EXERCISE SHEET II

The discussion about quantization in §4.1. of the notes bears great resemblance to the well-known PBW-theorem in Lie theory. The following exercise aims to make this more visible.

Exercise 1 (The Poincaré–Birkhoff–Witt theorem)**.** Let $\mathfrak{g}$ be a finite dimensional Lie algebra (over $\mathbb{R}$). Its *universal enveloping algebra* $\mathcal{U}(\mathfrak{g})$ is defined as the quotient of the tensor algebra $T(\mathfrak{g})$ by the two-sided ideal

$$I := \langle X \otimes Y - Y \otimes X - [X,Y],\ X, Y \in \mathfrak{g} \rangle .$$

a) Which algebra do you get when $\mathfrak{g}$ is abelian, i.e., $[\ ,\ ] = 0$? When is the $\mathcal{U}(\mathfrak{g})$ commutative?

b) Denote by F_k the vector subspace generated by elements of the form $X_1 \cdots X_l$, with $X_i \in \mathfrak{g}$, and $l \leq k$ i.e., products of up to k elements in $\mathfrak{g}$. Show that, with $F_0 = \mathbb{R}$, this turns $\mathcal{U}(\mathfrak{g})$ into a filtered algebra.

c) Show that the graded quotient of $\mathcal{U}(\mathfrak{g})$ is a commutative algebra and that

$$X_1 \cdots X_k \mapsto \frac{1}{k!} \sum_{\sigma \in S_k} X_{\sigma(1)} \cdots X_{\sigma(k)}$$

defines a map $\mathrm{Sym}(\mathfrak{g}) \to \mathcal{U}(\mathfrak{g})$ which descends to a morphism $\mathrm{Sym}(\mathfrak{g}) \to \mathrm{Gr}(\mathcal{U}(\mathfrak{g}))$ of commutative algebras. (Here, $\mathrm{Sym}(\mathfrak{g})$ denotes the symmetric algebra of the vector space underlying $\mathfrak{g}$, i.e., the algebra of polynomials on the dual vector space $\mathfrak{g}^*$.)

d) The Poincaré–Birkhoff–Witt theorem asserts that the morphism in c) is an isomorphism. (you don't have to prove this!) Use this to show that the commutator of elements in $\mathcal{U}(\mathfrak{g})$ induces a Lie bracket on $\mathrm{Sym}(\mathfrak{g})$. Show that with this bracket, $\mathrm{Sym}(\mathfrak{g})$ is a Poisson algebra. What is the bracket of linear elements in $\mathfrak{g} \subset \mathrm{Sym}(\mathfrak{g})$?

e) Let $\langle\ ,\ \rangle$ be an inner product on $\mathfrak{g}$ satisfying

$$\langle [X,Y], Z \rangle + \langle Y, [X,Z] \rangle = 0, \quad \text{for all } X, Y, Z \in \mathfrak{g}.$$

Show that the associated quadratic polynomial Poisson commutes with all other elements in $\mathrm{Sym}(\mathfrak{g})$.

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The following exercise shows one of the fundamental techniques to construct the heat kernel in a toy model:

Exercise 2. Let $V = \mathbb{R}^n$ and let $H \in \mathrm{End}(V)$ a matrix. We are looking for a solution to the "heat equation"

$$\frac{dK}{dt} + HK = 0, \quad K(0) = \mathrm{id}_V,$$

for $K : \mathbb{R}_{>0} \to \mathrm{End}(V)$.

a) Solve the ODE above directly.
b) Suppose now that we have already found an "asymptotic solution", i.e.,

$$\frac{dK}{dt} + HK = R_t, \quad \|R_t\| < Ct^\alpha, \text{ for some } \alpha > 0,$$

and $K(0) = \mathrm{id}_V$. Define

$$Q^k(t) := \int_{t\Delta^k} K_{t-t_k} R_{t_k - t_{k-1}} \cdots R_{t_2 - t_1} R_{t_1} dt_1 \cdots dt_k,$$

where $t\Delta^k = \{(t_1, \dots, t_k),\ 0 \le t_1 \le \dots \le t_k \le t\}$ denotes the rescaled k-simplex. We put $Q^0(t) = K(t)$. Show that

$$\left(\frac{d}{dt} + H\right) Q^k = R^{(k+1)} + R^{(k)},$$

where

$$R^{(k)}(t) = \int_{t\Delta^k} R_{t-t_k} R_{t_k - t_{k-1}} \cdots R_{t_2 - t_1} R_{t_1} dt_1 \cdots dt_k$$

Hint: use the following easy fact from calculus about convolution on $\mathbb{R}$:

$$\frac{d}{dt}\int_0^t f(t-s)g(s)ds = \int_0^t \frac{df}{dt}(t-s)g(s)ds + f(0)g(t).$$

c) Show that $\sum_{k\ge 0}(-1)^k Q^k(t)$ converges and solves the heat equation. How does this relate to a)?
d) As an application, let $H = H_0 + H_1$ and consider $K(t) = e^{-tH_0}$ as an asymptotic solution to the heat equation of H. Derive a series expansion of the form

$$e^{-t(H_0+H_1)} = e^{-tH_0} + \sum_{k\ge 1}(-1)^k I_k.$$

Give a formula for the I_k's.

Exercise 3 (The heat kernel on S^1)**.** In this exercise we consider the heat kernel for the standard Laplacian $\Delta_{S^1} = -d^2/dx^2$ on S^1. Here $x \in \mathbb{R}/2\pi\mathbb{Z}$ is the canonical global coordinate.

a) Use Fourier theory to determine the eigenfunctions and eigenvalues of the Laplacian and with this give a proof of Theorem 7 of the lecture notes. Use this to write down the heat kernel $K^{S^1}(x,y)$ and verify that it indeed satisfies the heat equation, as well as the initial condition

$$\lim_{t\downarrow 0}\int K_t^{S^1}(x,y)f(y)dy = f(x).$$

b) Prove that

$$K^{S^1}(x,y) = \sum_{n\in\mathbb{Z}} K_t^{\mathbb{R}}(x,y+2\pi n).$$

(Hint: use uniqueness of the heat kernel) Why would you expect such a result intuitively to be true?

c) Take the trace of the heat kernel and deduce the following result of Jacobi:

$$\sum_{n\in\mathbb{Z}} e^{-n^2 t} \sim \sqrt{\frac{\pi}{t}}, \quad t\to 0.$$

What happens for a circle of radius $\ell \in \mathbb{R}_+$? Relate to Weyl's asymptotic result.