dr] & & \\ & \mathsf{Bun}_\Gamma(M_1) \arrow[dl, "r_0"'] \arrow[dr, "r_1"] & & \mathsf{Bun}_\Gamma(M_2) \arrow[dl, "r_1"'] \arrow[dr, "r_2"] & \\ \mathsf{Bun}_\Gamma(\Sigma_0) & & \mathsf{Bun}_\Gamma(\Sigma_1) & & \mathsf{Bun}_\Gamma(\Sigma_1) \end{tikzcd}$$

All maps in this diagrams are functors, and the essential claim is that the rhombus in the middle is a pull-back:

$$\mathsf{Bun}_\Gamma(M_1 \cup_{\Sigma_1} M_2) \cong \mathsf{Bun}_\Gamma(M_1) \times_{\mathsf{Bun}_\Gamma(\Sigma_1)} \mathsf{Bun}_\Gamma(M_2). \tag{32}$$

(Check this for yourself!) The map $Z(M)$ assigned to a cobordism $M : \Sigma_0 \rightsquigarrow \Sigma_1$ is best described by giving its matrix coefficients: the entry mapping $\delta_{[}P_0] \in Z(\Sigma_0)$ to $\delta_{[P_1]} \in Z(\Sigma_1)$ is given by

$$Z(M)_{\delta_{[P_0]}\delta_{[P_1]}} := \sum_{\substack{[P]\in\mathcal{M}(M,\Gamma) \\ r_0([P])=P_0,\ r_1([P])=[P_1]}} \frac{1}{\#\mathrm{Aut}([P])}.$$

To check that this defines a TQFT, one uses (32). Notice that this TQFT is naturally defined over $\mathbb{Q}$, and doesn't use the orientation. There is a twisted version of of this TQFT, pointed out in the original paper [**?**], that uses as additional data a group cohomology class $\alpha \in H^n(\Gamma, \mathbb{Q})$. This twisted version is a true oriented TQFT.

8. Area-dependent theories

The first and easiest geometric structure we can add to our cobordisms is a volume form. Let M be a compact oriented manifold. As we have seen, a volume form is given in a local chart $(x_1, \ldots, x_n)$ by

$$\Omega = \rho(x) dx_1 \wedge \ldots \wedge dx_n,$$

with $\rho(x) > 0$. Therefore clearly two volume forms Ω_1, Ω_2 must be related by $\Omega_1 = f\Omega_2$ with f a positive real function. If $\phi : M \to M$ is an orientation preserving diffeomorphism, we can pull back the volume form Ω to obtain a new one $\phi^*\Omega$. Evidently we have that

$$\mathrm{Vol}(M,\Omega) := \int_M \Omega = \int_M \phi^*\Omega =: \mathrm{Vol}(M,\phi^*\Omega).$$

The following well-known theorem states the converse:

Theorem 8.1 (Moser, c.f. [Mo])**.** *Let Ω_1 and Ω_2 be two volume forms on a compact oriented manifold. There exists an oriented diffeomorphism $\phi : M \to M$ such that $\phi^*\Omega_1 = \lambda\Omega_2$ with positive constant λ equal to*

$$\lambda = \int_M \Omega_1 \Big/ \int_M \Omega_2.$$

Let us now consider the category $\mathsf{Bord}_n^{\mathrm{vol}}$ with the same objects as $\mathsf{Bord}_n^{\mathrm{or}}$, namely oriented closed manifolds of dimension $(n-1)$, but arrows given by diffeomorphism (relative to the boundary) classes of cobordisms *equipped with a volume form*. One easily checks that the proof of Theorem 7.3 is compatible with volume forms so that composition of arrows is well-defined. In fact we have that

$$\mathrm{Vol}(M\cup_f N, \Omega_M+\Omega_N) = \mathrm{Vol}(M,\omega_M) + \mathrm{Vol}(N,\Omega_N). \tag{33}$$

The only points is that this category is not a real category since it does not have unit arrows: we have to equip the cylinder $\Sigma\times I$ over any manifold Σ with a volume form and because of the additivity of volume in equation (33) we can't have a unit. Nevertheless, we can just as well do without units.

Before we give the definition, we have to think a bit about the target category. By contrast to TQFT's, it will be no longer the case that the quantum field theory we are aiming to define lives on a finite dimensional vector space. Therefore, it is natural to assume some topology on the vector space. Recall that a semi norm on a vector space V is a nonnegative function $p : V \to \mathbb{R}_+$ satisfying

- $p(\lambda v) = |\lambda| p(v)$, for all $\lambda\in\mathbb{R}$, $v\in V$,
- $p(v_1+v_2) \leq p(v_1)+p(v_2)$ for all $v_1, v_2 \in V$.

We see that if such a p has the property that $p(v) = 0 \Rightarrow v = 0$, it defines a norm. Suppose now that we have a family of semi norms $\{p_\alpha\}_{\alpha\in J}$ on V. These define a topology on V for which the basic vector space operations are continuous. A basis for this topology is given by the open subset $U_{J,\epsilon}(v)$, $\epsilon > 0$, $v\in V$ defined as

$$U_{J,\epsilon}(v) := \{w\in V,\ p_\alpha(v-w) < \epsilon,\ \text{for all } \alpha\in J\}.$$

A locally convex topological vector space is a vector space equipped with family of semi norms. Its dual $V^\vee$ consists of all continuous linear maps $V\to\mathbb{C}$.

Definition 8.2. A continuous linear map $T : V \to W$ between two locally convex topological vector spaces is called *trace class* (or *nuclear*) if it can be written as

$$T(v) = \sum_k \lambda_k \langle \alpha_k, v\rangle\, w_k,$$

where $\lambda_k \in \mathbb{C}$ form an absolutely summable sequence $\sum_k |\lambda_k| < \infty$, $\{\alpha_k\}$ is an equicontinuous sequence in $V^\vee$ and $\{w_k\}$ a sequence in W contained in a bounded subset.

Together with the linear maps that are continuous for the topology described above, they form a category $\mathsf{TVS}_\mathbb{C}$. A typical example of a locally convex topological vector spaces is given by $C^\infty(\mathbb{R}^n)$. In this case the family of semi norms is given by

$$p_{K,r}(f) := \sup_{x \in K} \left\{ \frac{\partial f}{\partial x^\alpha}, \; |\alpha| < r \right\},$$

for $K \subset \mathbb{R}^n$ compact and $r \in \mathbb{N}$.

Definition 8.3. An *Area-dependent quantum field theory* is a monoidal functor

$$Z : \mathsf{Bord}_n^{\mathrm{vol}} \to \mathsf{TVS}_\mathbb{C}.$$

We shall see that such theories, although not topological, resemble TQFT's in a number of ways. Let us first spell out the definition. An Area-dependent QFT will associate

- a locally convex topological vector space $Z(\Sigma)$ to each oriented closed $(n-1)$-dimensional manifold,
- A linear operator

$$Z(M,t) : Z(\partial M_{\mathrm{in}}) \to Z(\partial M_{\mathrm{out}}),$$

  associated to an oriented cobordism and a real positive number $t > 0$.

Again these should satisfy compatibility rules encoding the properties of a monoidal functor, the most important of which is given by the composition

$$Z(M_1 \cup_{\Sigma_1} M_2, t_1 + t_2) = Z(M_1, t_1) \circ Z(M_2, t_2). \tag{34}$$

Let us now explore some consequences of the axioms, analogous to our exploration of TQFT's.

Proposition 8.4. *The vector space $Z(\Sigma)$ associated to any closed oriented $(n-1)$-dimensional manifold comes equipped with the action of a one-parameter semigroup of trace-class operators.*

Proof. For any closed oriented $(n-1)$-dimensional manifold Σ define

$$S(t) := Z(I \times \Sigma, t).$$

Then the property (34) implies that this defines a one-parameter semigroup of operators. The axioms also imply that

$$Z(S^1 \times \Sigma, t) = \mathrm{Tr}\,(Z(I \times \Sigma, t)) \in \mathbb{C},$$

so that each $S(t),\ t > 0$ is trace class. This proves the proposition. $\square$

Although $\mathsf{Bord}_n^{\mathrm{vol}}$ does not have unit morphisms, we would like our functor to respect them if we add them formally. Therefore we require that:

$$S(t) \to 1, \quad \text{as } t \to 0, \text{ uniformly on compact subsets of } Z(\Sigma).$$

With the semigroup $\{S(t),\ t > 0\}$, let us now define the subspace $\check{Z}(\Sigma) \subset Z(\Sigma)$ by

$$\check{Z}(\Sigma) := \bigcup_{t>0} S(t)(Z(\Sigma)).$$

On the other hand, let us embed $Z(\Sigma)$ into a larger space $\hat{Z}(\Sigma)$ defined as

$$\hat{Z}(\Sigma) = \left\{ \{u_t\} \in \prod_{t>0} Z(\Sigma)_t,\ u_{t_1+t_2} = S(t_1)u_{t_2} \right\}$$

The embedding $Z(\Sigma) \hookrightarrow \hat{Z}(\Sigma)$ is given by mapping $u_0 \in Z(\Sigma)$ to the family $\{S(t)u_0\}_{t>0}$. We can give both $\check{Z}(\Sigma)$ and $\hat{Z}(\Sigma)$ natural topologies turning them into complete locally convex topological vector spaces, and we see that we have a chain of inclusions

$$\check{Z}(\Sigma) \subset Z(\Sigma) \subset \hat{Z}(\Sigma),$$

called a "rigging" of $Z(\Sigma)$. The following is analogous to Proposition 7.12 for TQFT's:

Proposition 8.5. *For an Area-dependent QFT, $\check{Z}(\Sigma)$ and $\hat{Z}(\bar{\Sigma})$ are in natural duality by a bilinear form*

$$\beta : \check{Z}(\Sigma) \otimes \hat{Z}(\bar{\Sigma}) \to \mathbb{C}.$$

Proof. As in the proof of Proposition 7.12, we consider the identity cobordism $\Sigma \times I$, but now viewed as a bordism from $\Sigma \coprod \bar{\Sigma}$ to $\emptyset$. Let $u := S(t)u_0 \in \check{Z}(\bar{\Sigma})$ and $v := \{v_t\}_{t>0} \in \hat{Z}(\Sigma)$. Then we define

$$\beta(u, v) := Z(I \times \Sigma, t - s)(u_0, v_s), \quad s > t.$$

It is easy to check that, using the axioms of an area-dependent QFT, this pairing is independent of the choices of s and t subject to the condition $s > t$. □

Definition 8.6. An Area-dependent QFT is said to be *unitary* if $Z(\bar{\Sigma}) = \overline{Z(\Sigma)}$.

The name is justified by the following:

Corollary 8.7. *For a unitary Area-dependent QFT the vector space $\check{Z}(\Sigma)$ carries a canonical pre-Hilbert space structure.*

Proof. □

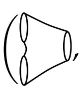
Examples. Let us first consider the $n = 1$ case, and we restrict to unitary theories. In this case, the QFT is determined by specifying a Hilbert space $\mathcal{H}$ for the oriented point $(pt., +)$. By Proposition 8.4, it comes equipped with an action of a one-parameter semigroup of trace class operators. Other than that, we easily see, as in §7.3 for TQFT's, that all other structure maps are defined using the semigroup and the Hilbert space structure. In other words: a 1-dimensional Area-dependent unitary QFT is nothing but a one-parameter semigroup acting on a Hilbert space by trace class operators. We have already seen that examples come from Quantum Mechanics, c.f. §5.4 ignoring the observables.

Next, we consider the $n = 2$ case. Recall that for a TQFT, a 2-dimensional theory is the same as giving a frobenius algebra. The following is the analogue of that statement:

Theorem 8.8 (c.f. [Se])**.** *A 2-dimensional Area-dependent QFT is the same as a commutative topological algebra A equipped with a non-degenerate trace $\theta : A \to \mathbb{C}$ and an approximate trace class unit, i.e., a family $\{\epsilon_t\}_{t>0}$ of elements in A with*

- $\lim_{t\to 0} \epsilon_t \cdot a = a = \lim_{t\to 0} a \cdot \epsilon_t$, *for all* $a \in A$,
- $\epsilon_{t_1} \cdot \epsilon_{t_2} = \epsilon_{t_1+t_2}$ *for all* $t_1, t_2 > 0$,
- *multiplication by ϵ_t is a trace class operator on A.*

Proof. Given a 2-dimensional Area-dependent QFT, we put $A := \check{Z}(S^1)$. Multiplication on A is defined by

$$(S_{t_1}u) \cdot (S_{t_2}v) = Z\Bigg($$

$$, t_1 + t_2\Bigg)(u \otimes v).$$

Again, one easily see that this is independent of the choice of $t_1, t_2 > 0$. □

8.1. **2D-Yang–Mills Theory.** The main example of a $2D$ Area-dependent quantum field theory is given by $2D$-Yang–Mills theory, as explained in a paper by Witten [**?**]. By Theorem 8.8, it suffices to define a commutative algebra with a non degenerate trace and an approximate unit. The data defining the theory are a compact Lie group G and a G-invariant metric on its Lie algebra $\mathfrak{g}$. This last ingredient is simply a real inner product $\langle\ ,\ \rangle$ on $\mathfrak{g}$ that is invariant under the adjoint action of G:

$$\left\langle \mathrm{Ad}_g(X), \mathrm{Ad}_g Y \right\rangle = \langle X, Y \rangle, \quad X, Y \in \mathfrak{g},\ g \in G.$$

Such a metric always exists and in fact is unique up to a scaling. For example, for $G = SU(N)$ one can take the trace in the standard representation:

$$\langle X, Y \rangle := \mathrm{Tr}_{\mathbb{C}^N}(XY), \quad X, Y \in \mathfrak{su}(N).$$

The *convolution algebra* of G is given by $C^\infty(G)$ equipped with the convolution product

$$(f_1 * f_2)(g) := \int_G f_1(h) f_2(gh^{-1}) d\mu(h), \tag{35}$$

where μ denotes the Haar measure, normalized to have total integral 1. One checks that this defines an associative algebra structure on $C^\infty(G)$, which is noncommutative if and only if G is nonabelian. Its *center* consists of the class functions on G, i.e., $f \in C^\infty(G)$ which satisfy

$$f(hgh^{-1}) = f(g), \quad \text{for all } g, h \in G.$$

We will write $\mathcal{A}$ for this algebra, i.e., that of class functions equipped with the convolution product. Typical examples of class functions come from characters of representations: if (V, π) is a finite dimensional representation of G, its character

$$\chi_V(g) := \mathrm{Tr}_V(\pi(g)), \quad g \in G,$$

is invariant under conjugation because of the trace property.

Theorem 8.9 (Peter–Weyl)**.** *The characters of irreducible representations of a compact Lie group G span a dense subset of its space of class functions.*

To have an area-dependent QFT, we need two more ingredients: a trace and an approximate unit. For the trace we shall take the functional

$$\tau(f) := f(1), \quad f \in \mathcal{A}.$$

Indeed one easily checks that this is indeed a trace with respect to the convolution product (35). The convolution algebra is nonunital because the unit would be a delta function at the unit element $e \in G$, which is not in $C^\infty(G)$. So, for an approximate unit, we need a family of elements approximating the delta function. A natural choice for this is to use the heat kernel. First remark that by left (or right) translation of the metric on $\mathfrak{g}$, we get a natural metric on G which is invariant under the right and left action of G on itself. In this way we can view G as a Riemannian manifold, and therefore the heat kernel $K_t^G(g,h)$ is well-defined. the fact that the metric is G-invariant translates into the property that

$$K_t^G(gl, hl) = K_t^G(g,h), \quad K_t^G(lg, lh) = K_t^G(g,h), \quad g, h, l \in G. \tag{36}$$

We then define $\epsilon_t(g) := K_t^G(g, e)$. This is the fundamental solution to the heat equation

$$\frac{\partial \epsilon_t}{\partial t} = \Delta \epsilon_t.$$

Let us now show that this is indeed an approximate unit in the algebra $\mathcal{A}$:

$$\begin{aligned}
\lim_{t\to 0}(\epsilon_t * f)(g) &= \lim_{t\to 0}\int_G K_t^G(h,e) f(gh^{-1}) d\mu(h) \\
&= \lim_{t\to 0}\int_G K_G^t(h^{-1}g,e) f(h) d\mu(h) && \text{(invariance of the Haar measure)} \\
&= \lim_{t\to 0}\int_G K_G^t(g,h) f(h) d\mu(h) && \text{(invariance of the heat kernel, eq. (36))} \\
&= f(g) && \text{(c.f., equation (22)).}
\end{aligned}$$

It is now not difficult to show that all the conditions of Theorem 8.8 are satisfied, i.e., that the data $(\mathcal{A}, \epsilon_t, \tau)$ define an area-dependent QFT.

Proposition 8.10. *The partition function $Z_t(M_g)$ of a closed surface of genus g is given by*

$$Z_t(M_g) = \sum_{[V]\in\hat{G}} \frac{e^{-t\lambda_V}}{(\dim V)^{2g-2}},$$

where λ_V is the eigenvalue of the Casimir operator on V.

Proof. Let us first recall the definition of the Casimir operator: choosing an orthonormal basis $X_i \in \mathfrak{g}$, the Casimir is defined as

$$C := \sum_i X_i \otimes X_i \in \mathcal{U}\mathfrak{g},$$

where $\mathcal{U}\mathfrak{g}$ denotes the universal enveloping algebra. The Casimir acts on any finite dimensional representation of V, and it commutes with the action of G. By Schur's Lemma it must therefore act by a scalar in any irreducible representation.

By left translation, the Casimir C defines a second order differential operator on G: this is exactly the Laplacian Δ! But then it follows that

$$(\Delta\chi_V)(g) = \mathrm{Tr}_V(Cg) = \lambda_V\chi_V(g),$$

and therefore, by equation (23), the heat kernel is given by

$$K_t^G(g,h) = \sum_{[V]\in\hat{G}} e^{-t\lambda_V}\chi_V(g)\otimes\bar{\chi}_V(h).$$

The key to the proof of the proposition is to find the Euler element θ_t associated to a torus with one hole and area $t > 0$. It is best to work in the basis given by characters χ_V, $[V] \in \hat{G}$. The Schur orthogonality relations state that the product in this basis is given by

$$\chi_V * \chi_W = \begin{cases} 0 & [V] \neq [W] \\ (\dim V)^{-1}\chi_V & [V] = [W]. \end{cases}$$

Furthermore, obviously $\tau(\chi_V) = \dim V$, so the characters χ_V form an orthonormal basis for the inner product given in Definition 7.17. With the previous formula for the heat kernel we therefore find

$$\theta_t(g) = \sum_{[v]\in\hat{G}} \frac{e^{-t\lambda_V}}{\dim(V)^2}\chi_V(g).$$

With this element we find that

$$Z_t(M_g) = \tau(\theta_{t_1} * \ldots * \theta_{t_g}), \quad t_1 + \ldots + t_g = t.$$

Combining the previous two formulas, we get the desired result. ∎

9. 2D-Conformal Field Theory

We now stick to the two-dimensional case. Then there is another interesting geometric structure that we can give to our bordisms: a *complex structure*. The quantum field theory resulting from this is called *conformal field theory* (CFT). Historically, this was the first quantum field theory axiomatized as a "representation" of a bordism category, and the idea is due to Graeme Segal [**?**]. It should be remarked however that this way of formulating CFT is quite different from the physicists way of doing it: they usually start by considering the vertex algebra of quantum fields.

9.1. **Segal's definition.** So we are lead to consider the category $\mathsf{Bord}_2^{\mathrm{conf}}$ which has objects equal to compact oriented 1 manifolds C (i.e., disjoint unions of circles) and morphisms from given by triples $(\varphi_1, \Sigma, \varphi_2)$, where Σ is a Riemann surface with boundary $\partial\Sigma = \varphi_1(C_1) \coprod \overline{\varphi_2(C_2)}$. We consider two such triples $(\varphi_1, \Sigma, \varphi_2)$ and $(\varphi_1', \Sigma', \varphi_2')$ as the same if there exists a boholomorphic map between the two commuting with the boundary parametrizations. We have to show that composition of morphisms is well-defined in this category. By the usual argument we can restrict to the (full) "subcategory" of $\mathsf{Bord}_2^{\mathrm{conf}}$ whose objects are disjoint unions $\coprod S^1$ of the standard circle embedded as $\{z \in \mathbb{C},\ |z| = 1\}$ in the complex plane. A morphisms between two such objects is then a Riemann surface Σ with parameterized boundaries, where the parameterization is given by maps $f_i : S^1 \to \Sigma$. The crucial statement is then:

Proposition 9.1. *Let Σ_1 and Σ_2 be two Riemann surfaces with boundary parameterized by $f_i : S^1 \to \Sigma_i$, $i = 1, 2$. There exists a unique conformal structure on the 2D-manifold $\Sigma_1 \cup_{f_1 \circ f_2^{-1}} \Sigma_2$ obtained by gluing Σ_1 to Σ_2.*

Proof. Suppose first that the boundary parameterizations of Σ_1 and Σ_2 are *analytic*, meaning that they extend to holomorphic maps

$$f_1 : A_{q_1} \to \Sigma_1, \quad f_2 : A_{1/q_2} \to \Sigma_2,$$

for some $q_1, q_2 \in (0, 1)$, where, for $q \in (0, 1)$ we have introduced the notation

$$A_q := \{z \in \mathbb{C},\ q < |z| < 1\}, \quad A_{1/q} := \{z \in \mathbb{C},\ 1 < |z| < 1/q\}$$

The fact that one extends to an annular region inside the unit disk and the other outside, has to do with the fact that one boundary is incoming and one outgoing, i.e., the orientation. A_q is biholomorphic to $A_{1/q}$ by means of the mapping $z \mapsto 1/z$, but this pas reverses the orientation. Looking at the proof of Theorem 7.3, we can indeed prove, using these holomorphic extensions as collars as in Lemma 7.4, that there exists a conformal atlas inducing the desired conformal structure.

The general case follows from the following classical theorem:

Theorem 9.2 ("Conformal welding")**.** *Given $f \in \mathrm{Diff}(S^1)$, there exist holomorphic functions g_+ and g_-, defined on $D_+ := \{z \in \mathbb{C},\ |z| \leq 1$ and $D_- := \{z \in \mathbb{C}, |z| \geq 1\} \cup \{\infty\}$ mapping onto complementary Jordan domains in $\mathbb{C}$ such that $g_-|_{S^1} = f \circ g_+|_{S^1}$.*

□

Remark 9.3 (Riemann surfaces and metrics)**.** It is good to remark that conformal structures and Riemannian metrics are closely related. More precisely, a conformal structure is determined by a *conformal equivalence class of metrics*:

Theorem 9.4. *Let (Σ, g) be a two-dimensional compact orientable riemannian manifold. Then there exists a unique conformal structure turning Σ into a Riemann surface, such that in a local holomorphic coordinate z, the metric takes the form*

$$g = \lambda(z, \bar{z}))dz \otimes d\bar{z}, \quad \lambda(z, \bar{z}) > 0. \tag{37}$$

Two metrics g and g' induce the same conformal structure if and only if they differ by a conformal factor, i.e., $g = e^f g'$ for some $f \in C^\infty(\Sigma, \mathbb{R})$. Conversely, any compact Riemann surface admits a metric which takes the form as above in a local holomorphic coordinate.

In such a metric, the Laplacian takes the form

$$\Delta_g = \frac{4}{\lambda^2} \frac{\partial}{\partial z} \frac{\partial}{\partial \bar{z}}.$$

The definition of CFT. We are now ready to state Segal's definition of a conformal field theory:

Definition 9.5. A conformal field theory is given by a projective monoidal functor

$$T : \mathsf{Bord}_2^{\mathrm{conf}} \to TVS_{\mathbb{C}}.$$

This definition suffers from the defect, by now familiar that both categories, domain and target of the functor, do not have unit morphisms, i.e., are not really categories. Therefore we spell out the definition: a CFT associates

- a Topological vector space $\mathcal{H}_S$ to a closed oriented smooth $1D$-manifold S,
- a ray of trace class operators

$$U_\Sigma : \mathcal{H}_{\partial\Sigma_{\mathrm{in}}} \to \mathcal{H}_{\partial\Sigma_{\mathrm{out}}},$$

subject to the usual axioms that we will not spell out. The fact that the functor is merely projective means that U_Σ is only defined up to a scalar multiple. This ambiguity is what physicists call the *conformal anomaly*. Recall that a trace class operator $T : \mathcal{H} \to \mathcal{H}$ can be written as

$$T = \sum_i \alpha_i \lambda_i e_i,$$

for $\{e_i\}$ a sequence of vectors in $\mathcal{H}$, $\{\alpha_i\}$ in $\mathcal{H}^\vee$ and $\{\lambda_i\}$ a sequence in $\mathbb{C}$ that is absolutely summable: $\sum_i |\lambda_i| < \infty$. Composition is given in the obvious way in this representation, and we see that if T_1 and T_2 are only defined up to scalar, the composition determines a unique ray of trace class operators.

Let $\mathcal{H}$ be the vector space associated to a circle S^1. As for an area dependent QFT, this space carries a (projective) representation of the semigroup $\mathcal{S}$ formed by all surfaces with parameterized boundaries that are topologically annuli. Let us analyze this semigroup a bit further. We have already seen the annuli A_q for $q \in (0,1)$. We can embed these in $\mathcal{S}$ by using the "standard parametrizations" of the boundaries. Even more, denote by A_q, $q \in \mathbb{C}_{<1}$ the annulus $_{|q|}$ equipped with the identity parameterization of the boundary $|z| = 1$ and the parameterization $\theta \mapsto qe^{i\theta}$ on the inner boundary. Gluing such surfaces leads to

$$A_{q_1} A_{q_2} = A_{q_1, q_2},$$

so $\mathbb{C}_{<1}$ embeds as a subsemigroup. Acting with this subsemigroup, we can define a rigging

$$\check{\mathcal{H}} \subset \mathcal{H} \subset \hat{\mathcal{H}},$$

as in section 8.8. Again, we call a CFT *unitary* if $\mathcal{H}_{\overline{S}} = \overline{\mathcal{H}}_S$.

Proposition 9.6. *The* partition function *of a CFT, defined as*

$$Z(\tau) := \mathrm{Tr}_{\mathcal{H}}(T_{e^{2\pi i z}}),$$

is modular, i.e., projectively invariant under the action of $PSL(2, \mathbb{Z})$.

Proof. Taking the trace of T_q corresponds to gluing the two boundaries of the annulus A_q together. This results in a complex torus with modular parameter $q = e^{2\pi i \tau}$. The projective linear action

$$\tau \mapsto \frac{a\tau + b}{c\tau + d}, \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in PSL(2, \mathbb{Z}),$$

produces an isomorphic complex torus. □

Example: Free bosonic CFT. Physiscists would introduce this example of a CFT by writing down the Lagrangian

$$S(\varphi) := \frac{1}{2} \int_\Sigma d\varphi \wedge * d\varphi,$$

for a scalar valued field φ on a Riemann surface Σ.

9.2. **The Heisenberg group.** Let V be a real vector space. we do not want to assume that V is finite dimensional, but when it is infinite dimensional it should have the structure of a locally convex topological vector space. We denote by $V^\vee$ its topological dual consisting of all *continuous* linear maps $V \to \mathbb{R}$.

Definition 9.7. An antisymmetric bilinear form $\omega : V \times V \to \mathbb{R}$ is called *weakly symplectic* if the induced map

$$\omega^\# : V \to V^\vee, \quad v \mapsto \iota_v \omega,$$

is injective. It is *strongly symplectic* when it is an isomorphism.

Remark 9.8. When V is finite dimensional, weakly and strongly symplectic forms coincide, and are just ordinary symplectic forms. For an infinite dimensional example, consider the vector space $V' = C^\infty(S^1)$, equipped with the antisymmetric pairing

$$\omega(f,g) := \int_{S^1} f dg, \quad f,g \in C^\infty(S^1). \tag{38}$$

The associated linear map $\omega^\# : V \to V^\vee$ has kernel given by

$$\omega^\#(f) = 0 \Longleftrightarrow \int_{S^1} g df = 0, \ \forall g \in C^\infty(S^1),$$

and therefore $df = 0$, i.e., f must be a constant. It follows that the quotient $V'/\mathbb{R}$ is weakly symplectic. Notice that it is definitely not strongly symplectic: the topological dual of $C^\infty(S^1)$ is the space of distributions on S^1.

From now on we shall always assume our symplectic form to be weak, and we drop the adjective "weak", referring to the pair (V,ω) as a symplectic vector space.

Definition 9.9. Let (V,ω) be a symplectic vector space. The *Heisenberg group* associated to (V,ω) is given as a set by $\tilde{V} := V \times \mathbb{T}$ with multiplication given by

$$(v_1,z_1)\cdot(v_2,z_2) := (v_1+v_2, z_1 z_2 e^{-i\omega(v_1,v_2)}).$$

Recall that a complex structure on V is an endomorphism $J : V \to V$ satisfying $J^2 = -1$.

Definition 9.10. A complex structure J on V is

1 *compatible* with ω is $\omega(Jv_1, Jv_2) = \omega(v_1,v_2)$ for all $v_1,v_2 \in V$,
2 *positive* if $\omega(Jv,v) > 0$, for all $v \in V$.

On the other hand, consider a subspace $W \subset V_\mathbb{C}$, where $V_\mathbb{C} := V \otimes \mathbb{C}$ is the complexification of V. We extend the symplectic form ω to a *complex* bilinear form on $V_\mathbb{C}$, denoted by $\omega_\mathbb{C}$.

Definition 9.11. A subspace $W \subset V_\mathbb{C}$ is called

- *isotropic* if $W \subset W^{\omega_\mathbb{C}}$,
- *maximal isotropic* (or *Lagrangian*) if $W = W^{\omega_\mathbb{C}}$,

- *positive* if the hermitian form $\langle\ ,\ \rangle$ on W defined by

$$\langle w_1, w_2\rangle := 2i\omega(\bar{w}_1, w_2) \tag{39}$$

  is positive definite.

Lemma 9.12. *There is a bijective correspondence between compatible positive complex structures on V and positive maximal isotropic subspaces $W \subset V_{\mathbb{C}}$.*

Proof. Given J, we get a decomposition $V_{\mathbb{C}} = W = W \oplus \overline{W}$, where W and $\overline{W}$ are the $+i$ and $-i$ eigenspaces of the complex linear extension of J. It is easy to check that W is maximal isotropic if and only if J is positive and compatible with ω. Conversely, given a positive, maximal isotropic subspace $W \subset V_{\mathbb{C}}$, remark that

$$W \cap \overline{W} = \{0\},$$

because any element in this intersection must have norm zero by (39). Therefore $W \oplus \overline{W} \subset V_{\mathbb{C}}$ is a symplectic subspace. But because W is maximal, we see that we must in fact have that $V_{\mathbb{C}} = W \oplus \overline{W}$. □

We now fix a choice of such a complex structure J (or equivalently, W), and use this to construct an representation of the Heisenberg group $\tilde{V}$. Since J is compatible with ω and positive, we consider the associated subspace W as a pre-Hilbert space with inner product (39). We consider the symmetric algebra $\mathrm{Sym}(W)$ of W, and extend the hermitian form by

$$\langle w_1 \cdot w_n, w_1' \cdot w_n'\rangle = \sum_{\sigma \in S_n} \left\langle w_1, w'_{\sigma(1)}\right\rangle \cdots \left\langle w_n, w'_{\sigma(n)}\right\rangle .$$

This defines an inner-product on $\mathrm{Sym}(W)$ for which the degree k and l parts are orthogonal if $k \neq l$. We denote by $\mathcal{H}_W$ the Hilbert space completion of $\mathrm{Sym}(W)$ in this inner product.

On $\mathcal{H}_W$ we have the "raising" and "lowering" operators that are used so often in physics: for $w \in W$, we define $a(w) : \mathrm{Sym}^k(W) \to \mathrm{Sym}^{k+1}(W)$ as multiplication by w. For $\bar{w} \in \overline{W}$, $a(\bar{w}) : \mathrm{Sym}^k(W) \to \mathrm{Sym}^{k-1}(W)$ is the derivation of $\mathrm{Sym}(W)$ fixed by its action on W as

$$a(\bar{w}_1)w_2 := \langle w_1, w_2\rangle$$

For a general element $v \in V$, we can uniquely write $v = v_+ + v_-$, with $v_+ \in W$ and $v_- \in \overline{W}$, and with this we define

$$a(v) := a(v_+) + a(v_-),$$

as an unbounded operator on $\mathcal{H}_W$ with domain given by $\mathrm{Sym}(W)$. On this domain, we clearly have the commutation relations

$$[a(v_1), a(v_2)] = 2i\omega(v_1, v_2). \tag{40}$$

Lemma 9.13. *For $w \in W$ and $\xi \in \mathrm{Sym}^k(W)$, we have the inequalities*

$$\|a(w)\xi\| \leq \sqrt{k+1}\,\|w\|\,\|\xi\|\,,$$
$$\|a(\bar{w})\xi\| \leq \sqrt{k}\,\|w\|\,\|\xi\|\,.$$

Therefore, for $v \in V$ we have

$$\|a(v)\xi\| \leq 2\sqrt{k+1}\,\|v\|\,\|\xi\|\,.$$

Proof. Write out the left hand side and use the Cauchy–Schwarz inequality. □

It follows from this Lemma that the series

$$\sum_{n=0}^{\infty} \frac{(ia(v))^n \xi}{n!}$$

converges in norm for all $\xi \in \mathrm{Sym}(W)$, and defines an operator $e^{ia(v)} : \mathrm{Sym}(W) \to \mathcal{H}_W$. We also have that

$$\begin{aligned} \left\langle e^{ia(v)}\xi_1, e^{ia(v)}\xi_2 \right\rangle &= \sum_{k,l=0}^{\infty} \left\langle \xi_1, \frac{(ia(v))^k}{k!} \frac{(ia(v))^l}{l!} \xi_2 \right\rangle \\ &= \sum_{n=0}^{\infty} \sum_{k+l=n} \left\langle \xi_1, \frac{(ia(v))^n}{k!l!} \xi_2 \right\rangle \\ &= \langle \xi_1, \xi_2 \rangle\,, \end{aligned}$$

which shows that the operator $e^{ia(v)}$ extends to a unitary operator on $\mathcal{H}_W$. By the Baker–Campbell–Hausdorff formula we have that

$$e^P e^Q = e^{[P,Q]/2} e^{P+Q},$$

whenever $[P, Q]$ commutes with P and Q. Therefore, by (40) we find

$$e^{ia(f)} e^{ia(g)} = e^{-i\omega(f,g)} e^{ia(f+g)},$$

so we have constructed a representation of the Heisenberg group of Definition 9.9!

Proposition 9.14. *The representation of the Heisenberg group $\tilde{V}$ on $\mathcal{H}_W$ is irreducible.*

Proof. Let $\Omega \in \mathcal{H}_W$ be the unique vector $1 \in \mathrm{Sym}^0(W)$ of lowest degree, called the "vacuum". We see from the description of $\mathcal{H}_W$ as the completion of the symmetric algebra of W that $W \cdot \Omega$ spans a dense subset of $\mathcal{H}_W$. Let $\mathcal{H}_0$ be a closed subspace of $\mathcal{H}_W$ stable under the action of $\tilde{V}$. Then it is clear that $\mathcal{H}$ must contain the vacuum Ω, because we can act with $\tilde{W}$ to lower degree till it becomes zero. But then, by acting with W, it is also clear that $\mathcal{H}_0$ must in fact coincide with $\mathcal{H}_W$. This proves the Proposition. □

Remark 9.15. In this construction of the representation of the Heisenberg group it was essential that the operator $e^{ia(v)}$ was norm-preserving for $v \in V$. If we take $v \in V_{\mathbb{C}}$, this is no longer the case, but we see by construction that in this case $e^{ia(v)} : \mathrm{Sym}(W) \to \mathcal{H}_W$ defines an unbounded operator satisfying

$$\left\langle \xi_1, e^{ia(v)}\xi_2 \right\rangle = \left\langle e^{-ia(\bar{v})}\xi_1, \xi_2 \right\rangle, \quad \text{for all } \xi_1, \xi_2 \in \mathrm{Sym}(W). \tag{41}$$

This extension of the representation of $\tilde{V}$ to its complexification will be used below.

Example 9.16. Let us return to our main example of an infinite dimensional symplectic manifold: $V := C^\infty(S^1, \mathbb{R})/\mathbb{R}$ equipped with the symplectic form (38). Clearly $V \otimes \mathbb{C} = C^\infty(S^1, \mathbb{C})/\mathbb{C}$, and there is a canonical decomposition $V \otimes \mathbb{C} = W_+ \oplus W_-$, where W_+ is spanned by the positive Fourier modes $e^{in\theta}$, $n > 0$ and W_- by the negative ones. Let us check that W_+ is isotropic:

$$\begin{aligned} \omega_{\mathbb{C}}(f, g) &= \int_{S^1} f dg \\ &= \int_D df \wedge dg = 0 \end{aligned}$$

because W_+ consists of functions f that extend holomorphically to the bounding unit disk $D := \{z \in \mathbb{C},\ |z| \leq 1\}$, and therefore $df = \partial f/\partial z dz$.

Since $\overline{W}_+ = W_-$, we have $V \otimes \mathbb{C} = W_+ \oplus \overline{W}_+$ and therefore W_+ must be maximal isotropic. Positivity amounts to the inequality

$$2i\omega(\bar{f}, f) = 2i \int_D \frac{\partial \bar{f}}{\partial \bar{z}} \frac{\partial f}{\partial z} d\bar{z} \wedge dz = 4 \int_D \bar{f}' f' dx \wedge dy \geq 0,$$

with equality if and only if $f' = 0$. Therefore W_+ satisfies all the criteria to define an irreducible representation of the Heisenberg group $\tilde{V}$ on the Hilbert space completion of $\mathrm{Sym}(W_+)$.

The main question now is how many different irreducible representations we get by this construction. The following is a classical theorem giving a complete answer:

Theorem 9.17 (Shale)**.** *The representations of $\tilde{V}$ on $\mathcal{H}_{W_0}$ and $\mathcal{H}_{W_1}$ corresponding to two complex structures J_0 and J_1 are isomorphic if and only if $J_0 - J_1$ is a Hilbert–Schmidt operator.*

Let us explain one of the key idea in the proof of this theorem: suppose we have two complex structures J_0 and J_1 with corresponding decompositions

$$V_{\mathbb{C}} = W_0 \oplus \overline{W}_0 = W_1 \oplus \overline{W}_1.$$

Write $\overline{W}_1$ as the graph of a linear map $\alpha : \overline{W}_0 \to W_0$. Then the fact that W_1 must be maximal isotropic and positive, c.f. Definition 9.11, translates to:

- α is symmetric: $\omega(\bar{w}_1, \alpha\bar{w}_2) = \omega(\bar{w}_2, \alpha\bar{w}_1)$,
- $1 - \bar{\alpha}\alpha$ is positive definite on $\overline{W}_0$.

Remark that for such an α, the element

$$g = \begin{pmatrix} 1 & \alpha \\ \bar{\alpha} & 1 \end{pmatrix}$$

is an invertible operator mapping W_0 to W_1. The corresponding complex structure is therefore given by

$$J_1 = g \begin{pmatrix} i & 0 \\ -i & 0 \end{pmatrix} g^{-1} = \begin{pmatrix} i\left(\frac{1+\bar{\alpha}\alpha}{1-\bar{\alpha}\alpha}\right) & \frac{-2i\alpha}{1-\bar{\alpha}\alpha} \\ \frac{-2i\bar{\alpha}}{1-\alpha\bar{\alpha}} & -i\left(\frac{1+\alpha\bar{\alpha}}{1-\alpha\bar{\alpha}}\right) \end{pmatrix}.$$

With this expression one can proof that if $J_1 - J_0$ is Hilbert–Schmidt, $\alpha : \overline{W}_0 \to W_0$ is Hilbert–Schmidt. Using the isomorphism

$$B_2(\overline{W}, W) \cong W \otimes W,$$

we obtain a symmetric vector $\alpha \in W \otimes W$, i.e., in $\mathrm{Sym}^2(W)$.

From the proof of Proposition 9.14 we see that $\mathcal{H}_{W_0}$ and $\mathcal{H}_{W_1}$ will be isomorphic if we can find a vector $\Omega_1 \in \mathcal{H}_{W_0}$ that is fixed under the action of $\overline{W}_1$. We now claim that this vector is given by $e^{\alpha/2} \in \mathcal{H}_{W_0}$.

Definition 9.18. A complex structure J on (V, ω) is called a *polarization*. Two polarizations belong to the same class if they satisfy the criterion of Theorem 9.17.

Lagrangian correspondences. For any symplectic vector space $V = (V, \omega)$, we denote by V^{op}, the *symplectic conjugate* given by the pair $(V, -\omega)$. Let us now consider two symplectic vector spaces V_1 and V_2.

Definition 9.19. A *Lagrangian correspondence* from V_1 to V_2 is a Lagrangian subspace $L \subset V_1 \oplus V_2^{\mathrm{op}}$.

Important examples come from invertible symplectic maps: if $f : V_1 \to V_2$ is a symplectic linear isomorphism, one easily checks that

$$L_f := \mathrm{graph}(f) \subset V_1 \oplus V_2^{\mathrm{op}}$$

is a Lagrangian subspace, hence a correspondence.

The most important point of the definition is the fact that it is possible to compose Lagrangian correspondences:

Proposition 9.20. *Let $L_1 \subset V_1 \oplus V_2^{\mathrm{op}}$ and $L_2 \subset V_2 \oplus V_1^{\mathrm{op}}$ be two Lagrangian correspondences. Then*

$$L_1 \star L_2 := \{(v_1, v_3) \in V_1 \oplus V_2^{\mathrm{op}},\ \exists v_2 \in V_2 \text{ with } (v_1, v_2) \in L_1,\ (v_2, v_3) \in L_2\},$$

is a Lagrangian correspondence from V_1 to V_2.

Proof. □

We now turn to the quantization of this category of Lagrangian correspondences. For this we consider *complex* Lagrangian correspondences in the complexification of symplectic vector spaces, i.e.,

$$L \subset (V_2 \otimes \mathbb{C}) \oplus (V_1 \otimes \mathbb{C})^{\mathrm{op}}. \tag{42}$$

Remark 9.21. Let $W \subset V \otimes \mathbb{C}$ be a positive Lagrangian subspace. Then $\overline{W}$ is a negative Lagrangian subspace, which is exactly positive for the symplectic conjugate $(V \otimes \mathbb{C})^{\mathrm{op}}$. With this we easily see that there is a canonical isomorphism

$$\mathcal{H}_{\overline{W}} \cong \overline{\mathcal{H}}_W,$$

as representations of the Heisenberg group of V^{op}.

Let V_1 and V_2 be two polarized symplectic vector spaces, and we write $\mathcal{H}_{V_1}$ and $\mathcal{H}_{V_2}$ for the corresponding irreducible representations of the associated Heisenberg groups. The direct sum $V_1 \oplus V_2$ carries a canonical polarization. Combining the previous remark with Theorem 9.17, we see that a Lagrangian correspondence as in (42) determines a ray

$$T_L \subset \mathcal{H}_{V_2} \otimes \overline{\mathcal{H}}_{V_1}.$$

This is nothing but a Hilbert–Schmidt operator $T_L : \mathcal{H}_1 \to \mathcal{H}_2$, defined uniquely up to a scalar. The crucial point of this construction is the following:

Theorem 9.22. *Let $L_1 \subset (V_2 \otimes \mathbb{C}) \oplus (V_1 \otimes \mathbb{C})^{\mathrm{op}}$ and $L_2 \subset (V_3 \otimes \mathbb{C}) \oplus (V_2 \otimes \mathbb{C})^{\mathrm{op}}$ be Lagrangian correspondences. Then the equality*

$$T_{L_1} \circ T_{L_2} = T_{L_1 \star L_2}$$

holds up to a scalar.

What is meant is that both T_{L_1} and T_{L_2} are defined up to a scalar, and for any choice for both of them, their composition lies in the ray determined by $L_1 \star L_2$.

Proof. Let $\{e_i\}$, $\{f_i\}$ and $\{g_i\}$ be orthonormal bases of $\mathcal{H}_{V_1}$, $\mathcal{H}_{V_2}$ and $\mathcal{H}_{V_3}$. With these bases we write

$$T_{L_1} = \sum_i \lambda_i e_i \otimes f_i, \quad T_{L_2} = \sum_i \mu_i f_i \otimes g_i,$$

where $\{\lambda_i\}$ and $\{\mu_i\}$ are square summable sequences in $\mathbb{C}$. By the (sketch of the) proof of Shale's theorem we know that T_{L_1} and T_{L_2} are uniquely characterized, up to a scalar, by the property that they are fixed by the action of $\overline{L}_1$, resp. $\overline{L_2}$. It therefore suffices to show that $T_{L_1} \circ T_{L_2}$ is fixed by $\overline{L_1 \star L_2}$. Let $(v_1, v_3) \in \overline{L_1 \star L_2}$. Using property (41) we then have

$$\begin{aligned} v_1 T_{L_1} \circ T_{L_2} v_3 &= \sum_i \lambda_i \mu_i \langle f_i, f_i \rangle \, v_1 e_i \otimes v_3 g_i \\ &= \sum_i \lambda_i \mu_i \langle \bar{v}_2 f_i, v_2 f_i \rangle \, v_1 e_i \otimes v_3 g_i, \quad \text{for } v_2 \in V_2 \text{ as in Prop. 9.20.} \\ &= c T_{L_1} \circ T_{L_2}, \quad c \in \mathbb{C}. \end{aligned}$$

This proves the theorem. □

9.3. **The CFT of free bosons.** We shall now use the previous theory of the Heisenberg group to construct a CFT. First, we shall associate a Hilbert space to a circle as follows: consider the vector space $\Omega^0(S^1) \oplus \Omega^1(S^1)$ equipped with the symplectic form

$$\omega((f_1, \alpha_1), (f_2, \alpha_2)) = \int_{S^1} (f_1 \alpha_2 - f_2 \alpha_1).$$

On a one-dimensional manifold we have an exact sequence of the form

$$0 \longrightarrow \mathbb{R} \longrightarrow \Omega^0(S^1) \xrightarrow{d} \Omega^1(S^1) \xrightarrow{\int_{S^1}} \mathbb{R} \longrightarrow 0.$$

With this, we define the symplectic vector space

$$V(S^1) := \Big(\Omega^0(S^1)/\mathbb{R} \Big)) \oplus \ker \left(\int_{S^1} : \Omega^1(S^1) \to \mathbb{R} \right).$$

Lemma 9.23. *There exists a canonical linear symplectic isomorphism*

$$V(S^1) \cong V \oplus V^{\mathrm{op}},$$

where V is the symplectic vector space of Remark 9.8.

Proof. Consider the map $V(S^1) \to V \oplus V^{\mathrm{op}}$ defined by

$$(f, \alpha) \mapsto \frac{1}{\sqrt{2}} (f + d^{-1}\alpha, f - d^{-1}\alpha).$$

(Remark that d^{-1} is unambiguously defined by the exacts sequence above.) This is obviously a linear isomorphism. Let us check the symplectic form:

$$\begin{aligned} &\frac{1}{2} \omega_{V \oplus V^{\mathrm{op}}} \Big((f_1 + d^{-1}\alpha_1, f_1 - d^{-1}\alpha_1), (f_2 + d^{-1}\alpha_2, f_2 - d^{-1}\alpha_2) \Big) \\ &\qquad = \frac{1}{2} \int_{S^1} \Big((f_1 + d^{-1}\alpha_1) d(f_2 + d^{-1}\alpha_2) - (f_1 - d^{-1}\alpha_1) d(f_2 - d^{-1}\alpha_2) \Big) \\ &\qquad = \int_{S^1} (f_1 \alpha_2 - f_2 \alpha_1). \end{aligned}$$

This proves that the map is compatible with the symplectic form. □

Let Σ be a Riemann surface with boundary $\partial\Sigma$. Consider the following subspace $W_\Sigma \subset V(\partial\Sigma) \otimes \mathbb{C}$:

$$W_\Sigma := \{ (\varphi, *d\varphi) \ \varphi \text{ is harmonic on } \Sigma \}.$$

Recall, c.f. [Fo, §22], that for $f \in \Omega^0(\partial\Sigma)$, the associated Dirichlet problem

$$\Delta\varphi = 0, \quad \varphi|_{\partial\Sigma} = f,$$

has a unique solution. In other words, a harmonic function is uniquely determined by its boundary values. Therefore we can really consider W_Σ to be a subspace of $V(\partial\Sigma)$.

Proposition 9.24. *W_Σ is a positive maximal isotropic subspace of $V_{\partial\Sigma} \otimes \mathbb{C}$.*

Proof. Let us first check that W_Σ is isotropic: let φ_1, φ_2 be harmonic on Σ. Then we have that

$$\begin{aligned}\omega((\varphi_1, i * d\varphi_1), (\varphi_2, i * d\varphi_2)) &= i \int_{\partial\Sigma} (\varphi_1 * d\varphi_2 - \varphi_2 * d\varphi_1), \\ &= i \int_\Sigma (d\varphi_1 \wedge *d\varphi_2 - d\varphi_2 \wedge *d\varphi_1) \\ &= 0,\end{aligned}$$

So W_Σ is indeed isotropic. Next, let $(f, \alpha) \in W_\Sigma^\omega$. For φ harmonic on Σ, we then compute

$$\begin{aligned}0 &= \int_{\partial\Sigma} (\varphi\alpha - if * d\varphi) \\ &= \int_\Sigma d(\varphi\alpha - if * d\varphi) \\ &= \int_\Sigma (d\varphi \wedge \alpha + \varphi d\alpha - df \wedge i * d\varphi) \\ &= \int_\Sigma d\varphi \wedge (\alpha - i * df) + \int_\Sigma \varphi d\alpha.\end{aligned}$$

From this it should follow that $d\alpha = 0$ and $\alpha = i * df$, which combines to $d * df = 0$. This implies that f is harmonic and proves that W_Σ is maximal isotropic.

Finally, we need to check positivity:

$$\begin{aligned}-2i\omega((\bar\varphi, \overline{i * d\varphi}), \varphi, i * d\varphi)) &= 2 \int_{\partial\Sigma} (\bar\varphi * d\varphi + \varphi * d\bar\varphi) \\ &= 2 \int_\Sigma d\varphi \wedge d\bar\varphi \geq 0.\end{aligned}$$

□

Example 9.25. Let us consider the unit disk $D := \{z \in \mathbb{C},\ |z| \leq 1\}$ bounding the unit circle. Harmonic functions on D are given by $z^n,\ \bar z^m, m, n \geq 0$. Then we have

$$i * dz^n = dz^n, \quad i * d\bar z^m = -d\bar z^m,$$

since $*(udz + vd\bar z) = -iudz + ivd\bar z$. Using the isomorphism of Lemma 9.23 we therefore see that the Lagrangian subspace $W_D \subset V \oplus V^{\mathrm{op}}$ is given by

$$W_D = \mathrm{span}_{\mathbb{C}}\{(e^{in\theta}, 0), (0, e^{-im\theta}),\ m, n > 0\}.$$

Comparing with Example 9.16, we see that this is nothing but $W_D = W_+ \oplus W_-$. The associated irreducible representation of the Heisenberg group $\widetilde{V(S^1)} \cong \widetilde{V} \times \widetilde{V}^{\mathrm{op}}$ is defined on the Hilbert space $\mathcal{H}_{W_+} \otimes \overline{\mathcal{H}}_{W_+}$.

Proposition 9.26. *The polarization class of $V(S)$ induced by two Riemann surfaces Σ_1 and Σ_2 with the same boundary $\partial\Sigma_1 \cong S \cong \partial\Sigma_2$ is the same.*

Proof. We skip this one. $\square$

Theorem 9.27. *Let Σ_1 and Σ_2 be two Riemann surfaces with $\partial\Sigma_{\mathrm{out}} \cong S \cong \partial\Sigma_{\mathrm{in}}$. Then the associated Lagrangian subspaces are related by:*

$$L_{\Sigma_1 \cup_S \Sigma_2} \cong L_{\Sigma_1} \star L_{\Sigma_2}.$$

Proof. Let us sketch the argument: there is an obvious sequence

$$0 \to \mathrm{Harm}(\Sigma_1 \cup_S \Sigma_2) \to \mathrm{Harm}(\Sigma_1) \oplus \mathrm{Harm}(\Sigma_2) \to \Omega^0(S).$$

Here the third map is given by $(\varphi_1, \varphi_2) \mapsto \varphi_1|_{\partial\Sigma_1} - \varphi_2|_{\partial\Sigma_2}$. By the unique solubility of the Dirichlet problem, one shows that the sequence is exact. This implies that the second map $\varphi \mapsto (\varphi|_{\Sigma_1}, \varphi|_{\Sigma_2})$ induces the desired isomorphism. $\square$

To construct the CFT we consider the following associations:

- to a closed one-manifold S we associate the Hilbert space $\mathcal{H}_S$ which carries the unique irreducible representation of the Heisenberg group $\widetilde{V(S^1)}$. The polarization class that underlies the definition of this representation is obtained by mapping S diffeomorphically onto a disjoint union of S^1's and using Example 9.25. Strictly speaking this association is not quite canonical, as it depends on a chosen diffeomorphism. This is called a *rigging* of a 1-manifold, and is related to the conformal anomaly. We shall ignore this aspect and simply consider the case $S = S^1$.
- To a Riemann surface Σ with boundary $\partial\Sigma = \partial\Sigma_{\mathrm{in}} \coprod \partial\Sigma_{\mathrm{out}}$ we associate a ray

$$Z(\Sigma) \subset \overline{\mathcal{H}}_{\partial\Sigma_{\mathrm{in}}} \otimes \mathcal{H}_{\partial\Sigma_{\mathrm{out}}},$$

  determined by the Lagrangian correspondence of Proposition 9.24.

Combining Theorem 9.27 with Theorem 9.22, one now shows that the fundamental composition property when gluing surfaces holds true: we have constructed a CFT.

Proposition 9.28. *The partition function of the CFT constructed above is given by*

$$Z(q) = \prod_{m,n \geq 0} \frac{1}{(1-q^n)} \frac{1}{(1-\bar{q}^m)}.$$

Notice that this is, up to a factor $|q|^{1/12}$, just $\eta(\tau)\overline{\eta(\tau)}$, where $q = e^{2\pi i \tau}$ and $\eta(\tau)$ is the well-known Dedekind eta-function, a modular form of weight $1/2$.

APPENDIX A. SOME RIEMANNIAN GEOMETRY

Let M be a smooth n-dimensional manifold. A *Riemannian metric* on M is given by an inner product $g_x : T_xM \times T_xM \to \mathbb{R}$ which depends smoothly on $x \in M$. More precisely, it is given by a smooth symmetric tensor field in $T^*M \otimes T^*M$ which defines a positive definite inner product on each tangent space T_xM. By a standard partition of unity argument, one proves that each manifold can be equiped with a Riemannian metric. In local coordinates $(x^1, \ldots x^n) : U \to \mathbb{R}^n$ we can write

$$g(x) = g_{ij}(x) dx^i \otimes dx^j.$$

Recall the notion of a connection ∇ on a vector bundle $E \to M$: this is a linear map

$$\nabla : \Gamma(M; E) \to \Gamma(M; E \otimes T^*M)$$

satisfying the Leibniz rule:

$$\nabla(fs) = f\nabla s + df \otimes s, \quad f \in C^\infty(M),\ s \in \Gamma(M; E).$$

We write ∇_X for the contraction of ∇ with a vector field X.

Theorem A.1. *Given a Riemannian metric g, there exists a unique connection on TM, called the* Levi–Civita connection $\nabla : \Gamma(M; TM) \to \Gamma(M; TM \otimes T^*M)$ *satisfying*

i) (compatibility with the metric)

$$Zg(X,Y) = g(\nabla_Z X, Y) + g(X, \nabla_Z Y), \quad \textit{for all vector fields } X, Y, Z \in \mathfrak{X}(M),$$

ii) (Torsion-free)

$$\nabla_X Y - \nabla_Y X = [X, Y].$$

This connection is uniquely determined by the equation

$$\begin{aligned} g(\nabla_X Y, Z) = \frac{1}{2}\big(&Xg(Y,Z) - Zg(X,Y) + Yg(Z,X) \\ &- g(X,[Y,Z]) + g(Z,[X,Y]) + g(Y,[Z,X])\big). \end{aligned} \tag{43}$$

Proof (sketch): First check that properties $i)$ and $ii)$ imply (43). Then prove that the right hand side, for fixed X, Y is tensorial in Z in the sense that it is $C^\infty(M)$-linear. Conclude that we can write $\omega(Z)$ for the right hand side, for a unique one-form ω. Using the nondegeneracy of g, define $g(\nabla_X Y, Z) = \omega(Z)$. Then check that the ∇ thus defined is indeed a connection. Finally, the metric and torsion free property follow from (43). This same equation also shows that ∇ is unique. □

In local coordinates the connection writes out as

$$\nabla_{\partial/\partial x^i} \sum_j X^j \frac{\partial}{\partial x^j} = \sum_j \frac{\partial X^j}{\partial x^i}\frac{\partial}{\partial x^j} + \sum_{jk} \Gamma^j_{ik} X^k \frac{\partial}{\partial x^j},$$

where Γ^j_{ik} are called the *Christoffel symbols*. They are obtained from the metric by the formula

$$\Gamma^k_{ij} = \frac{1}{2}\sum_l g^{kl}\left(\frac{\partial g_{il}}{\partial x^j} + \frac{\partial g_{jl}}{\partial x^i} - \frac{\partial g_{ij}}{\partial x^l}\right),$$

where g^{ij} denotes the components of the inverse of the matrix (g_{ij}).

The *curvature* of g is given by the tensor $R \in \Gamma(M, \wedge^2 T^*M \otimes \mathrm{End}(TM))$ defined by

$$R(X,Y)(Z) := [\nabla_X, \nabla_Y]Z - \nabla_{[X,Y]}Z.$$

Given a piecewise C^1-path $\gamma : [t_0, t_1] \to M$ in M, we can define its *length* as

$$L(\gamma) := \int_{t_0}^{t_1} \left\| \frac{d\gamma}{dt}(t) \right\| dt.$$

With this notion of length we can define a metric given by

$$d_M(x_0, x_1) := \inf_\gamma L(\gamma),$$

where γ varies over all piecewice C^1-paths from x_0 to x_1.

Theorem A.2 (Hopf–Rinow)**.** *A Riemannian manifold is complete as a metric space if and only if it is geodesically complete, i.e., geodesics are defined for all $t \in \mathbb{R}$.*

Appendix B. The spectral theorem and one-parameter groups

B.1. **Self-adjoint operators.** For this appendix, [RSI, Ch. VIII] is an excellent reference. Recall that an *unbounded operator* A on a Hilbert space $\mathcal{H}$ is a linear operator $A : D(A) \to \mathcal{H}$ defined on a domain $D(A) \subset \mathcal{H}$. In the sequel we will assume that this domain $D(A)$ is dense in $\mathcal{H}$, as is often the case. We call B an *extension of* A if $D(A) \subset D(B)$ and $B|_{D(A)} = A$. The fundamental idea is to study unbounded operators A by means of their graph

$$G(A) = \{(x, Ax),\ x \in D(A)\}.$$

A is said to be *closed* if $G(A)$ is a closed subset of $\mathcal{H} \oplus \mathcal{H}$, and *closable* if the closure $\overline{G(A)}$ of $G(A)$ is graph of an operator $\bar{A}$ (extending A).

If we let $D(A^*)$ be the set of $v \in \mathcal{H}$ for which there exists an $w \in \mathcal{H}$ with the property that

$$\langle Au, v\rangle = \langle u, w\rangle, \quad \text{for all } u \in D(A),$$

then the above equation defines a linear map $A^* : D(A^*) \to \mathcal{H}$, called the *adjoint*, if we set $A^*w = v$. (For this it is essential that $D(A)$ is dense!) The domain of the adjoint $D(A^*)$ need not be dense and in fact one can prove that it is if and only if A is closable, in which case $\bar{A} = (A^*)^*$.

Definition B.1. A densely defined operator $A : D(A) \to \mathcal{H}$ is called

i) *symmetric* if

$$\langle Au, v\rangle = \langle u, Av\rangle, \quad \text{for all } u, v \in D(A),$$

ii) *selfadjoint* if $A = A^*$, i.e., symmetric and $D(A) = D(A^*)$,
iii) *essentially selfadjoint* if it is closable and its closure is selfadjoint.

Remark B.2. With the fact mentioned just above the definition, one easily shows that a symmetric operator is closable and that a selfadjoint operator is closed. Another way to phrase that A is essentially selfadjoint is to say that $D(A^*)$ is dense and that A^* is selfadjoint.

The main point about these definitions is that one usually is confronted (e.g. in quantum mechanics) with only a symmetric operator acting on some natural domain. A general symmetric operator need not have a selfadjoint extension, and when it has, this extension may not be unique. There are two natural closed extensions of a symmetric operator:
i) A_{min} obtained by taking the closure, called the *minimal extension*.
ii) $A_{max} := A^*$, the *maximal extension*.
One has that $A_{min} \subset A_{max}$ and any selfadjoint extension $\bar{A}$ is inside this inclusion: $A_{min} \subset \bar{A} \subset A_{max}$. A natural way to show that an operator is essentially selfadjoint is therefore to show that $A_{min} = A_{max}$.

B.2. **Friedrichs' extension.** Friedrichs' method gives a canonical selfadjoint extension of a nonnegative symmetric operator. A densely defined operator $A : D(A) \to \mathcal{H}$ is nonnegative if the associated quadratic function is nonnegative:

$$\langle u, Au \rangle \geq 0, \quad \text{for all } u \in D(A).$$

We introduce the following sesquilinear form on $D(A)$:

$$Q(u,v) := \langle u, v \rangle + \langle u, Av \rangle .$$

Since $Q(u,u) \geq \|u\|^2$, Q defines an inner product, and we write V for the completion of $D(A)$ with respect to this norm. We have $V \subset \mathcal{H}$. Define $D(\bar{A})$ by

$$D(\bar{A}) := \{u \in V,\ v \mapsto Q(u,v) \text{ is bounded linear on } \mathcal{H}\}.$$

By Riesz' Theorem, there exists a $\bar{A}u \in \mathcal{H}$ such that $Q(u,v) = \langle \bar{A}u, v \rangle$. This defines the extension $\bar{A}$ of A. It is selfadjoint.

B.3. **The spectral theorem.**

Theorem B.3 (Spectral Theorem)**.** *Let $A : D(A) \to \mathcal{H}$ be a selfadjoint operator. There is a unique $*$-homomorphism $\phi \mapsto \phi(A)$ from the bounded Borel functions on $\mathbb{R}$ into $B(\mathcal{H})$ such that*
i) $\|\phi(A)\| \leq \|\phi\|_\infty$,
ii) *If $\phi_n \to \phi$ point wise and the sequence $\{\|\phi_n\|_\infty\}_n$ is bounded, then $\phi_n(A) \to \phi(A)$ strongly.*
iii) *If $Au = \lambda u \Longrightarrow \phi(A)u = \phi(\lambda)u$,*

iv) *if $\phi \geq 0$, then $\phi(A) \geq 0$.*

B.4. **One-parameter groups and Stone's theorem.** With the spectral theorem, we can get dynamics once we have a selfadjoint operator.

Definition B.4. A *strongly continuous one-parameter group of unitaries* is a unitary representation of $\mathbb{R}$ on $\mathcal{H}$ defined by a homomorphism $U : \mathbb{R} \to U(\mathcal{H})$ which is continuous for the strong operator topology.

Proposition B.5. *Let $A : D(A) \to \mathcal{H}$ be a selfadjoint operator. The operator*

$$U(t) := e^{\sqrt{-1}tA}$$

is unitary and defines a strongly continuous one-parameter group of unitaries. Furthermore,

$$\lim_{t\to 0} \frac{(U(t)-1)u}{t} = \sqrt{-1}Au, \quad \textit{for all } u \in D(A).$$

Remark B.6. When A is *bounded* and selfadjoint, the series

$$e^{\sqrt{-1}tA} = \sum_{k=0}^{\infty} \frac{(\sqrt{-1}t)^k A^k}{k!}$$

converges in norm, and defines a one-parameter group of unitaries that is *norm continuous*. It is important to remark that for unbounded A, the operator $e^{\sqrt{-1}tA}$ can't be defined directly by the series above, we have to use the spectral theorem.

Stone's theorem gives the converse to this construction:

Theorem B.7 (Stone's theorem)**.** *Given a strongly continuous one-parameter group of unitaries $U(t)$, there is a selfadjoint operator A such that $U(t) = e^{\sqrt{-1}tA}$.*

The theory for semigroups is very similar:

Definition B.8. A map $\mathbb{R}_{\geq 0} \to B(\mathcal{H})$, $t \mapsto S(t)$ is called a *strongly continuous semigroup of operators* if

i) $S(0) = 1$,
ii) $S(t_1)S(t_2) = S(t_1 + t_2)$, for all $t_1, t_2 \geq 0$.

If $\|S(t)\| \leq 1$ we say $S(t)$ forms a semigroup of contractions.

Proposition B.9 (c.f. [RSII, p.242])**.** *Let $A \geq 0$ be a nonnegative selfadjoint operator. Then the family of bounded operators*

$$S(t) := e^{-tA},$$

defined by means of the spectral theorem, is a strongly continuous semigroup of contractions satisfying

$$\lim_{t\downarrow 0} \frac{(S(t)-1)u}{t} = -Au, \quad \textit{for all } u \in D(A).$$

Remark B.10. Similar remarks as in the unitary case apply to the case that A is bounded: in this case

$$e^{-tA} = \sum_{k=0}^{\infty} \frac{(-t)^k A^k}{k!}$$

converges in norm and the resulting semigroup is even continuous in norm.

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