Mock exam Topology in Physics 2018

May 24, 2018

This is the **mock exam** for the course topology in physics academic year 2017/2018. Some remarks beforehand:

- This mock exam was made with short notice and may therefore contain some typos and we apologize for those in advance, we also guarantee that the actual exam will have no such mistakes and will of course be present at this exam.
- This mock exam should give an indication of the type/style and difficulty of questions on the exam, we do not guarantee that the exam covers the exact same topics as this mock exam. Any topics discussed in the relevant lectures of the course (depending on whether you take the 6 EC or 8 EC version) could be on the exam
- While the mock exam does not have a marked problem that can be skipped for the 6 EC version of the course the actual exam will have such a problem.
- While the mock exam does not have an indication of the amount of points that can be earned per problem the actual exam will have such an indication for each subproblem.

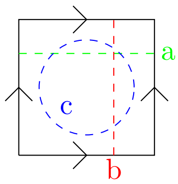
You can ask questions about the problems on the mock exam during the next lecture on the 28th of May.

Quickfire Questions

We will start the exam off with a lightning round. This means you do **not** need to motivate your answers for these quickfire questions.

i) Ind $\begin{pmatrix} 5 & 10 & 3 & 7 \\ 29 & 31 & 11 & 9 \\ 101 & 97 & 4 & 13 \end{pmatrix} = ?.$

ii) Consider the following picture of a 2-torus as a unit square in the usual (x, y) plane with opposite ends identified.



Over which of the dashed loops does the integral of dx vanish?

iii) Which of the following integrals in Maxwell theory is a topological invariant:

$$\frac{1}{4\pi}\int_{S^2} \vec{E}\cdot d\vec{\sigma}, \quad \frac{1}{4\pi}\int_{S^2} \vec{B}\cdot d\vec{\sigma}.$$

Problem 1: The Dirac monopole

In this exercise, we consider three-dimensional space with a single point removed: $M = \mathbb{R}^3 \setminus (0,0,0)$.

a. Using de Rham's theorem, explain why the second de Rham cohomology group of M is nontrivial: $H^2_{dR}(M) \neq \{0\}$.

Instead of rectangular coordinates x, y and z, we will use polar coordinates

$$\begin{aligned} x &= r\sin\theta\cos\phi \\ y &= r\sin\theta\sin\phi \\ z &= r\cos\theta. \end{aligned} \tag{1}$$

b. Show that in polar coordinates, the volume 3-form $\Omega = dx \wedge dy \wedge dz$ can be written as $\Omega = r^2 \sin\theta dr \wedge d\theta \wedge d\phi$.

One way to construct a nontrivial element of $H^2_{dR}(M)$ is as follows. Begin by writing

$$\Omega = dr \wedge \omega, \tag{2}$$

where ω is of course a 2-form. Now let S^2 be a round 2-sphere of radius R around the origin.

c. Argue that

$$\int_{S^2} \omega = 4\pi R^2, \tag{3}$$

that is: ω is the volume form on the S^2.
Hint: you don't need a lengthy computation involving sines and cosines; you can for example start by integrating Ω over a shell of thickness δr around the S^2 and then let $\delta r \to 0$.

Now, we remove the r-dependence by defining

$$F = \frac{\omega}{4\pi r^2} \tag{4}$$

d. Show that F is a closed form on M, but that it is not exact. Could we also have viewed F as a closed form on $\mathbb{R}^3$? Why (not)?

As you know, the field configuration F describes the Dirac magnetic monopole. To describe it mathematically, one can use the exact sequence

$$0 \xrightarrow{f_0} H^1_{\mathrm{dR}}(M) \xrightarrow{f_1} \Omega^1(M)/d\Omega^0(M) \xrightarrow{f_2} \Omega^2_{\mathrm{cl}}(M) \xrightarrow{f_3} H^2_{\mathrm{dR}}(M) \xrightarrow{f_4} 0. \tag{5}$$

e. Describe the maps f_i appearing in this sequence. (You do ***not*** have to prove that the sequence is exact!)

Since in our setup, $H^1_{\mathrm{dR}}(M) = \{0\}$, one obtains from this sequence that

$$H^2_{\mathrm{dR}}(M) \cong \frac{\Omega^2_{\mathrm{cl}}(M)}{\Omega^1(M)/d\Omega^0(M)}. \tag{6}$$

f. Using this identity and the field strength we have studied in this exercise, explain why "gauge fields modulo gauge transformations" is not always sufficient to describe a physical setup.
Note: no computations are required; an explanation of the physical interpretation of the above identity and how the Dirac monopole example fits in there suffices.

Problem 2: The Signature

Recall that given an oriented manifold M of dimension $n = 4k$ we can define the signature $Sign(M)$ as follows. First we note that there is a bilinear pairing

$$Q\colon H^{2k}_{\mathrm{dR}}(M) \times H^{2k}_{\mathrm{dR}}(M) \longrightarrow \mathbb{C}$$

given by

$$Q(\alpha, \beta) = \int_M \alpha \wedge \beta.$$

Then given a basis $(\alpha_i)_{i=1}^r$ of $H^{2k}_{\mathrm{dR}}(M)$ we set

$$Sign(M) = \#\{1 \leq i \leq r \mid Q(\alpha_i, \alpha_i) > 0\} - \#\{1 \leq i \leq r \mid Q(\alpha_i, \alpha_i) < 0\}$$

and note that this is independent of the chosen basis. Above we have implicitely assumed that differential forms are defined with *complex* values.

Recall that $\mathbb{CP}$ is the 4 dimensional manifold of (complex) lines in $\mathbb{C}^3$. As a set this is given by equivalence classes $[(z_1, z_2, z_3)]$ where $(z_1, z_2, z_3) \sim (w_1, w_2, w_3)$ if there is $0 \neq \lambda \in \mathbb{C}$ such that $\lambda z_i = w_i$ for $i = 1, 2, 3$. In the lectures we considered the atlas given by the three charts

$$U_i := \{[(z_1, z_2, z_3)] \mid z_i \neq 0\}$$

and we considered $U_1 \simeq \mathbb{C}^2$ by the map $\phi_1([(z_1, z_2, z_3)]) = (\frac{z_2}{z_1}, \frac{z_3}{z_1})$ and similar for U_2 and U_3.

a. Use the Mayer–Vietoris sequence to determine the cohomology of $\mathbb{CP}^2$.

b. Show that $|Sign(\mathbb{CP}^2)| = 1$.

Suppose M is equipped with a metric g and recall the Hodge star operator $\star\colon \Omega^p(M) \to \Omega^{n-p}(M)$. Recall in particular the fact that $\star^2\alpha = (-1)^{np+p}\alpha$ on p-forms α. Also set

$$\delta\alpha = (-1)^{np+n+1} \star d\star$$

and recall that it is the dual of d.

c. Consider the operator $\epsilon\colon \Omega^p(M) \to \Omega^{n-p}(M)$

$$\epsilon\alpha = i^{2k+p(p-1)} \star \alpha.$$

How does ϵ define a grading

$$\Omega^\bullet(M) = \Omega^+(M) \oplus \Omega^-(M)$$

such that $d + d*$ maps $\Omega^+(M)$ into $\Omega^-(M)$?
Note: Really use a property of ϵ, we are not looking for the grading $\Omega^{even}(M) \oplus \Omega^{odd}(M)$.

It turns out that the operator

$$S := d + d^* : \Omega^+(M) \longrightarrow \Omega^-(M)$$

has Ind $S = \mathit{Sign}\ M$. Recall that the Atiyah–Singer index theorem states that

$$\text{Ind } S = (-1)^n \int_M \text{Ch } \left(\bigwedge^+ T^*M - \bigwedge^- T^*M\right) \frac{\text{Td }((TM)_{\mathbb{C}})}{\text{e }(TM)}.$$

If we pick a connection on the tangent bundle so that the corresponding connection on the complexified tangent bundle may be diagonalized with eigenvalues $2\pi x_1, -2\pi x_1, \ldots, 2\pi x_n, -2\pi x_n$ we may express the integrand

$$(-1)^n \text{Ch } \left(\bigwedge^+ T^*M - \bigwedge^- T^*M\right) \frac{\text{Td }((TM)_{\mathbb{C}})}{\text{e }(TM)}$$

as a symmetric polynomial in the x_i. We will spare you the formal power series yoga which shows that this polynomial agrees in the nth degree with the polynomial

$$\prod_{i=1}^{n} \frac{x_i}{\tanh x_i}$$

corresponding to the so-called L-class of the tangent bundle TM, denoted $L(TM)$. Thus, the Atiyah–Singer index theorem gives

$$\mathit{Sign}(M) = \int_M L(TM).$$

d. Argue that for vector bundles E and F we have

$$L(E \oplus F) = L(E) \wedge L(F).$$

e. Use the fact in d. and the fact that

$$H^\bullet_{\text{dR}}(S^4) = \begin{cases} \mathbb{R} & \text{if } \bullet = 0, 4 \\ 0 & \text{otherwise} \end{cases}$$

to argue the value of $|\mathit{Sign}(S^4 \times \mathbb{CP}^2)|$.

Problem 3: Spinors in odd dimensions

We consider the Clifford algebra $\mathrm{Cliff}_n(\mathbb{R})$ generated by $\psi_i,\ i = 1, \ldots, n$ satisfying

$$\begin{aligned} \psi_i\psi_j &= -\psi_j\psi_i, \quad i \neq j, \\ \psi_i^2 &= -1. \end{aligned}$$

We assume n to be **odd**.

a) Show that the element

$$\eta := \psi_1 \cdots \psi_n$$

lies in the center of $\mathrm{Cliff}_n(\mathbb{R})$, i.e., commutes with all elements.

b) By adding one extra generator ψ_{n+1} we can embed $\mathrm{Cliff}_n(\mathbb{R}) \subset \mathrm{Cliff}_{n+1}(\mathbb{R})$. Let V now be a vector space carrying a representation of $\mathrm{Cliff}_{n+1}(\mathbb{R})$, e.g. the spinor representation. We now consider the action of $\mathrm{Cliff}_n(\mathbb{R})$ on V via the embedding above. Use the element η to decompose $V = V^+ \oplus V^-$ into two subspaces that are invariant under the action of $\mathrm{Cliff}_n(\mathbb{R})$.

c) Is the Dirac operator

$$D = \sum_i \psi_i \frac{\partial}{\partial x^i},$$

acting on functions in $C^\infty(\mathbb{R}^n, V)$ an even or odd operator with respect to the grading in b)?