EXERCISESET 1, TOPOLOGY IN PHYSICS

- The hand-in exercise is the exercise 1.
- Please hand it in electronically at `topologyinphysics2019@gmail.com` (1 pdf!)
- Deadline is Wednesday February 13, 23.59.
- Please make sure your name and the week number are present in the file name.

Exercise 1: Computing $H^\bullet_{dR}(S^n)$.

Definition 0.1 (Homotopy Equivalence)**.**
A (smooth) homotopy equivalence between two manifolds M and N is given by a pair of smooth maps

$$f\colon M \longrightarrow N \quad \text{and} \quad g\colon N \longrightarrow M$$

such that $f \circ g$ is smoothly homotopic to Id_N and $g \circ f$ is smoothly homotopic to Id_M.

Note that homotopy equivalence defines an equivalence relation on smooth manifolds, which we denote $\sim_h$.

i): Show that $N \sim_h M$ implies that $H^\bullet_{\mathrm{dR}}(N) \simeq H^\bullet_{\mathrm{dR}}(M)$.

ii): How do we use the result of i) in the Poincaré lemma?

iii): Using the definition of $H^\bullet_{\mathrm{dR}}$ in terms of differential forms show that $H^\bullet_{\mathrm{dR}}(M \coprod N) \simeq H^\bullet_{\mathrm{dR}}(M) \oplus H^\bullet_{\mathrm{dR}}(N)$.

iv): You will now compute the cohomology of the n-sphere by decomposing it into two opens sets and applying the Mayer–Vietoris sequence.

- **a):** Use the description of H^0_{dR} (or the definition) to show that we have $H^0_{\mathrm{dR}}(S^0) = \mathbb{R}^2$ and $H^0_{\mathrm{dR}}(S^n) = \mathbb{R}$ for $n > 0$.
- **b):** Find two open subsets U and V of S^n such that $U \cap V \sim_h S^{n-1}$ (also show why they are homotopy equivalent).
- **c):** Apply the Mayer–Vietoris sequence to find that $H^1_{\mathrm{dR}}(S^n) = \mathbb{R}^{\delta_{1n}}$.
- **d):** Apply the Mayer–Vietoris sequence and the result of c) to compute the comohology of S^n for any $n \geq 0$ as

$$H^k_{\mathrm{dR}}(S^n) = \mathbb{R}^{\delta_{k0}+\delta_{kn}}. \tag{1}$$

Exercise 2: Computing $H^\bullet_{dR}(\mathbb{T}^2)$. In this exercise we will compute the cohomology of the 2-torus $\mathbb{T}^2$. We consider the flat model of the 2-torus as the space $\mathbb{R}^2/\mathbb{Z}^2$, i.e. we consider the plane and identify points (x_1, y_1) and (x_2, y_2) if $x_1 - x_2$ and $y_1 - y_2$ are both integers.

i): Show that $\mathbb{T}^2$ is given by considering the square $[0,1] \times [0,1] \subset \mathbb{R}^2$ and identifying the points $(0,t)$ with $(1,t)$ for $t \in [0,1]$ as well as identifying the points $(s,0)$ with $(s,1)$ for $s \in [0,1]$.

ii) [Bonus]: What does this model of $\mathbb{T}^2$ have to do with snake?

iii): Compute the cohomology of $\mathbb{T}^2$ by decomposing it into two open subsets U_{outer} and U_{middle} such that you already know the cohomology of U_{middle}, U_{outer} and $U_{middle} \cap U_{outer}$ and applying the Mayer–Vietoris sequence.

Exercise 3: The Hopf invariant. If we consider the n-sphere S^n as embedded in $\mathbb{R}^{n+1}$ as the submanifold defined by

$$\sum_{i=1}^{n+1} (x^i)^2 = 1,$$

we can write its volume form $\omega \in \Omega^n(S^n)$ as

$$\omega := \sum_{i=1}^{n+1} (-1)^{i+1} x^i dx^1 \wedge \ldots \wedge \widehat{dx^i} \wedge \ldots \wedge dx^{n+1}$$

This is a closed form generating the cohomology of S^n in degree n as in (1). We now consider a smooth map $f : S^{2n-1} \to S^n$.

i) Show that $f^*\omega$ is exact: $f^*\omega = d\alpha$ for some $\alpha \in \Omega^{n-1}(S^{2n-1})$.

ii) Show that the integral

$$H(f) := \int_{S^{2n-1}} \alpha \wedge d\alpha$$

is independent of the choice of "potential" α: it only depends on the map f.

iii) Show that the integral above is zero for odd n.

iv) Now you will show that the Hopf invariant $H(f)$ is a homotopy invariant. So consider two maps $f_i \colon S^{2n-1} \to S^n$ for $i = 0, 1$ and a homotopy $F \colon S^{2n-1} \times [0,1] \to S^n$ between them. Note that this means that if $\iota_i \colon S^{2n-1} \to S^{2n-1} \times [0,1]$ denotes the inclusion at an endpoint for $i = 0, 1$ respectively, then $F \circ \iota_i = f_i$.

  a) Show that $F^*\omega = d\alpha$ for some $\alpha \in \Omega^{n-1}(S^{2n-1} \times [0,1])$.
  b) Show that $f_i^*\omega = d\alpha_i$ for $\alpha_i = \iota_i^*\alpha$ the restriction of α to the endpoint $S^{2n-1} \times \{i\}$ for $i = 0, 1$. Conclude that we may use α_i to compute $H(f_i)$.
  c) Show that $d\alpha \wedge d\alpha = 0$.
  d) Show that $H(f_0) = H(f_1)$. (*hint: Stokes' theorem*)