LECTURE 1: COHOMOLOGY OF MANIFOLDS

1. Manifolds and Topology

Many of the spaces we encounter in physics are very nice: they are manifolds, meaning that we know how to do differential calculus on them in accordance with the fact that typically physics is described by means of *(partial) differential equations*. However, we sometimes encounter objects or quantities that depend on a much coarser structure, namely the underlying *topology* of the manifold. For example, consider the electric and magnetic fields $\mathbf{E}$ and $\mathbf{B}$ satisfying the Maxwell equation on a domain in $\mathbb{R}^3$. For any embedding surface Σ, the *electric and magnetic flux* through Σ are defined as

$$\frac{1}{2\pi}\int_\Sigma \mathbf{E}\cdot d\mathbf{n}, \quad \frac{1}{2\pi}\int_\Sigma \mathbf{B}\cdot d\mathbf{n}.$$

We then have:

- Both fluxes remain constant if we deform the surface Σ a little bit: for two surfaces Σ and a nearby Σ' there is a three-dimensional domain N with boundary $\partial N = \Sigma \amalg \Sigma'$ and by the divergence theorem
$$\frac{1}{2\pi}\int_\Sigma \mathbf{E}\cdot d\mathbf{n} - \frac{1}{2\pi}\int_{\Sigma'} \mathbf{E}\cdot d\mathbf{n} = \frac{1}{2\pi}\int_N \nabla\cdot\mathbf{E}dV = 0,$$
by the Maxwell equations, if we assume that N does not contain any charges. (For the magnetic flux this does not matter.)
- The electric flux depends on the charge distribution via Maxwell's equations, but the magnetic flux doesn't: it is an integer, which depends only on the topology of the configuration! We can't understand this yet, but we will identify the magnetic flux later as a *Chern number*.

In the next lecture we shall rephrase Maxwell theory in the language of differential forms and understand the first item in terms of *de Rham cohomology*. For now we take from this little discussion the idea that topological invariants are *rigid under deformations*.

Mathematically, what is going on is that we can always "forget" that a manifold has a smooth structure so that we have an "inclusion"

$$\{\text{Manifolds}\} \subset \{\text{Topological spaces}\}.$$

The idea that we can deform spaces in topology is captured in the notion of *homotopy*:

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Definition 1.1. Let $f_0, f_1 : M \to N$ be two continuous (resp. smooth) maps between topological spaces (resp. manifolds). We see that f_0 and f_1 are (smoothly) homotopic if there exists a (smooth) map $F : M \times [0,1] \to N$ such that $F(x,0) = f_0(x)$ and $F(x,1) = f_1(x)$ for all $x \in M$.

Remark 1.2. From now on, when dealing with manifolds, we shall always use smooth homotopies without saying explicitly so, and write $f_0 \sim_h f_1$ for two such homotopic maps. There are approximation theorem stating that any two smooth maps that are continuously homotopic, are also smoothly homotopic, so the distinction does not matter.

We can now say what it means for two manifolds M and N to be homotopic: it means that there are maps $f : M \to N$ and $g : N \to M$ such that $f \circ g \sim_h \mathrm{id}_N$ and $g \circ f \sim_h \mathrm{id}_M$. Intuitively it means that we can deform M into N "without creating or destroying holes". It defines an equivalence relation between manifolds, written $M \sim N$ and a topological invariant is a gadget that is invariant under homotopy.

As an example, consider the *homotopy groups* of a space M with a base point $x_0 \in M$:

$$\pi_k(M, x_0) := \left\{ \gamma : [0,1]^k \to M,\ \gamma(\partial[0,1]^k) = x_0 \right\} / \sim_h .$$

For $k = 1$ these are just homotopy classes of loops starting and ending in x_0. There is a group structure on $\pi_k(M, x_0)$, for $k = 1$ given by concatenation of loops. It is not difficult to see that a map $f : M \to N$ mapping $f(x_0) = y_0 \in N$ induces a map $f_* : \pi_k(M, x_0) \to \pi_k(N, y_0)$ and because we are taking homotopy classes in the definition of π_k, we have that for $f, g : M \to N$,

$$f \sim_h g \implies f_* = g_* : \pi_k(M, x_0) \to \pi_k(N, y_0).$$

It then follows that the homotopy groups are indeed a topological invariant. However, they are notoriously difficult to compute! This is partly so because in its definition, we have not used the fact that M is a smooth manifold! The basic idea of *differential topology* is to use the smooth structure on a manifold to study its underlying topology. A prime example of a *differential* topological invariant are the de Rham cohomology groups.

2. The de Rham complex

As before, we let M be a manifold. We now consider the system of differential forms (of arbitrary degree) together with the exterior differential:

$$C^\infty(M) \xrightarrow{d} \Omega^1(M) \xrightarrow{d} \Omega^2(M) \xrightarrow{d} \dots \tag{1}$$

A differential form $\alpha \in \Omega^k(M)$ for which $d\alpha = 0$, is called *closed*. If there exists a form $\beta \in \Omega^{k-1}(M)$ such that $d\beta = \alpha$, α is called exact. We have seen that $d \circ d = 0$, so being exact implies being closed. With this property, the system (1) is an example of a *cochain* *complex*: we have $\mathrm{Im}\{d\colon \Omega^{k-1}(M) \to \Omega^k(M)\} \subset \ker\{d\colon \Omega^k(M) \to \Omega^{k+1}(M)\}$. The de Rham *cohomology groups* measure to what extend closedness fails to imply exactness:

$$H^k_{dR}(M) := \ker\{d\colon \Omega^k(M) \to \Omega^{k+1}(M)\}/\mathrm{Im}\{d\colon \Omega^{k-1}(M) \to \Omega^k(M)\}.$$

The main point of *de Rham's theorem* (see Theorem 4.2 below) is that these groups are *topological invariants* of the underlying topological space. For now, let us collect a few properties of these groups.[1]

- The assignment $M \mapsto H^\bullet_{\mathrm{dR}}(M)$ associates a (graded) vector space to a manifold.
- The wedge product of differential forms induces a product

$$\wedge\colon H^p_{\mathrm{dR}}(M) \times H^q_{\mathrm{dR}}(M) \to H^{p+q}_{\mathrm{dR}}(M),$$

  because $d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^{|\alpha|}\alpha \wedge d\beta$.
- A smooth map $f\colon M \to N$ induces a map $f^*\colon H^\bullet_{\mathrm{dR}}(N) \to H^\bullet_{\mathrm{dR}}(M)$, because of the fact that the pull-back of differential forms is compatible with the exterior differential: $f^* d\alpha = d f^* \alpha$ for all $\alpha \in \Omega^k(N)$. This map is compatible with the wedge product.

The following property is less straightforward, but all the more important:

Theorem 2.1 (Homotopy invariance of de Rham cohomology)**.** *Let $f_0, f_1\colon M \to N$ be two smooth maps that are smoothly homotopic. Then they induce the same map on the level of de Rham cohomology groups:*

$$[f_0^*] = [f_1^*]\colon H^\bullet_{\mathrm{dR}}(N) \to H^\bullet_{\mathrm{dR}}(M).$$

Proof. The fact that f_0 and f_1 are smoothly homotopic means that there exists a smooth map $F\colon M \times [0,1] \to N$ with $F(x,0) = f_0(x)$ and $F(x,1) = f_1(x)$ for all $x \in M$. With this homotopy F we shall construct a map $H\colon \Omega^k(M) \to \Omega^{k-1}(N)$ satisfying[2]

$$(\star) \qquad f_0^*\alpha - f_1^*\alpha = (d \circ H + H \circ d)\alpha, \quad \text{for all } \alpha \in \Omega^k(N).$$

This property implies that indeed $[f_0^*] = [f_1^*]$.

To construct H, observe that a k-form on $M \times [0,1]$ decomposes as

$$\beta = \beta^0 + dt \wedge \beta^1, \quad \beta \in \Omega^k(M \times [0,1]),$$

where t is the coordinate on $[0,1]$, $\beta^0 = \sum \beta^0_{i_1,\ldots,i_k}(x,t)dx^{i_1} \wedge \ldots \wedge dx^{i_k}$ (in local coordinates) a k-form which does not contain dt and $\beta^1 = \sum \beta^1_{i_1,\ldots,i_{k-1}}(x,t)dx^{i_1} \wedge \ldots \wedge dx^{i_{k-1}}$

[1] For the mathematically minded: These three properties can be rephrased by saying that de Rham cohomology defines a contravariant functor from the category of smooth manifolds (with morphisms given by smooth maps) to the category of graded algebras.

[2] In homological algebra, H is called a *chain homotopy* between $[f_0^*]$ and $[f_1^*]$.

a $k-1$ form. We can define the *fiber integral* along the projection $M \times [0,1] \to M$ by integrating the dt-component over $[0,1]$:

$$\int_0^1 (\iota_{\partial/\partial t}\beta)dt = \int_0^1 \beta^1 dt.$$

This defines a map $\int_{[0,1]} : \Omega^k(M \times [0,1]) \to \Omega^{k-1}(M)$ and Stokes' theorem gives the property

$$d\int_{[0,1]} \beta + \int_{[0,1]} d\beta = \beta|_{M\times\{1\}} - \beta|_{M\times\{0\}}.$$

(This is easily seen using the fact that the exterior derivative on $M \times [0,1]$ is given by $d_t + d$, where d_t is the derivative in the t variable and d the exterior derivative on M.)

With the fiber integral, we now define H by

$$H(\alpha) = \int_{[0,1]} F^*\alpha, \quad \alpha \in \Omega^k(N).$$

The property ($\star$) now follows from the the above version of Stokes' theorem together with the fact that the exterior derivative is compatible with the pull-back along F. □

An important Corollary of this theorem is the Poincaré Lemma: recall that a domain $U \in \mathbb{R}^n$ is called *star-shaped* if there is a point $x_0 \in U$ such that for any other point $x \in U$, the straight line $tx + (1-t)x_0$ connecting x_0 and x is in U. For example $\mathbb{R}^n$ itself is star-shaped.

Corollary 2.2 (Poincaré Lemma)**.** *Let $U \subset \mathbb{R}^n$ be a star-shaped domain. Then:*

$$H^\bullet_{\mathrm{dR}}(U) = \begin{cases} \mathbb{R} & \bullet = 0 \\ 0 & \bullet > 0. \end{cases}$$

In the end this is a statement about solutions to certain systems of PDE's: Given a k-form $\alpha \in \Omega^k(M)$ on a manifold M, a necessary condition for the equation $\alpha = d\beta$ to have a solution $\beta \in \Omega^{k-1}(M)$, is that α is closed: $d\alpha = 0$. The Poincaré Lemma says that for M a star-shaped domain in $\mathbb{R}^n$, this condition is also sufficient: any closed form is always exact. Since any point $x \in M$ in a manifold has a neighborhood that is star-shaped, this means that *locally* any closed form on a manifold is also exact. The answer to the question whether this is *globally* true however depends on the global topology of M.

3. Singular (co)homology of manifolds

Recall that the standard k-dimensional simplex $\Delta^k \subset \mathbb{R}^{k+1}$ is defined as the convex subset satisfying the equation

$$\Delta^k := \{(t_0, \ldots .t_k) \in \mathbb{R}^{k+1}, \sum_{i=0}^k t_i = 1,\ t_i \geq 0.\}$$

The boundary of Δ^k consists of $k+1$ copies of the $(k-1)$-dimensional simplex by putting $t_i = 0$, $i = 0, \ldots, k$. We write $d_i : \Delta^{k-1} \hookrightarrow \Delta^k$, $i = 0, \ldots, k$ for the corresponding inclusion. A *smooth singular k-simplex* is a smooth map $\sigma : \Delta^k \to M$, where smooth means that we can extend σ to a small open neighborhood of Δ^k in $\mathbb{R}^{k+1}$. We write $S_k^\infty(M)$ for the vector space (over $\mathbb{R}$) spanned by all smooth singular k-simplices. So an element in $S_k^\infty(M)$ is given by a finite sum $\sum_i \lambda_i \sigma_i$ with $\lambda_i \in \mathbb{R}$ and σ_i smooth singular k-simplices. There is an operator $\partial : S_k^\infty(M) \to S_{k-1}^\infty(M)$ given on simplices by

$$\partial\sigma := \sum_{i=0}^{k} (-1)^i \sigma \circ d_i,$$

i.e. this operator restricts a smooth singular k-simplex σ, a map from Δ^k to M, to its $k+1$ boundary faces equal to Δ^{k-1}, with a sign. Exactly because of this sign, one checks that $\partial \circ \partial = 0$, i.e. the system

$$\ldots \xrightarrow{\partial} S_2^\infty(M) \xrightarrow{\partial} S_1^\infty(M) \xrightarrow{\partial} S_0^\infty(M)$$

forms a *chain complex*.[3] This time, for a chain complex, we should take its *homology*:

$$H_k^{\text{sing}}(M,\mathbb{R}) := \ker\{\partial\colon S_k^\infty(M) \to S_{k-1}^\infty(M)\}/\mathrm{Im}\{\partial\colon S_{k+1}^\infty(M) \to S_k^\infty(M)\}.$$

To get an idea what these groups measure, consider $H_0^{\text{sing}}(M,\mathbb{R})$: a singular 0-simplex $\sigma : \Delta^0 \to M$ is just a point in M, and any such simplex is automatically closed since $S_{-1}^\infty(M,\mathbb{R}) = 0$. If two points $x, y \in M$ are in the same path connected component of M, any path $\gamma\colon [0,1] \to M$ from $\gamma(0) = x$ to $\gamma(1) = y$ defines a 1-simplex $\gamma\colon \Delta^1 \to M$ such that $\partial\gamma = x - y$, showing that they induce the same homology class. In other words: $H_0^{\text{sing}}(M,\mathbb{R})$ measures the number of path connected components of M.

Example 3.1 (The fundamental class of an oriented manifold)**.** Let M be a smooth n-dimensional manifold. It was proved by Whitehead in 1940 that M can be triangulated: we can write M as a finite union of smooth singular n-simplices $\sigma_i\colon \Delta^n \to M$ for $i = 1, \ldots, p$ such that any boundary face of σ_i is a boundary face of exactly one other simplex σ_j, $j \neq i$. Consider the combination

$$\sum_{i=1}^{p} \pm\sigma_i \in S_n^\infty(M,\mathbb{R}).$$

If we can consistently put the $\pm$-signs so that each boundary face appears with both a $+$ and a $-$ sign when taking ∂ of this expression, we get an n-cycle and therefore a class in $H_n^{\text{sing}}(M,\mathbb{R})$. This can be done exactly when M is *orientable* and the resulting class of an oriented manifold M is called the *fundamental class*, written $[M] \in H_n^{\text{sing}}(M,\mathbb{R})$.

[3]The difference between a *chain* complex and a *cochain* complex is that in the former the differential has degree -1, whereas in the latter it has degree $+1$.

If we want to have a *cochain* complex, we should take the dual:

$$S^k_\infty(M) := \mathrm{Hom}_{\mathbb{R}}(S^\infty_k(M), \mathbb{R}).$$

Now the differential ∂ dualizes to a degree *increasing* operator $d\colon S^k_\infty(M) \to S^{k+1}_\infty(M)$ by

$$d\varphi(\sigma) := \varphi(\partial\sigma), \quad \varphi \in S^k_\infty(M),\ \sigma \in S^\infty_{k+1}(M).$$

Clearly $d \circ d = 0$, so that we have a cochain complex, and its cohomology

$$H^k_{\mathrm{sing}}(M, \mathbb{R}) := \ker\{d\colon S^k_\infty(M) \to S^{k+1}_\infty(M)\}/\mathrm{Im}\{d\colon S^{k-1}_\infty(M) \to S^k_\infty(M)\}$$

4. The de Rham theorem

Given a k-form on M and a smooth singular k-simplex $\sigma\colon \Delta^k \to M$ we can integrate:

$$\langle \alpha, \sigma \rangle := \int_{\Delta^k} \sigma^*\alpha. \tag{2}$$

Notice that it is important that we use *smooth* singular simplices to be able to pull-back the differential form to Δ^k. Stokes' theorem now gives us:

Lemma 4.1. *For $\alpha \in \Omega^{k-1}(M)$ and $\sigma \in S^\infty_k(M)$ we have the equality*

$$\langle d\alpha, \sigma \rangle = \langle \alpha, \partial\sigma \rangle.$$

We can therefore reinterpret the pairing (2) as a map

$$\Psi : \Omega^\bullet(M) \longrightarrow S^\bullet_\infty(M)$$

satisfying $d \circ \Psi = \Psi \circ d$. Such a map is called a *morphism of cochain complexes*. The fact that Ψ is compatible with the differentials on both sides implies that it induces a map on cohomology:

$$[\Psi] : H^\bullet_{\mathrm{dR}}(M) \longrightarrow H^\bullet_{\mathrm{sing}}(M, \mathbb{R}).$$

Theorem 4.2 (de Rham's theorem)**.** *The map $[\Psi]$ is an isomorphism.*

We will not give the full proof of the theorem, but only sketch the main idea. An important ingredient in the proof is the following crucial property satisfied by de Rham cohomology:

Theorem 4.3 (Mayer–Vietoris)**.** *Suppose that $M = U \cup V$ is covered by two open subsets. Then there exists a long exact sequence*[4]

$$\ldots \longrightarrow H^k_{\mathrm{dR}}(U \cup V) \longrightarrow H^k_{\mathrm{dR}}(U) \oplus H^k_{\mathrm{dR}}(V) \longrightarrow H^k_{\mathrm{dR}}(U \cap V) \longrightarrow H^{k+1}_{\mathrm{dR}}(U \cup V) \longrightarrow \ldots$$

[4] A *long exact sequence* is a complex with zero cohomology. In other words: each composition of maps is zero and the kernel of each map equals the image of the map preceding it.

Proof. Given U and V, we have maps on the level of differential forms

$$\begin{array}{ccccc} \Omega^k(U\cup V) & \to & \Omega^k(U)\oplus\Omega^k(V) & \to & \Omega^k(U\cap V) \\ \alpha & \mapsto & (\alpha|_U,\alpha|_V) & & \\ & & (\beta_U,\beta_V) & \mapsto & \beta_U|_{U\cap V}-\beta_V|_{U\cap V} \end{array}$$

Remark that the composition of these maps is zero, and that $(\beta_U,\beta_V)\in\Omega^k(U)\oplus\Omega^k(V)$ mapping to zero in $\Omega^k(U\cap V)$ means it comes from a form $\beta\in\Omega^k(U\cup V)$. Applying cohomology, we get

$$H^k_{\mathrm{dR}}(U\cup V)\to H^k_{\mathrm{dR}}(U)\oplus H^k(V)_{\mathrm{dR}}\to H^k_{\mathrm{dR}}(U\cap V).$$

Let us now construct a map $H^k(U\cap V)\to H^{k+1}(U\cup V)$. For this we choose a function $\chi_U\in C^\infty(U)$ which is ≤ 1 on U and equal to 1 on $U\backslash(U\cap V)$. Then $\chi_V:=1-\chi_U$ is equal to 1 on $V\backslash(U\cap V)$ and we have $\chi_U+\chi_V=1$. Given a closed differential form $\omega\in\Omega^k(U\cap V)$, let

$$(d\chi_U\wedge\omega,d\chi_V\wedge\omega)\in\Omega^k(U)\oplus\Omega^k(V).$$

Then on $U\cap V$ we have

$$\begin{aligned} d\chi_U\wedge\omega-d\chi_V\wedge\omega &= d(\chi_U-\chi_V)\wedge\omega \\ &= d(1)\wedge\omega \\ &= 0, \end{aligned}$$

and therefore these two forms glue together to a closed form of degree $k+1$ on $U\cup V$. We will skip the proof that the sequence is exact. $\square$

Remark 4.4. Those who know a bit of homological algebra will recognize the snake Lemma in the proof above: The core of the argument is to show that the sequence

$$0\to\Omega^\bullet(U\cup V)\to\Omega^\bullet(U)\oplus\Omega^\bullet(V)\to\Omega^\bullet(U\cap V)\to 0$$

is exact. This short exact sequence of complexes induces the long exact sequence in cohomology. Remark that the choice of the function χ_U is irrelevant: choosing another χ'_U results in a closed differential form which differs from the one constructed above by an exact form. (Try to prove this!)

The proof of de Rham's Theorem now amounts to proving first that singular cohomology $H^\bullet_{\mathrm{sing}}(M,\mathbb{R})$ satisfies the same properties as the properties of de Rham cohomology we have just outlined: functoriality, homotopy invariance and the existence of Mayer–Vietoris sequences. With that, by choosing an open cover of the manifold by open sets that are homeomorphic to a star-shaped domain, the proof is reduced to proving the de Rham isomorphism for such domains, which is done by the Poincaré Lemma. For the full details, see [2, §V.9].

Example 4.5. The Mayer–Vietoris property, together with homotopy invariance is extremely useful in computations. As an example let us compute the cohomology of $\mathbb{P}^n$. The inclusion $\mathbb{C}^n \hookrightarrow \mathbb{C}^{n+1}$ given by $(z^0, \ldots, z^{n-1}) \mapsto (z^0, \ldots, z^{n-1}, 0)$ induces an inclusion $\mathbb{P}^{n-1} \subset \mathbb{P}^n$. The complement $U := \mathbb{P}^n \backslash \mathbb{P}^{n-1}$ is isomorphic to $\mathbb{C}^n$ via the map

$$(z^0, \ldots, z^n) \mapsto (\frac{z^0}{z^n}, \ldots, \frac{z^{n-1}}{z^n}).$$

On the other hand, define $V := \mathbb{P}^n \backslash \{[0, \ldots, 0, 1]\}$. Then $U \cap V \cong \mathbb{C}^n \backslash \{0\} \sim S^{2n-1}$, and the map $F : V \times [0, 1] \to V$ defined by

$$F([z^0, \ldots, z^n], t) = [z^0, \ldots, z^{n-1}, tz^n],$$

defines a contraction $V \sim \mathbb{P}^{n-1}$. The Mayer–Vietoris sequence, together with the homotopy invariance of de Rham cohomology leads to the exact sequence

$$\ldots \longrightarrow H^k_{\mathrm{dR}}(\mathbb{P}^n) \longrightarrow H^k_{\mathrm{dR}}(\mathbb{P}^{n-1}) \longrightarrow H^k_{\mathrm{dR}}(S^{2n-1}) \longrightarrow H^{k+1}_{\mathrm{dR}}(\mathbb{P}^n) \longrightarrow \ldots$$

Because

$$H^k_{\mathrm{dR}}(S^{2n-1}) = \begin{cases} \mathbb{R} & k = 2n-1 \\ 0 & k \neq 2n-1 \end{cases},$$

and $H^{2n-1}_{\mathrm{dR}}(\mathbb{P}^{n-1}) = 0$ (recall that $\mathbb{P}^{n-1}$ is $2n-2$-dimensional), the sequence above breaks up into

$$\begin{aligned} 0 \to H^k_{\mathrm{dR}}(\mathbb{P}^n) \to H^k_{\mathrm{dR}}(\mathbb{P}^{n-1}) \to 0, &\quad \text{for } k < 2n-1 \\ 0 \to H^{2n-1}_{\mathrm{dR}}(\mathbb{P}^{n-1}) \to 0 & \\ 0 \to \mathbb{R} \to H^{2n}_{\mathrm{dR}}(\mathbb{P}^n) \to 0 & \end{aligned}$$

From this we see, by induction that for $0 \leq k \leq 2n$:

$$H^k_{\mathrm{dR}}(\mathbb{P}^n) = \begin{cases} \mathbb{R} & k = \text{even} \\ 0 & k = \text{odd}. \end{cases}$$

5. ČECH COHOMOLOGY

As is clear from Example 4.5, the Mayer–Vietoris principle, together with homotopy invariance, makes cohomology very computable. However, after a few examples, one gets the feeling many of the steps in the computations are very similar, and it should be possible to formalize these. This is done by giving an alternative, very small cochain complex, called the *Čech complex* associated to an open cover, which computes the cohomology. We shall explain this now. Consider the following:

Definition 5.1. Given an n-dimensional manifold M, a *good covering* of M is given by a collection $\mathcal{U} := \{U_i\}_{i \in I}$ of open subsets satisfying

i) $M = \bigcup_{i \in I} U_i$,

ii) each finite intersection $U_{i_1,\ldots,i_k} := U_{i_1} \cap \ldots \cap U_{i_k}$ is diffeomorphic with $\mathbb{R}^n$.

For such a good covering, we define the following cochain complex:

$$C^k(\mathcal{U}, \mathbb{R}) := \prod_{U_{i_1,\ldots,i_k} \neq \emptyset} \mathbb{R}.$$

(So for each nonempty k-fold intersection, we add a copy of $\mathbb{R}$.) Let us write $\eta = (\eta_{i_1\ldots i_k} \in \mathbb{R})_{i_1,\ldots,i_k \in I}$ for the elements in this complex associated to such an intersection. The combinatorics of the covering defines a differential $\delta : C^k(\mathcal{U}, \mathbb{R}) \to C^{k+1}(\mathcal{U}, \mathbb{R})$ by

$$(\delta\eta)_{i_1\ldots i_{k+1}} := \sum_{i=1}^{k+1} (-1)^{j+1} \eta_{i_1\ldots\hat{i_j}\ldots i_{k+1}},$$

where ˆ means omission from the argument. One easily checks that $\delta \circ \delta = 0$, so that we have a cochain complex of which we can take the cohomology.

Theorem 5.2. *For a good covering, the cohomology of the complex $(C^\bullet(\mathcal{U}, \mathbb{R}), \delta)$ is isomorphic to the de Rham cohomology (and by de Rham's theorem, singular cohomology).*

The proof is basically given by applying the Mayer–Vietoris principle together with homotopy invariance, see e.g. [1, Chapter II].

Concluding, we now have three ways to compute the cohomology of a manifold:

i) using the de Rham complex,
ii) using the singular cohomology cochain complex
iii) using the Čech complex of a good open cover.

As an example, let us consider the cohomology of the circle S^1.

i) First, the de Rham complex looks like

$$0 \longrightarrow C^\infty(S^1) \xrightarrow{d} \Omega^1(S^1) \longrightarrow 0.$$

Fix a coordinate $s \in \mathbb{R}/\mathbb{Z}$ so that a one-form looks like $g(s)ds$. Clearly, the exterior differential $d : C^\infty(S) \to \Omega^1(S^1)$ is given by

$$df = \frac{df}{ds} ds,$$

and is not surjective: for a given one-form $g(s)ds$ with $g \in C^\infty(S^1)$, we have to find an anti-derivative of $g(s)$. But the formula

$$f(s) := \int_0^s g(t)dt,$$

is only well-defined on S^1 if $\int_0^1 g(t)dt = 0$. We therefore see that we can extend the de Rham complex to the left and right to the sequence

$$0 \longrightarrow \mathbb{R} \longrightarrow C^\infty(S^1) \xrightarrow{d} \Omega^1(S^1) \xrightarrow{\int_{S^1}} \mathbb{R} \longrightarrow 0,$$

which can be checked to be exact. We therefore find that

$$H^0_{dR}(S^1) = \mathbb{R}, \quad H^1_{dR}(S^1) = \mathbb{R}. \tag{3}$$

A generator for $H^1_{dR}(S^1)$ is given by the one-form ds. Remark that s is not really a function, since it is multi-valued. However ds is a well-defined one-form. Since $\int_0^1 ds = 1$, it represents a nontrivial cohomology class.

ii) To illustrate the singular approach, let us compute the singular *homology*, not cohomology. The singular chain complex looks like

$$\dots \xrightarrow{\partial} S_2^\infty(S^1) \xrightarrow{\partial} S_1^\infty(S^1) \xrightarrow{\partial} S_0^\infty(S^1)$$

A singular 0-chain is just a finite number of points, since $\Delta^0 = 1 \in \mathbb{R}$. For a singular 0-chain consisting of two points $x_0, x_1 \in S^1$, one can clearly find a map $\sigma : \Delta^1 \to M$ which connects the two points. We therefore have that $H_0^{\text{sing}}(S^1) = \mathbb{R}$, generated by a single point $x_0 \in S^1$. For a singular 1-chain $\sigma : \Delta^1 \to S^1$ to satisfy $\partial\sigma = 0$, σ should map the beginning and end point of Δ^1 to the same point. If σ wraps around the circle once, there can never be a 2-chain $\tau \in S_2^\infty(S^1)$ such that $\partial\tau = \sigma$, if it doesn't there is. We therefore find that $H_1^{\text{sing}}(S^1) = \mathbb{R}$ with generator the cycle that wraps around the circle once.

iii) For the Čech complex, we need a good cover of S^1. For this we choose the open cover consisting $\mathcal{U}$ of 3 intervals overlapping near the end-points. The Čech complex associated to this cover is then given by

$$C^0(\mathcal{U}, \mathbb{R}) = \mathbb{R}^3, \quad C^1(\mathcal{U}, \mathbb{R}) = \mathbb{R}^3, \quad C^2(\mathcal{U}, \mathbb{R}) = 0.$$

The differential $\delta : \mathbb{R}^3 \to \mathbb{R}^3$ is given by

$$\delta(x, y, z) = (x - y, y - z, z - x).$$

From this we recover the result (3).

References

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