NOTES ON CHERN–SIMONS THEORY

1. Preliminaries

1.1. **Connections on principal bundles.** Throughout, we fix a compact Lie group G with Lie algebra $\mathfrak{g}$. Let $\pi : P \to M$ be a principal G-bundle over a smooth manifold M. For a point $p \in P$, we have an exact sequence of vector spaces

$$0 \longrightarrow \mathfrak{g} \xrightarrow{\rho_p} T_pP \xrightarrow{d_p\pi} T_{\pi(p)}M \longrightarrow 0,$$

where ρ_p denotes the infinitesimal action of G on P. The following is elementary:

Lemma 1.1. *The following are equivalent:*

i) *a section* $\sigma_p : T_{\pi(p)}M \to T_pP$ *of* $d_p\pi$, *i.e.*, $d_p\pi \circ \sigma_p = 1$,
ii) *a map* $\tau_p : T_pP \to \mathfrak{g}$ *such that* $\tau_p \circ \rho_p = 1$,
iii) *a subspace* $H_p \subset T_pM$ *such that* $T_pP = \mathfrak{g} \oplus H_p$.

We refer to any of these equivalent choices as a *splitting* of the short exact sequence. Varying over $p \in P$ we get a short exact sequence of vector bundles over P:

$$0 \longrightarrow \mathfrak{g} \times P \xrightarrow{\rho} TP \xrightarrow{d\pi} \pi^*TM \longrightarrow 0. \tag{1.1}$$

The group G naturally acts on these bundles making the maps above equivariant:

- trivially on π^*TM,
- by the derivative $dR_g : TP \to TP$ of the right action R_g on P,
- by $\mathrm{Ad}_g \times R_g$ on $\mathfrak{g} \times P$. (This follows from the fact that $\mathfrak{g}$ is identified with the vector space of *left* invariant vector fields on G).

Definition 1.2. *A connection on* P *is a* G*-equivariant splitting of* (1.1)

By point of view ii) in Lemma 1.1 above, a connection is therefore given by a Lie algebra valued 1-form $A \in \Omega^1(P, \mathfrak{g})$ such that

$$\iota_{\xi_P} A = \xi, \quad \text{for all } \xi \in \mathfrak{g},$$
$$R_g^* A = \mathrm{Ad}_g(A) \quad \text{for all } g \in G.$$

Here, ξ_P is the generating vector field of $\xi \in \mathfrak{g}$. Equivalently, we can think of a connection as a vector subbundle $H \subset TP$ satisfying $R_g^* H = H$ for all $g \in G$.

Remark 1.3 (Linear connections on vector bundles)**.** Recall that for a representation V of G, we obtain an associated vector bundle

$$E(V) := (P \times V)/G,$$

where G acts diagonally on $P \times V$. A (local) section s of this bundle is given by a G-equivariant map $s : P \to V$. Below we will use the associated bundle construction mostly for the adjoint representation $\mathrm{Ad} : G \to GL(\mathfrak{g})$, and we use the notation $\mathrm{ad}(P)$ for this vector bundle.

Next, for a vector bundle $E \to M$, introduce the space of E-value differential forms

$$\Omega^k(M, E) = \Gamma^\infty(M, \Lambda^k T^*M \otimes E).$$

Of course, $\Omega^0(M, E)$ is just the space of smooth sections of E. With this, we can define a *linear connection* on E as a degree 1 differential operator:

Definition 1.4. *A* linear connection *on a vector bundle* E *is a map*

$$\nabla : \Omega^0(M, E) \to \Omega^1(M, E)$$

satisfying the Leibniz rule

$$\nabla(fs) = df \otimes s + f\nabla(s), \tag{1.2}$$

for any $f \in C^\infty(M)$ *and* $s \in \Gamma^\infty(M, E)$.

The two notions of connections on principal and vector bundles correspond to each other under the associated vector bundle construction:

Lemma 1.5. *Let* $\pi : P \to M$ *be a principal* G*-bundle equipped with a connection* $\sigma : \pi^*TM \to TP$*. For any representation* V *of* G*, the the formula*

$$(\nabla_X s)(p) := ds(\sigma_p(X_{\pi(p)})),$$

where $s : P \to V$ *is a* G*-equivariant map, defines a linear connection on* $E(V)$*.*

Example 1.6. *The trivial bundle* $P = G \times M \to M$ *has a canonical connection given by* $H_{g,x} := T_x M \subset TP$.

Lemma 1.7. *Every principal* G*-bundle admits a connection. Furthermore, the space of all connections* $\mathrm{Conn}(P)$ *is an affine space modeled on* $\Omega^1(M, \mathrm{ad}(P))$*.*

Proof. For the first statement one uses local triviality of the P to exhibit the existence of a local connection. Then one uses a partition of unity to construct a global one. For the second statement, let $A_1, A_2 \in \Omega^1(P, \mathfrak{g})$ be two connections on P. The difference $A_1 - A_2 \in \Omega^1(P, \mathfrak{g})$ is:

- G-invariant because both A_1 and A_2 are,
- G-basic: $\iota_{\xi_P}(A_1 - A_2) = 0$ for all $\xi \in \mathfrak{g}$.

It therefore descend to a 1-form on the quotient $(P \times \mathfrak{g})/G$, but this is exactly the adjoint bundle $\mathrm{ad}(P)$. □

Finally, let us discuss the *curvature* of a connection A: From the point of view i) of Lemma 1.1 as a splitting $\sigma : \pi^*TM \to TP$, it is clear that with a connection we can lift vector fields $X \in \mathfrak{X}(M)$ to $\tilde{X} \in \mathfrak{X}(P)$. The curvature is the obstruction to this lift being a morphism of Lie algebras with respect to the Lie bracket of vector fields:

$$R(X, Y) := [\tilde{X}, \tilde{Y}] - \widetilde{[X, Y]}, \quad X, Y \in \mathfrak{X}(M).$$

The right hand side of this equation lies in the kernel of $d_p\pi$ for each $p \in P$, so by the exact sequence (1.1), is an element in $\mathfrak{g}$. This means that the curvature is an element $R \in \Omega^2(M, \mathrm{ad}(P))$. The connection is called *flat* if $R = 0$.

Proposition 1.8. *Let $A \in \Omega^1(P, \mathfrak{g})$ be a connection on a principal G-bundle $P \to M$.*

i) $R(A) = dA + [A, A]$
ii) (Bianchi identity) $\nabla R = 0$, *where ∇ is the induced linear connection on* $\mathrm{ad}(P)$.

1.2. **Chern–Weil theory.** Chern–Weil theory gives an explicit construction of characteristic classes as closed differential forms associated invariant polynomials.

Definition 1.9. *Let G be a Lie group. An* invariant polynomial *is a polynomial $F : \mathfrak{g} \to \mathbb{C}$ invariant under the adjoint action:*

$$F(\mathrm{Ad}_g(\xi)) = F(\xi), \quad \xi \in \mathfrak{g}, \ g \in G.$$

The space of invariant polynomials $I_{\mathrm{inv}}(G)$ is given by the invariant part of the symmetric algebra $I_{\mathrm{inv}}(G) = \mathrm{Sym}(\mathfrak{g}^*)^G$, and is graded according to the degree of a monimial. When thinking of a symmetric tensor, we often write $F(\xi, \ldots, \xi)$ for the value of the polynomial F in ξ. A famous theorem of Chevalley asserts that when G is reductive $I_{\mathrm{inv}}(G)$ is a polynomial algebra with $\mathrm{rank}(G)$ generators.

Given an invariant polynomial F of degree k, and a connection A with curvature R on a principal bundle $P \to M$, we consider

$$F(R) \in \Omega^{2k}(M).$$

This requires some explanation: recall that $R(A) \in \Omega^2(M, \mathrm{ad}(P))$. The vector bundle $\mathrm{ad}(P)$ obviously has fibers $\mathrm{ad}(P)_x \cong \mathfrak{g}$ for each $x \in M$, but it is important to realize that in general there is no *canonial* isomorphism. However, and two isomorphisms differ by the action of G, so the differential form $F(R)$ is unambiguously defined.

Theorem 1.10 (Chern–Weil)**.** *Let F be an invariant polynomial of degree k.*

i) *For any connection A on P with curvature R the form $F(R) \in \Omega^{2k}(M)$ is closed:*
$$dF(R) = 0.$$

ii) *The de Rham cohomology class $[F(R)] \in H^{2k}_{dR}(M)$ does not depend on the choice of a connection A.*

Proof. i) follows from the Bianchi identity.

For ii), let A_0, A_1 be two connections on P with curvature R_0 and R_1. Because the space of connections on P is affine, we can consider the convex combination $A^{\mathrm{aff}} := tA_1 + (1-t)A_0$ interpolating between A_0 and A_1. We now view A^{aff} as a connection on the principal G-bundle $P \times [0,1] \to M \times [0,1]$, where $t \in [0,1]$ is the coordinate on the unit interval. This connection has curvature $R^{\mathrm{aff}} \in \Omega(M \times [0,1], \mathrm{ad}(P)$, and we can therefore consider $F(R^{\mathrm{aff}}) \in \Omega^{2k}_{\mathrm{cl}}(M \times [0,1])$. With this we define

$$L(A_0, A_1) := \int_0^1 F(R^{\mathrm{aff}})dt \in \Omega^{2k-1}(M). \tag{1.3}$$

The notation above is slightly imprecise: $F(R^{\mathrm{aff}})$ is itself a differential form over $M \times [0,1]$, and we mean that one integrates the components containing dt. The integral should be understood as a fiber integral over the fibration $M \times [0,1] \to M$, resulting in a differential form on M. For such a fiber integral Stokes' theorem gives

$$d\int_0^1 \alpha = \int_0^1 d\alpha - \alpha_{t=1} + \alpha_{t=0}, \quad \alpha \in \Omega^p(M \times [0,1]).$$

(Remark that in this formula the d on the left hand side refers to the exterior differential on $M \times [0,1]$ whereas on the right hand side it is the one on M.) In our case this leads to

$$dL(A_0, A_1) = F(R_1) - F(R_0),$$

proving the theorem. □

The so-called *transgression form* $L(A_0, A_1)$ defined in equation (1.3) is going to play an important role in the following. Let us remark that it has the following property:

Proposition 1.11. *For three connections A_0, A_1 and A_2, the following equality holds true:*

$$L(A_0, A_1) + L(A_1, A_2) = L(A_0, A_2).$$

Proof. The three connections A_0, A_1, A_3 define a connection A^{aff} on the bundle $P \times \Delta^2 \to M \times \Delta^2$, where Δ is the 2-simplex. The Proposition then follows from Stokes' theorem for the fiber integral over $M \times \Delta^2 \to M$ applied to the characteristic form $F(R^{\text{aff}})$. $\square$

Remark 1.12. Recall the following definition, motivated by the proof of de Rham's theorem:

Definition 1.13. *Let $\alpha^p_{cl}(M)$ be a closed differential form of degree p on a smooth manifold M. Its* period *group is defined as the subgroup of $\mathbb{R}$ (or $\mathbb{C}$ when α is $\mathbb{C}$-valued) defined by the integrals*

$$\int_{\Delta^p} \sigma^* \alpha,$$

over all smooth singular chains $\sigma \in S^\infty_p(M)$ of degree p.

By Stokes' theorem, the values of these integrals only depends on the homology class $[\sigma] \in H^{\text{sing}}_p(M)$ and the de Rham cohomology class $[\alpha] \in H^p_{dR}(M)$. If the period group $\mathrm{Per}(\alpha)$ of a form α is a subgroup of $\mathbb{Z}$, we say that α is *integral*.

Definition 1.14. *An invariant polynomial $F \in I_{\text{inv}}(G)$ is called* integral *if for all principal G-bundles P equipped with a connection A with curvature R, the differential form $F(R)$ has integral periods.*

Example 1.15. *For $G = GL(n, \mathbb{C})$, let F_k be the k-th symmetric polynomial in the eigenvalues of matrices in $\mathfrak{gl}(n, \mathbb{C})$. Then*

$$I_{\text{in}}(GL(n, \mathbb{C})) \cong \mathbb{C}[F_1, \ldots, F_n].$$

The associated characteristic class

$$c_k(P) := \left[\frac{1}{(2\pi\sqrt{-1})^k} F_k(R) \right] \in H^{2k}_{dR}(M),$$

is called the k-th Chern class. *It is a fundamental fact in the theory of characteristic classes that these classes are integral.*

2. The Chern–Simons action

2.1. **Definition.** Now we let G be a *compact* Lie group. Chern–Simons theory depends on the following datum of an invariant integral inner product on the Lie algebra $\mathfrak{g}$:

Definition 2.1. *We denote by $B : \mathfrak{g} \times \mathfrak{g} \to R$ the choice of an inner product on $\mathfrak{g}$ satisfying the following properties:*

- *it is Ad-invariant:*

$$B(\mathrm{Ad}_g(\xi), \mathrm{Ad}_g(\eta)) = B(\xi, \eta), \quad \xi, \eta \in \mathfrak{g},\ g \in G.$$

- *it is integral in the sense of Definition 1.14 when we view* B *as an invariant quadratic polynomial.*

Example 2.2. *Such inner products exist: for* $G = SU(N)$ *we can take*

$$B(X, Y) := \mathrm{Tr}_{\mathbb{C}^N}(XY).$$

This choice of inner product gives the second Chern class of vector bundles, c.f. Example 1.15. More generarlly, we can take kB *with* $k \in \mathbb{N}$.

We now assume that M is a closed 3-manifold and that $P \to M$ is a trivializable principal G-bundle. (Remark that since $\pi_2(G) = 0$ for a compact Lie group, if we assume that G is simply connected, any principal G-bundle is automatically trivial.) We therefore explicitly trivialize P by choosing a global section $s : M \to P$. Since this gives an isomorphism $P \times M \times G$, this equips P with a flat connection as in Example 1.6. We write A_s for this connection. Now let A be any other connection and consider the transgression form $L(A, A_s)$ (1.3) for the characteristic form defined by B in Chern–Weil theory. The Chern–Simons action is defined as

$$CS(A) := \int_M L(A, A_s). \tag{2.1}$$

Implicit in the notation is:

Lemma 2.3. $CS(A) \in \mathbb{R}/\mathbb{Z}$ *is independent of the choice of section* s.

Proof. Let s' be another section. This introduces another flat connection $A_{s'}$ on P. By Proposition 1.11 above we have

$$L(A, A_s) = L(A, A_{s'}) - L(A_s, A_{s'}).$$

For the construction of the last term on the right hand side, we have to equip $P \times [0, 1]$ with the connection $A^{\mathrm{aff}}_{s,s'} := tA_{s'} + (1-t)A_s$. The curvature $R^{\mathrm{aff}}_{s,s'}$ of this connection is given by

$$\begin{aligned} R^{\mathrm{aff}}_{s,s'} = & dt \wedge (A_{s'} - A_s) + t dA_{s'} + (1-t) dA_s \\ & + t^2 [A_{s'}, A_s] + (1-t)^2 [A_s, A_s] + 2t(1-t)[A_{s'}, A_s]. \end{aligned}$$

At $t = 0$ and at $t = 1$ this expression reduces to the same form $dt \wedge (A_{s'} - A_s)$, because both A_s and $A_{s'}$ are flat. We can therefore view the fiber integral over $[0, 1]$ as being over S^1 and with this

$$\int_M \int_0^1 B(R^{\mathrm{aff}}_{s,s'}, R^{\mathrm{aff}}_{s,s'}) = \int_{M\times S^1} B(R^{\mathrm{aff}}_{s,s'}, R^{\mathrm{aff}}_{s,s'}) \in \mathbb{Z},$$

by the integrality assumption on B. □

Because of this Lemma the exponent

$$\exp\left(2\pi\sqrt{-1}CS(A)\right)$$

is well-defined. Let us now compute the Chern–Simons action explicitly. For this we consider $P = M \times G$ so that $A_s = 0$, and we consider the connection $A^{\mathrm{aff}} = tA$ on the bundle $P \times [0,1] \to M \times [0,1]$. This connection has curvature

$$R^{\mathrm{aff}} = dt \wedge A + tdA + t^2[A, A].$$

With this we compute

$$\begin{aligned} L(A, 0) &= \int_0^1 B(R^{\mathrm{aff}}, R^{\mathrm{aff}})dt \\ &= \int_0^1 2B(dt \wedge A, tdA + t^2[A, A]) \\ &= B(A, dA + \frac{2}{3}B(A, [A, A]). \end{aligned}$$

We therefore find the expression

$$CS(A) = \int_M \left(B(A, dA) + \frac{2}{3}B(A, [A, A])\right).$$

Lemma 2.4. *The critical points of the Chern–Simons action are the flat connections.*

Proof. Consider a small variation $A + t\alpha$, $t \in (-\epsilon, \epsilon)$ of a connection A. Then

$$\begin{aligned} \left.\frac{d}{dt}\right|_{t=0} CS(A + t\alpha) &= \int_M (B(\alpha, dA) + B(A, d\alpha) + 2B(\alpha, [A, A])) \\ &= 2\int_M (B(\alpha, dA) + B(\alpha, [A, A])) + \int_M dB(\alpha, A) \\ &= 2\int_M B(\alpha, R), \end{aligned}$$

where we have used Stokes' theorem to go to the third line. For a critical point, this derivative should be zero for all $\alpha \in \Omega^1(M, \mathfrak{g})$. We therefore see that the critical points are exactly the flat connections. □

2.2. **The Witten invariant of 3-manifolds.** Let M be a closed 3-manifold. Assume we have chosen a B as above. Then kB satisfies the same integrality assumption, and can also be uses in the definition of the Chern–Simons action. The Witten invariant of a three manifold is defined by the formal path integral

$$Z_k(M) := \int_{\mathrm{Conn}(P)} \exp\left(2\pi\sqrt{-1}kCS(A)\right) DA.$$

Indeed, the Chern–Simons action does not make use of a metric, so this number only depends on the underlying manifold structure of M. But how can we make mathematical sense of this? One thought would perhaps be to consider a perturbative expansion around classical solutions (i.e., flat connections) using Feynman diagrams. However it turns out that the critical points of the Chern–Simons action are highly degenerate due to the presence of a large symmetry group that we shall discuss in the next subsection. A better approach is therefore to use the structure of a TQFT: we should cut up our 3-manifold M along a compact oriented surface Σ and construct a the vector space associated to Σ to compute the invariant $Z_k(M)$. This is closer to a *Hamiltonian approach* to Chern–Simons theory that we shall discuss in the next section.

2.2.1. *The Jones Polynomial.* We can add observables to the path integral above. Let $G = SU(N)$, and consider an embedded curve $\gamma : S^1 \to S^3$, i.e., a knot $K \subset S^3$. The associated Wilson loop is the function on $\mathrm{Conn}(P)$ defined by

$$W_\gamma(A) := \mathrm{Tr}_{\mathbb{C}^N}(\mathrm{Hol}_\gamma(A)),$$

where $\mathrm{Hol}_\gamma(A) \in G$ is the *holonomy* of A along γ, defined as follows: along γ we try to gauge transform the connection A to the trivial connection by solving the ODE

$$\mathrm{Ad}_{\varphi(t)}(A(t)) + \frac{d\varphi}{dt}\varphi^{-1}, \quad \varphi(0) = e.$$

This initial value problem has a unique solution leading to an element $\varphi(t) \in PG$, the space of paths in G. The value $\gamma(1) \in G$ is the holonomy. Varying the base point this is well defined up to conjugacy and because of the trace the resulting function is unambiguous. Witten's remarkable claim is that the formal integral

$$V_K(q = e^{2\pi\sqrt{-1}/(N+k)}) := \int_{\mathrm{Conn}(P)} W_K(A) \exp\left(2\pi\sqrt{-1}kCS(A)\right) DA, \tag{2.2}$$

gives the Jones polynomial!

2.3. **The Gauge group.** Recall that a morphism of principal G-bundles $P \to M$ and $P' \to M'$ is a smooth equivariant map $P \to P'$. It therefore induces a map $M \to M'$. The *gauge group* $\mathrm{Aut}(P)$ is defined as

$$\mathrm{Aut}(P) := \{\text{morphisms } \varphi : P \to P \text{ covering the identity map on } M\}.$$

Indeed such maps are automatically invertible.

The gauge group $\mathrm{Aut}(P)$ has a natural action on the space of connections $\mathrm{Conn}(P)$ by pull-back: for a connection $A \in \Omega^1(P, \mathfrak{g})$ one easily verifies that φ^*A is also a connection.

When $P = M \times G$ is the trivial bundle, the gauge group takes the simple form $\mathrm{Aut}(P) = C^\infty(M, G)$ acting on P via

$$\varphi \cdot (x, g) = (x, \varphi(x)g).$$

Indeed $\varphi : M \to G$ should act from the left to be able to commute with the right action. For the trivial bundle $\mathrm{Conn}(P) = \Omega^1(M, \mathfrak{g})$ and the action of $\mathrm{Aut}(P)$ can be written down explicitly as

$$\varphi \cdot A = \mathrm{Ad}_\varphi(A) + d\varphi\varphi^{-1}. \tag{2.3}$$

This new connection has curvature

$$R(\varphi^*A) = \mathrm{Ad}_\varphi(R(A)). \tag{2.4}$$

We now want to consider the behavior of the Chern–Simons action under the action of the gauge group.

Lemma 2.5. *$CS(\varphi^*A) - CS(A) \in \mathbb{Z}$.*

Proof. Recall the definition of the Chern–Simons action (2.1). Using invariance of B we claim that

$$L(\varphi^*A, A_s) = -L(A, \varphi^*A_s) = -L(A, A_{\varphi\cdot s}),$$

where $\varphi \in \mathrm{Aut}(P)$ acts by changing the section s. The Lemma therefore follows directly from (even better: is equivalent to) Lemma 2.3. □

We therefore see that $\exp 2\pi\sqrt{-1}CS(A)$ is invariant under the action of $\mathrm{Aut}(P)$ and therefore descends to the quotient space $\mathrm{Conn}(P)/\mathrm{Aut}(P)$.

3. THE MODULI SPACE OF FLAT CONNECTIONS

3.1. **Some more symplectic geometry: reduction.** Let (X, ω) be a symplectic manifold, and consider a smooth (left)-action of a Lie group K on X. We say that the action of K is *symplectic* if $k^*\omega = \omega$, $\forall k \in K$. Infinitesimally this means that $L_{\xi_X}\omega = 0$, where ξ_X is the generating vector field of the action of K associated to a Lie algebra element $\xi \in \mathfrak{k}$. Because ω is a closed form, this implies by Cartan's formula that $\iota_{\xi_X}\omega$ is closed. When it is also exact, we say the action is *Hamiltonian*. This means that there exists a smooth map $J : X \to \mathfrak{k}^*$ such that

$$d\langle J, \xi\rangle = d\iota_{\xi_X}\omega, \quad \text{for all} \xi \in \mathfrak{k}. \tag{3.1}$$

We shall assume that J is K-equivariant, where K acts on $\mathfrak{k}^*$ by the coadjoint action, the dual of the adjoint. The map J is called the moment (or momentum) map and gives the conserved quantities for the infinitesimal action of $\mathfrak{k}$ via Noether's theorem. In this situation *symplectic reduction* is a way to construct a new symplectic space, where the action of K is modded out.

Theorem 3.1 (Symplectic reduction)**.** *Assume that K acts properly and free, and that $0 \in \mathfrak{k}^*$ is a regular value of J so that that $J^{-1}(0) \subset X$ is smooth submanifold. Then the quotient space*

$$X_{\mathrm{red}} := J^{-1}(0)/K$$

carries a unique symplectic form ω_{red} satisfying $\pi^*\omega_{\mathrm{red}} = i_0^*\omega$, where $\pi : J^{-1} \to X_{\mathrm{red}}$ is the quotient map, and $i_0 : J^{-1}(0) \hookrightarrow X$ the inclusion.

Proof. The assumptions guarantee that X_{red} is a smooth manifold. Consider the 2-form $\omega_0 := i_0\omega$ on $J^{-1}(0)$. Clearly, it is K-invariant. Because of the moment map conditon (3.1) we see that

$$\iota_\xi\omega_0 = i_0^*\iota_\xi\omega = i_0^* d\langle J, \xi\rangle = di_0^*\langle J, \xi\rangle = 0.$$

We therefore find that it is K-basic as well, and therefore descends to the quotient X_{red}. We only are left to prove that ω_{red} is nondegenerate. Let $x \in J^{-1}(0)$, and consider the maps

$$\mathfrak{k} \xrightarrow{\rho_x} T_xX \xrightarrow{d_xJ} \mathfrak{k}^*. \tag{3.2}$$

The kernnel of d_xJ is exactly the annihilator of the image of ρ_x. Therefore on the quotient

$$T_{[x]}X_{\mathrm{red}} = \ker(d_xJ)/\mathrm{im}(\rho_x),$$

ω_{red} is exactly nondegenerate! $\square$

3.2. **The Atiyah–Bott construction.** As before we have fixed the Lie group G with an invariant integral inner product B. Let Σ be an oriented closed 2-manifold. We consider the space of connections $\mathrm{Conn}(P) = \Omega^1(\Sigma, \mathfrak{g})$ on the trivial G-bundle $P := \Sigma \times G$. Because Σ is two-dimensional, we can define the following bilinear form on $\Omega^1(\Sigma, \mathfrak{g})$:

$$\omega(\alpha, \beta) := \int_\Sigma B(\alpha, \beta), \quad \alpha, \beta \in \Omega^1(\Sigma, \mathfrak{g}).$$

Here the notation $B(\alpha, \beta)$ means that we combine the inner product on $\mathfrak{g}$ with the wedge product of α and β as 1-forms. Therefore, ω is antisymmetric, and one easily observes that it is also nondegenerate in the sense that

$$\omega(\alpha, \beta) = 0, \ \forall \beta \implies \alpha = 0.$$

It is therefore a symplectic form!

We now consider the action of the infinite dimensional group $\mathrm{Aut}(P)$ given by (2.3) on the infinite dimensional symplectic manifold $\mathrm{Conn}(P)$. We view $\mathrm{Aut}(P)$ as a Lie group with Lie algebra $\Omega^0(\Sigma, \mathfrak{g})$ and exponential map given by the pointwise (in Σ) exponential on G.

Lemma 3.2. *The generating vector field of the action* $\mathrm{Aut}(P)$ *is given by*

$$\nabla_A \xi, \quad \xi \in \Omega^0(\Sigma, \mathfrak{g}),$$

where ∇_A *denotes the covariant derivative on* $\mathrm{ad}(P) = \Sigma \times \mathfrak{g}$ *induced by a connection* A.

Proof. This is a small computation: write $\varphi_t := e^{t\xi}$ for $\xi \in C^\infty(\Sigma, \mathfrak{g})$, and compute the derivative with respect to t:

$$\left.\frac{d}{dt}\right|_{t=0} \varphi_t^* A = [A, \xi] + d\xi =: \nabla_A \xi.$$

This proves the Lemma. □

Proposition 3.3. *The action of* $\mathrm{Aut}(P)$ *on* $\mathrm{Conn}(P)$ *is Hamiltonian with momentum map given by minus the curvature of a connection:*

$$J(A) := -R(A).$$

Proof. Recall that the curvature $R(A)$ of a connection $A \in \mathrm{Conn}(P)$ is an element in $\Omega^2(\Sigma, \mathfrak{g})$ which we view as being dual to the Lie algebra $\Omega^0(\Sigma, \mathfrak{g})$ via the pairing

$$(\xi, \gamma) \mapsto \int_\Sigma B(\xi, \gamma), \quad \xi \in \Omega^0(\Sigma, \mathfrak{g}), \ \gamma \in \Omega^2(\Sigma, \mathfrak{g}).$$

Since

$$R(A + t\alpha) = R(A) + t\nabla_A \alpha + \mathcal{O}(t^2), \quad \alpha \in \Omega^1(\Sigma, \mathfrak{g}),$$

the derivative of the curvature map $A \mapsto R(A)$ is given by $\nabla_A : \Omega^1(\Sigma, \mathfrak{g}) \to \Omega^2(\Sigma, \mathfrak{g})$. With this we can check the momentum map condition (3.1). Let $\xi \in \Omega^1(\Sigma, \mathfrak{g})$ and $\alpha \in \Omega^1(\Sigma, \mathfrak{g})$:

$$\begin{aligned} \iota_\xi \omega(\alpha) &= \int_\Sigma B(\nabla_A \xi, \alpha) \\ &= -\int_\Sigma B(\xi, \nabla_A \alpha) \\ &= \langle \xi, d_A J(\alpha) \rangle . \end{aligned}$$

□

Corollary 3.4. *The symplectic quotient*

$$\mathcal{M}(\Sigma, G) := \mathrm{Conn}(P)_{\mathrm{flat}}/\mathrm{Aut}(P),$$

carries a symplectic form.

There is a very explicit description of this moduli space of flat connections:

Theorem 3.5. *There is an isomorphism*

$$\mathcal{M}(\Sigma, G) \cong \mathrm{Hom}(\pi_1(\Sigma), G)/G.$$

Proof. Exercise. □

Remark 3.6. In general, $\mathcal{M}(\Sigma, G)$ will not be smooth because $\mathrm{Aut}(P)$ does not quite act freely. We see this in the model above as the fact that the conjugacy action of G is not free. On the level of tangent spaces, a little thinking reveals that in the gauge theoretic picture the maps in equation (3.2) are given by

$$\Omega^0(\Sigma, \mathfrak{g}) \xrightarrow{\nabla_A} \Omega^1(\Sigma, \mathfrak{g}) \xrightarrow{\nabla_A} \Omega^2(\Sigma, \mathfrak{g}).$$

When $A \in \mathrm{Conn}(P)$ is flat, this is a complex: $\nabla_A^2 = 0$, and the tangent space of of $\mathcal{M}(\Sigma, G)$ is given by the first "de Rham cohomology group"

$$T_{[A]}\mathcal{M}(\Sigma, G) \cong \ker(\nabla_A)/\mathrm{im}(\nabla_A) = H^1_{dR}(\Sigma, \mathfrak{g}).$$

But this is only true in the *smooth points* where $H^0_{dR}(\Sigma, \mathfrak{g}) = H^2_{dR}(\Sigma, \mathfrak{g}) = 0$. In the following we will ignore singularities from the discussion and simply pretend $\mathcal{M}(\Sigma, G)$ is a smooth space.

3.3. **Prequantization.** To construct the quantum Chern–Simons theory, we need to quantize the symplectic space $\mathcal{M}(\Sigma, G)$. For this we use geometric quantization, so we need to find a prequantum line bundle. For this we use the fact that $\mathcal{M}(\Sigma, G)$ is constructed from the infinite dimensional symplectic space $\mathrm{Conn}(P)$ by symplectic reduction. As a vector space, there is a (up to isomorphism) unique prequantum line bundle (L, ∇, h) on $\mathrm{Conn}(P)$, where

- $L := \mathrm{Conn}(P) \times \mathbb{C}$ is the trivial line bundle,
- $\nabla := d + 2\pi\sqrt{-1}\theta$, with θ the 1-form on $\mathrm{Conn}(P)$ given by
$$\theta_A(\alpha) := \int_\Sigma B(A, \alpha), \quad A \in \mathrm{Conn}(P),\ \alpha \in \Omega^1(\Sigma, \mathfrak{g}),$$
- h is the standard hermitian metric on $\mathbb{C}$.

Indeed, if we compute the first Chern form of this bundle we find
$$c_1(L, \nabla) = d\theta = \omega,$$
as required for a prequantum line bundle. The aim is now to define a lift of the action of $\mathrm{Aut}(P)$ on $\mathrm{Conn}(P)$ to this prequantum line bundle preserving the connection and the hermitian metric. Then we can define a line bundle
$$L_{\mathrm{red}} := L|_{\mathrm{Conn}(P)_{\mathrm{flat}}} / \mathrm{Aut}(P),$$
with induced metric and connection. One easily checks that this defines a prequantum line bundle on $\mathcal{M}(\Sigma, G)$. So the question is really how to define the action of $\mathrm{Aut}(P)$ on L. For the answer to this question, we reconsider the Chern–Simons action.

Consider a 3-manifold with boundary $\partial M = \Sigma$, equipped with the trivial G-bundle $M \times G$.

Lemma 3.7. *Any connection* $A \in \mathrm{Conn}(P)$ *and gauge transformation* $\varphi \in \mathrm{Aut}(P)$ *can be extended to* M.

Proof. A connection $A \in \mathrm{Conn}(P) = \Omega^1(\Sigma, \mathfrak{g})$ is nothing but a differential form with values in the Lie algebra $\mathfrak{g}$, and therefore a section of a vector bundle. Therefore, a partition of unity argument shows that we can extend an element in $\Omega^1(\Sigma, \mathfrak{g})$ to one in $\Omega^1(M, \mathfrak{g})$. For the gauge group $\mathrm{Aut}(P) = C^\infty(\Sigma, G)$, first remark any element in the connected component of the unit can be written as $\varphi = e\xi$, with $\xi \in \Omega^0(\Sigma, \mathfrak{g})$. As before, as a section of a vector bundle, ξ will extend to an element in $\Omega^0(M, \mathfrak{g})$ and this defines an extension by taking the exponential. Because $\pi_2(G) = 0$ for any compact Lie group, $\pi_0(\mathrm{Aut}(P)) = 0$, so the previous argument suffices for the proof. □

Proposition 3.8. *Let* $\varphi \in \mathrm{Aut}(P)$ *and* $A \in \mathrm{Conn}(P)$. *For* φ' *and* A' *extending* φ *and* A, *the element in the circle group*

$$\Theta(\varphi, A) := \exp 2\pi\sqrt{-1}\left(CS((\varphi')^*A') - CS(A')\right),$$

only depends on the restrictions (φ, A) *of* (φ', A') *to the boundary, and satisfies the cocycle condition*

$$\Theta(\varphi_1, A)\Theta(\varphi_2, \varphi_1^*A) = \Theta(\varphi_1\varphi_2, A)$$

Proof. Consider two extensions (φ', A') and (φ'', A'') over 3-manifolds M' and M'' of the pair (φ, A). We can then glue M' and M'' over Σ to obtain a closed 3-manifold $M^{(3)} := M' \cup_\Sigma \overline{M}''$. Also the connections A' and A'' glue to a global connection $A_{(3)}$ and a gauge transformation $\varphi_{(3)}$. By Lemma 2.5 we see that

$$1 = \exp 2\pi\sqrt{-1}\left(CS(\varphi_{(3)}^*A_{(3)}) - CS(A_{(3)})\right) = \Theta(\varphi', A')\Theta(\varphi'', A'')^{-1}.$$

This proves the first claim. The cocycle property is immediate from the definition of $\Theta(\varphi, A)$. □

We are now in a position to define the lifting of the action to L. Let $\varphi \in \mathrm{Aut}(P)$ and $(A, z) \in \mathrm{Conn}(P) \times \mathbb{C} =: L$. Define

$$\varphi \cdot (A, z) := (\varphi^*A, \Theta(\varphi, A)z).$$

The cocycle property of Θ ensures that this indeed defines a group action. Furthermore, since $\Theta(\varphi, A) \in \mathbb{T}$, this action obviously preserves the hermitian metric on L. We therefore only need to check that this action preserves the connection ∇. For this we first need the following

Lemma 3.9. *The derivative of the cocycle* Θ *is given by:*

$$\left.\frac{d}{dt}\right|_{t=0} \Theta(\varphi, A + t\alpha) = \Theta(\varphi, A)2\pi\sqrt{-1}\int_\Sigma B(\mathrm{Ad}_\varphi(\alpha), d\varphi\varphi^{-1}).$$

Proof. Reconsidering the proof of Lemma 2.4, and taking into account that now $\partial M \neq \emptyset$, we easily derive that

$$\left.\frac{d}{dt}\right|_{t=0} CS(A' + t\alpha') = 2\int_M B(\alpha', R') + \int_\Sigma B(\alpha, A).$$

Likewise, a small computation shows that for the other term

$$\begin{aligned}\left.\frac{d}{dt}\right|_{t=0} CS((\varphi')^*(A' + t\alpha')) &= \left.\frac{d}{dt}\right|_{t=0} CS((\varphi')^*A' + t\mathrm{Ad}_{\varphi'}(\alpha')) \\ &= 2\int_M B(\mathrm{Ad}_{\varphi'}\alpha', \mathrm{Ad}_{\varphi'}(R')) + \int_\Sigma B(\mathrm{Ad}_\varphi(\alpha), \varphi^*A).\end{aligned}$$

Substracting the two, and taking into account that B is invariant, the result now follows. $\square$

Let $s : \mathrm{Conn}(P) \to \mathbb{C}$ be a section of L. With the lift of the action, $\mathrm{Aut}(P)$ acts on s by

$$(\varphi \cdot s)(A) = \Theta(\varphi, A)s(\varphi^* A).$$

Let us now verify that this action commutes with taking the covariant derivative. Recall that $\nabla = d + 2\pi\sqrt{-1}\theta$, so the commutator between d and the action of φ is given by the derivative of Θ computed in the Lemma above as $2\pi\sqrt{-1}(\varphi^*\theta - \theta)$. On the other hand, since this is exactly minus the commutator between $2\pi\sqrt{-1}\theta$ and the action of φ, the result follows.