EXERCISE SHEET I: NOETHER'S THEOREM

Noether's theorem describes the principle that (infinitesimal) symmetries lead to conservation laws. Here we shall proves two versions of the theorem, one in Hamiltonian mechanics, the other in the Lagrangian approach. We begin with the Lagrangian approach.

Exercise 1. Let $L : TM \to \mathbb{R}$ be a Lagrangian function. Let $x = (x^1, \ldots, x^n)$ be local coordinates on M. We denote the induced local coordinates on TM by $(x^1, \ldots, x^n, \xi^1, \ldots, \xi^n)$

a) Show that the function E_L and 1-form α_L locally given by

$$E_L := \sum_i \xi^i \frac{\partial L}{\partial \xi^i} - L, \quad \alpha_L := \sum_i \frac{\partial L}{\partial \xi^i} dx^i$$

are globally well-defined, i.e., independent of the choice of local coordinates.

b) Show that with these definitions, the Euler–Lagrange equations for L can be written as

$$\iota_{\ddot{\gamma}} d\alpha_L = -dE_l(\dot{\gamma})$$

c) A diffeomorphism $\Phi : M \to M$ is said to be a *symmetry* if its tangent map $T\Phi : TM \to TM$ preserves L: $L \circ T\Phi = L$. Show that $T\Phi$ preserves E_L and ω_L:

$$(T\Phi)^* E_L = E_L, \quad (T\Phi)^* \alpha_L = \alpha_L.$$

d) A vector field X on M is said to be an *infinitesimal symmetry* if its local flow is a symmetry. Denote by X' its tangent lift, i.e., the infinitesimal generator of the tangent lift of the local flow. Show that $\iota_{X'} \alpha_L$ is a *first integral*, i.e., constant along solutions to the Euler–Lagrange equations. This is Noether's theorem.

Exercise 2. Recall that the basic set-up of Hamiltonian mechanics is given by a symplectic manifold (M, ω) together with a smooth function H, the *Hamiltonian*. A diffeomorphism $\Phi : M \to M$ is said to be a *symmetry* if it preserves the symplectic form and the Hamiltonian:

$$\Phi^* \omega = \omega, \quad \Phi^* H = H.$$

A vector field X is called an *infinitesimal symmetry* if its local flow Φ_t^X is a symmetry.

a) Let X be an infinitesimal symmetry. Show that when $H^1(M, \mathbb{R}) = 0$, X is the Hamiltonian vector field of a function $f \in C^\infty(M)$ which Poisson-commutes with H:

$$\{H, f\} = 0.$$

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b) Show that f is constant along the flow lines given by solutions to the Hamilton equations: f is a *first integral* (conserved quantity).

c) Suppose that M is a cotangent bundle: $M = T^*N$ and that $\Phi : M \to M$ is the cotangent lift $\Phi := T^*\varphi$ of a diffeomorphism $\varphi : N \to N$. Show that Φ preserves the Liouville form $\Phi^*\theta = \theta$, and therefore also the symplectic form.

d) Now let Y be a vector field on N whose cotangent lift of the local flow is a symmetry. Write down a first integral for Y.