EXERCISE SHEET III

Exercise 1. We consider an n-dimensional TQFT.

a) Let Σ be an $(n-1)$-dimensional oriented manifold and let $\varphi : \Sigma \to \Sigma$ be an orientation preserving diffeomorphism. Denote by C_φ the cobordism given by $\Sigma \times I$, where we identify the outgoing boundary with Σ via the identity map, and the incoming boundary using φ. Prove that in the category Bord_n^{or} we have

$$C_{\varphi_1} \circ C_{\varphi_2} = C_{\varphi_1 \circ \varphi_2}.$$

b) Prove that two diffeomorphisms that are smoothly homotopic induce the same cobordism class. Use this fact to show that the vector space $Z(\Sigma)$ of a TQFT carries a representation of the group $\pi_0(\mathrm{Diff}^+(\Sigma))$ (this is the mapping class group) .

Exercise 2. Consider the TQFT associated to a finite group. Work out the details of this TQFT in 2 dimensions. (What is the corresponding Frobenius algebra?)

Exercise 3 (Principal bundles)**.**

a) Let $P \to M$ be a principal bundle equipped with a *flat* connection A. Show that the corresponding horizontal distribution in TP is integrable. Use Frobenius' theorem to turn P (by changing the topology) into a covering of M. With this, show that A determines a homomorphism

$$\pi_1(M) \to G, \quad \gamma \mapsto \mathrm{Hol}_\gamma(A).$$

(Hol stands for *holonomy*).

b) Use the correspondence in a) to show that

$$\begin{aligned} \mathcal{M}(M,G) :&= \{\text{flat connections}\}/\{\text{gauge transformations}\} \\ &\cong \mathrm{Hom}(\pi_1(M), G)/G. \end{aligned}$$

c) When $G = \mathbb{T}$, show that $\mathcal{M}(M,G) \cong H^1(M,\mathbb{T})$. Given a flat line bundle (L,∇) use the Čech complex together with local trivializations, to construct an explicit cocycle representing the class in $H^1(M,\mathbb{T})$.

Exercise 4. Let Σ be an orientable 2D surface of genus g. Given a compact Lie group G, we have constructed in the lectures the moduli space of flat connections on a trivial principal G-bundle:

$$\mathcal{M}(\Sigma, G) \cong \mathrm{Hom}(\pi_1(\Sigma, x_0), G)/G.$$

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As in the lectures, we will ignore the unpleasant fact that this space has singularities.

We can construct observables (=smooth functions) on $\mathcal{M}(\Sigma, G)$ as follows: Let γ be a closed curve in Σ and choose a finite dimensional representation V of G. For any flat connection A on P, the *Wilson loop* is defined as

$$W_\gamma(A) := \mathrm{Tr}_V(\mathrm{Hol}_\gamma(A)),$$

where $\mathrm{Hol}_\gamma(A) \in G$ denotes the holonomy of A along γ.

a) Show that W_γ is invariant under the action of the gauge group and therefore defines a function on $\mathcal{M}(\Sigma, G)$. Show that when γ is contractible, $W_\gamma = 0$.

b) Argue that, when $\gamma_1 \cap \gamma_2 = \emptyset$, we have

$$\{W_{\gamma_1}, W_{\gamma_2}\} = 0.$$

c) Let $G = SU(2)$ with

$$B(X, Y) := \mathrm{Trace}(XY).$$

Show that, by choosing suitable loops γ, $\mathcal{M}(\Sigma, SU(2))$ carries an *integrable system* in the following sense:

Definition 1. An *(completely) integrable system* on a smooth $2n$-dimensional symplectic manifold (X, ω) is an n-tuple of smooth functions $f_1, \ldots, f_n \in C^\infty(X)$, satisfying $df_1(x) \wedge \ldots \wedge df_n(x) \neq 0$, for almost all each $x \in X$, as well as

$$\{f_i, f_j\} = 0,$$

for all $i, j \in \{1, \ldots, n\}$.

Exercise 5 (Abelian Chern–Simons theory)**.** In this exercise we consider Chern–Simons theory for the circle group $\mathbb{T}$. Before getting started, let's recall some standard Gaussian integrals. Let $\langle x, Qx\rangle$ be a nondegenerate quadratic form on $\mathbb{R}^n$ and let $v \in \mathbb{R}^n$ be a fixed vector. Define

$$Z(v) := \int_{\mathbb{R}^n} e^{-\frac{\sqrt{-1}}{2}\langle x, Qx\rangle + \sqrt{-1}\langle v, x\rangle} dx$$

By completing the square, one easily proves that

$$\frac{Z(v)}{Z(0)} = e^{-\frac{\sqrt{-1}}{2}\left\langle x, Q^{-1}x\right\rangle}$$

a) Identify $\mathrm{Conn}(P) \cong \Omega^1(M)$ by writing a connection as $\sqrt{-1}\alpha$, $\alpha \in \Omega^1(M)$. Show that for a curve γ,

$$\mathrm{Hol}_\gamma(A) = e^{\sqrt{-1}\int_\gamma \alpha}.$$

b) Use this to write the Witten invariant (see equation (2.2.) of the notes) for a link $L = K_1 \cup K_2$ in $\mathbb{R}^3$ as

$$Z(K_1, K_2) = \int_{\Omega^1(\mathbb{R}^3)/d\Omega^0(\mathbb{R}^3)} \exp\left(\frac{\sqrt{-1}}{4\pi}\int_{\mathbb{R}^3} \alpha \wedge d\alpha + \sqrt{-1}\int_{K_1} \alpha + \sqrt{-1}\int_{K_2} \alpha\right) D\alpha.$$

(don't worry about the normalization of the Chern–Simons action).

c) Write the last two terms in the exponent as

$$\int_{K_i} \alpha = \int_{\mathbb{R}^3} \alpha \wedge \delta_{K_i},$$

where δ_{K_i} is a "δ-type 2-form" supported at K_i, $i = 1, 2$. (This is known as a *de Rham current*.) With this, we can view $\int_{\mathbb{R}}^3 \alpha \wedge d\alpha$ as a nondegenerate quadratic form on $\Omega^1(\mathbb{R}^3)/d\Omega^0(\mathbb{R}^3)$ and formally apply the Gaussian integral formula above to obtain

$$\frac{Z(K_1, K_2)}{Z(\emptyset)} = \exp\left(-\frac{\sqrt{-1}}{4\pi} \sum_{i,j} \int_{\mathbb{R}^3} \delta_{K_i} \wedge d^{-1}\delta_{K_j}\right)$$

d) Consider the following 2-form on $\mathbb{R}^3$:

$$\omega(x) = \frac{1}{4\pi} \frac{x_1 dx_2 \wedge dx_3 + x_2 dx_3 \wedge dx_1 + x_3 dx_1 \wedge dx_2}{||x||^3}$$

Show that ω satisfies

$$d\omega = \delta_0.$$

With this, show that

$$d_x \int_{K_i} \omega(x - y) dy = \delta_{K_i}.$$

With this, arrive at the final result that

$$\log \frac{Z(K_1, K_2)}{Z(\emptyset)} = \sqrt{-1}\pi \sum_{i,j} \int_{x \in K_i, y \in K_j} \omega(x - y)$$

This is the Gauss linking number.

Exercise 6. The goal of this exercise is to understand better why the model explained §8.1 of the Lecture notes is the same as $2D$ Yang–Mills theory.

a) Let (Σ, g) be a two-dimensional riemannian manifold, and fix a compact Lie group with an invariant inner product $\langle\ ,\ \rangle$ on its Lie algebra $\mathfrak{g}$. We consider the trivial principal G-bundle $G \times \Sigma$. On this bundle, a connection is given by $\alpha \in \Omega^1(\Sigma, \mathfrak{g})$, a $\mathfrak{g}$-valued one-form. Its *curvature* is given by

$$F_\alpha := d\alpha + \frac{1}{2}[\alpha, \alpha] \in \Omega^2(\Sigma, \mathfrak{g}).$$

Here the bracket means taking the Lie bracket on $\mathfrak{g}$ combined with the wedge of the one-forms: this is a symmetric operation.)

The action of $2D$ Yang–Mills is given by the function $S : \Omega^1(\Sigma, \mathfrak{g}) \to \mathbb{R}$ defined by

$$S(\alpha) := \frac{1}{2} \int_\Sigma \langle F_\alpha, *F_\alpha \rangle .$$

Explain why this action only depends on the induced volume form of the metric, and therefore it is to be expected that $2D$ Yang–Mills theory is an area dependent QFT.

b) We consider a lattice approximation of the theory. For this we pick a triangulation of the surface Σ, and we write Σ_0, Σ_1 and Σ_2 for the set of 0, 1 and 2 simplices. Instead of the connection itself, we now consider its holonomy along the 1-simplices which gives a map $h : \Sigma_1 \to G$. To a 2-simplex $\sigma = [v_0, v_1, v_2]$ in the triangulation we then associate

$$K_h(\sigma) := h(v_0 v_1) h(v_1 v_2) h(v_2 v_0),$$

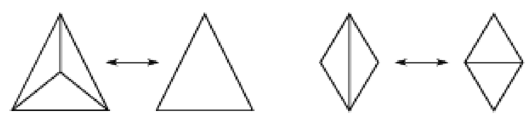
and this defines a map $K_h : \Sigma_2 \to G$. We now consider the following expression[1]:

$$e^{-S(h)} := \prod_{\sigma \in \Sigma_2} \epsilon_{t(\sigma)}(K_h(\sigma)),$$

where $t(\sigma)$ is the area of σ and ϵ_t is as in the lecture notes. Proof that the approximation to the path integral

$$\int_{G^{\times |\Sigma_1|}} e^{-S(h)} dh$$

is independent of the triangularization. *Hint: it is known that any two triangulations of a manifold are related by a finite sequence of moves, the so-called Pachner moves. In two dimensions, there are two of them, namely:*



c) Can you show that the partition function in b) coincides with the one coming from the model in §8.1. of the lecture notes?

[1]Using the asymptotics for the heat kernel for $t \to 0$, one can show that this expression approximates the exponential of the action in equation a) as the triangulation gets finer. If you have the courage, show this yourself.