LECTURE 5: CONNECTIONS

1. Introduction: The mathematical framework of gauge theories

Gauge theories such are extremely important in physics. Examples are given by Maxwell's theory of electromagnetism but also the Yang–Mills theories describing the strong and weak forces in the standard model. Central ingredient in this is the choice of a compact Lie group G that physicist call the *gauge group*. For Maxwell theory this group is abelian, namely $G = U(1)$, but for other important examples this group must be chosen to be *nonabelian*, e.g. $G = SU(N)$. Mathematically, the framework for such gauge theories is given by the theory of *principal bundles*. Each ingredient of the physical theory has a solid mathematical meaning in the theory of principal bundles. The table below gives a "dictionary" to translate from physics to mathematics, and back:

Physics	Mathematics
Gauge group	Structure group G of a principal bundle $P \to M$
Gauge field	Connection A on P
Field strength	Curvature $F(A)$
Gauge transformation	Bundle isomorphism $\varphi : P \to P$
Matter fields in a representation V of G	Sections of the associated vector bundle $(P \times V)/G$

The most important ingredient of gauge theories is of course the gauge field itself, which, mathematically, corresponds to connections on a principal bundle. The goal of this lecture is to explain what a connection precisely is. We explain this concept on a principal bundle, as well as on a vector bundle, in view of the last item when we want to introduce matter fields.

2. Connections

In the previous lecture we have discussed the fundamental dichotomy

$$\boxed{\text{Principal bundles}} \underset{\text{Frame bundle}}{\overset{\text{Associated bundle}}{\rightleftarrows}} \boxed{\text{Vector bundles}}$$

Let us briefly recall the associated bundle construction: suppose $\pi : P \to M$ is a principal G-bundle, and let V be a representation of G. (This means that G acts linearly on V, i.e., we are given a homomorphism $G \to GL(V)$.) The associated vector bundle is defined as the quotient of $P \times V$ under the diagonal action of G: $(P \times V)/G$. This

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construction of the associated bundle may seem abstract, but its (local) *sections* have a very concrete description: they are given by maps $s : P \to V$ that are G-equivariant:

$$s(pg) = g \cdot s(p), \quad \text{for all } g \in G,$$

where G acts on the right via the homomorphism to $GL(V)$. If we think of P as being defined by an open covering $\{U_\alpha\}_{\alpha \in I}$ together with transition functions $\varphi_{\alpha\beta} : U_\alpha \cap U_\beta \to G$, then the transition functions of the associated bundle are given by $\tilde{\varphi}_{\alpha\beta} : U_\alpha \cap U_\beta \to GL(V)$ by composing with the homomorphism $G \to GL(V)$. In the following we shall need this construction for the *adjoint representation* of G on its Lie algebra $\mathfrak{g}$: this is a vector bundle with typical fiber $\mathfrak{g}$ that we denote by $\mathrm{ad}(P) \to M$.

2.1. **Connections on principal bundles.** Let $\pi : P \to M$ be a principal G-bundle. We denote the right action of $g \in G$ on P by $R_g : P \to P$. Since G is a *Lie group*, elements in a neighborhood of the identity element can be written as $g = e^{\xi}$ with $\xi \in \mathfrak{g}$, the Lie algebra of G. Therefore the action on P has generators

(1) $$\xi_P(p) := \left.\frac{d}{dt}\right|_{t=0} R_{e^{t\xi}}(p), \quad \text{for all } \xi \in \mathfrak{g}.$$

This defines a vector field for any Lie algebra element $\xi \in \mathfrak{g}$. This correspondence defines a map $\rho : \mathfrak{g} \to \mathfrak{X}(M)$ called the *infinitesimal action*.[1]

Definition 2.1. A *connection* on a principal G-bundle $P \to M$ is a $\mathfrak{g}$-valued 1-form $A \in \Omega^1(P, \mathfrak{g})$ satisfying

(2a) $$\iota_{\xi_P} A = \xi, \quad \text{for all } \xi \in \mathfrak{g},$$

(2b) $$R_g^* A = \mathrm{Ad}_{g^{-1}}(A), \quad \text{for all } g \in G.$$

At first sight, this definition looks pretty artificial, so let us find out what a connection does. Consider a point $p \in P$. The correspondence (1) defines an injective map $\mathfrak{g} \subset T_p P$ consisting of vectors that point in the direction of the G-action. Because the projection $\pi : P \to M$ is the map that mods out this action, we see that

$$T_p\pi(\xi_p) = 0, \quad \text{for all } \xi \in \mathfrak{g}.$$

It is then not difficult to see that because G acts transitively along the fibers of $\pi : P \to M$, that this defines an isomorphism

$$\mathfrak{g} \cong \ker(T_p\pi).$$

We can summarize this little discussion by the statement that the sequence of linear maps

(3) $$0 \longrightarrow \mathfrak{g} \longrightarrow T_p P \xrightarrow{T_p\pi} T_{\pi(p)} M \longrightarrow 0,$$

is *exact*. Indeed:

[1] This map is in fact a morphism of Lie algebras: $[\xi, \eta]_P = [\xi_P, \eta_P]$, for all $\xi, \eta \in \mathfrak{g}$.

1) it is exact at the first node $\mathfrak{g}$ because the map $\mathfrak{g} \to T_pP$ defined in (1) is injective as remarked above (the action of G on P is *free*).
2) it is exact at the second node T_pP because $\ker(T_p\pi) = \mathfrak{g}$ because the G-action is *transitive* along the fibers of $\pi : P \to M$.
3) it is exact at the third node $T_{\pi(p)}M$ because the surjectivity of π implies that $T_p\pi : T_pP \to T_{\pi(p)}M$ is surjective.

We know from linear algebra that there is an isomorphism of vector spaces

$$T_pP \cong \ker(T_p\pi) \oplus \mathrm{Im}(T_p\pi) = \mathfrak{g} \oplus T_{\pi(p)}M,$$

but there is no *canonical* choice for such an isomorphism: we are working with abstract vector spaces, the injection $\mathfrak{g} \subset T_pP$ is given, but to get the desired isomorphism, we have to make additional *choices*.

Given a connection $A \in \Omega^1(P, \mathfrak{g})$, define the map

$$T_pP \to \mathfrak{g} \oplus T_{\pi(p)}M, \quad \text{by} \quad X \mapsto (A_p(X), T_p\pi(X)).$$

Condition (2a) guarantees that this is an isomorphism, so we see that a connection is exactly the additional data needed to define the isomorphism! It splits the tangent space T_pP into a *vertical subspace* $\mathfrak{g}$ and a *horizontal subspace* $T_{\pi(p)}M$.

Example 2.2. Consider the Lie group G itself as a principal G-bundle over a point. (The point is considered as a 0-dimensional manifold: its tangent space is just the trivial vector space.) The Lie group G acts on itself by the right action

$$R_g : h \mapsto hg,$$

with inverse $R_g^{-1} = R_{g^{-1}}$. We can then define the so-called *right Maurer–Cartan* form $\theta \in \Omega^1(G, \mathfrak{g})$ by

$$\theta_g(X) := T_gR_{g^{-1}}(X), \quad \text{for all } X \in T_gG.$$

Indeed $T_gR_{g^{-1}} : T_gG \to T_eG = \mathfrak{g}$, so this is a $\mathfrak{g}$-valued 1-form. By definition, the generating vector field on G is given by

$$\xi_g := \left.\frac{d}{dt}\right|_{t=0} ge^{t\xi}, \quad \text{for all } \xi \in \mathfrak{g}.$$

We now check that

$$\theta_g(\xi_g) = T_gR_{g^{-1}}(\xi_g) = \left.\frac{d}{dt}\right|_{t=0} (e^{t\xi}) = \xi, \quad \text{for all } \xi \in \mathfrak{g},$$

so θ defines a connection on G! (The second condition (2b) is trivial because the base-manifold is a point.)

For matrix groups (such as $G = SU(N)$), we can simply write $\theta = dgg^{-1}$. Let us use this notation to verify that θ satisfies the so-called *Maurer–Cartan* equation:

$$d\theta + \frac{1}{2}[\theta, \theta] = 0. \tag{4}$$

We start by deriving the following useful relation:

$$0 = d(1) = d(gg^{-1}) = dgg^{-1} + gd(g^{-1}),$$

which gives $d(g^{-1}) = -g^{-1}dgg^{-1}$ (Remark that G may be non-abelian, so we should be careful in distinguishing the left and right action on itself!) With this relation we see that

$$d(dgg^{-1}) = -dgg^{-1} \wedge dgg^{-1},$$

which shows that (4) holds true. We will see in the next section that the Maurer–Cartan equation can be interpreted as saying that θ has *zero curvature*.

Choosing local trivializations $s_\alpha : U_\alpha \to P$, see Lemma **??**, we can pull-back the connection form to M: $A_\alpha := s_\alpha^* A \in \Omega^1(U_\alpha, \mathfrak{g})$. These 1-forms are called the *local connection* 1*-forms*. These are only defined locally. On the overlaps we now have

Lemma 2.3. *On $U_\alpha \cap U_\beta$ the two local connection* 1*-forms are related by*

$$A_\alpha = \varphi_{\alpha\beta} A_\beta \varphi_{\alpha\beta}^{-1} + d\varphi_{\alpha\beta}\varphi_{\alpha\beta}^{-1},$$

where $\varphi_{\alpha\beta} : U_{\alpha\beta} \to G$ is the transition function.

Proof. First recall the following method from differential geometry to compute the derivative of a smooth map $f : M \to N$. For a tangent vector $X \in T_x M$, we can find a smooth curve $\gamma : (-\varepsilon, \varepsilon) \to M$ with $\gamma(0) = x$ and $\dot{\gamma}(0) = X$. Then the tangent to f is given by

$$T_x f(X) = \frac{d}{dt}(f(\gamma(t)))\Big|_{t=0}.$$

This is very useful to compute the pull-back of a 1-form $\omega \in \Omega^1(N)$ since

$$(f^*\omega)_x(X) = \omega_{f(x)}(T_x f(X)).$$

Now, on the overlap $U_\alpha \cap U_\beta$ the two local trivializations are related by the cocycle $\varphi_{\alpha\beta}$, i.e., $s_\alpha = s_\beta \cdot \varphi_{\alpha\beta}$. We are going to apply the above method to compute $T_x s_\alpha : T_x M \to T_{s_\alpha(x)} P$. For the following computation we assume that we are working with a matrix group such as $G = SU(N)$.

$$\begin{aligned}
T_x s_\alpha(X) &= \frac{d}{dt}\Big|_{t=0} s_\alpha(\gamma(t)) \\
&= \frac{d}{dt}\Big|_{t=0} \left(s_\beta(\gamma(t))\varphi_{\alpha\beta}(\gamma(t))\right) \\
&= \frac{d}{dt}\Big|_{t=0} \left(s_\beta(\gamma(t))\right) \varphi_{\alpha\beta}(x) + \frac{d}{dt}\Big|_{t=0} s_\beta(x)\varphi_{\alpha\beta}(\gamma(t)) \\
&= T_x s_\beta(X)\varphi_{\alpha\beta}(x) + s_\alpha(x)\varphi_{\alpha\beta}^{-1} d\varphi_{\alpha\beta}(X).
\end{aligned}$$

If we now evaluate the connection on this expression we find

$$\begin{aligned} A_{\alpha,x}(X) &= A_{s_\alpha(x)}(T_x s_\alpha(X)) \\ &= A_{s_\alpha(x)}(T_x s_\beta(X)\varphi_{\alpha\beta}(x)) + A_{s_\alpha(x)}(s_\alpha(x)\varphi_{\alpha\beta}^{-1} d\varphi_{\alpha\beta}(X)) \\ &= \varphi_{\alpha\beta}(x) A_{s_\beta(x)}(T_x s_\beta(X))\varphi_{\alpha\beta}^{-1}(x) + \varphi_{\alpha\beta}^{-1} d\varphi_{\alpha\beta}(X), \end{aligned}$$

where we have used that $\varphi_{\alpha\beta}^{-1} d\varphi_{\alpha\beta}(X) \in \mathfrak{g}$ and used the two properties (2a) and (2b). This proves the statement. □

Remark 2.4. This Lemma explains how the physicists usually work with gauge fields: they are local, Lie algebra valued 1-forms A_α defined on open patches $U_\alpha \subset M$, subject to the funny transformation rule stated in the Lemma for some $\varphi_{\alpha\beta} : U_\alpha \cap U_\beta \to G$, called a *gauge transformation*.

For the following consequence of this Lemma, define the *adjoint bundle* $\mathrm{ad}(P) \to M$ as the vector bundle associated to the adjoint representation $\mathrm{Ad}_g : \mathfrak{g} \to \mathfrak{g}$. This is a vector bundle with typical fiber $\mathfrak{g}$.

Corollary 2.5. *Given two connections A and B, their difference $A - B \in \Omega^1(M, \mathrm{ad}(P))$.*

Proof. Use local trivializations $s_\alpha : U_\alpha \to P$ as in the previous Lemma. Define $\theta_\alpha := A_\alpha - B_\alpha$, a $\mathfrak{g}$-valued 1-form on U_α. Then we find that on the overlaps

$$\theta_\alpha = \varphi_{\alpha\beta}^{-1} \theta_\beta \varphi_{\alpha\beta}.$$

This means that the θ_α glue together to form a 1-form with values in $\mathrm{ad}(P)$. □

This Corollary shows that the space of all connections on a principal bundle $P \to M$ is an *affine space* modeled on $\Omega^1(M, \mathrm{ad}(P))$. A typical example of an affine space is given by considering a line L in $\mathbb{R}^2$ that does not pass through the origin: L is not a vector space under addition, but it is true that the difference between any two of its points lie in a fixed vector space, namely the line L' parallel to L that does pass through the origin.

2.2. **Connections on vector bundles.** Let M be a smooth manifold, and $E \to M$ a smooth vector bundle. We denote by $\Omega^k(M; E)$ the space of differential k-forms on M with values in E:

$$\Omega^k(M; E) := \Gamma^\infty(M, E \otimes \bigwedge^k T^* M)$$

The following definition is fundamental:

Definition 2.6. A *connection* on E is a linear map

$$\nabla : \Gamma^\infty(M; E) \to \Omega^1(M; E),$$

satisfying the Leibniz rule

$$\nabla(fs) = f\nabla(s) + df \otimes s,$$

with $f \in C^\infty(M)$ and $s \in \Gamma^\infty(M; E)$.

In short: a connection on a vector bundle $E \to M$ is a gadget which allows us to take "directional derivatives" of smooth sections of E along vector fields on M. For a vector field $X \in \mathfrak{X}(M)$, we shall write $\nabla_X : \Gamma^\infty(M;E) \to \Gamma^\infty(M;E)$ for this directional derivative: $\nabla_X(s) := \iota_X(\nabla s)$. If we want to stipulate for which bundle exactly ∇ is a connection, we shall write ∇^E.

Lemma 2.7. *The space of connections on a vector bundle E is an affine space modeled on $\Omega^1(M, \mathrm{End}(E))$.*

Proof. Let ∇ and ∇' be two connections on E. It follows form the Leibniz rule that

$$(\nabla - \nabla')fs = f(\nabla - \nabla')s, \quad \text{for all } f \in C^\infty(X),\ s \in \Gamma(M;E).$$

The operator $\nabla - \nabla' : \Gamma^\infty(M;E) \to \Omega^1(M;E)$ is therefore $C^\infty(M)$-linear, and it follows that $\nabla - \nabla' \in \Omega^1(M;\mathrm{End}(E))$ □

Remark 2.8.

i) For a trivial vector bundle $E = M \times \mathbb{C}^r$ we always have the trivial connection given by the de Rham operator d extended to vector valued functions. By the Lemma above, any other connection can be written as $\nabla = d + A$ with $A \in \Omega^1(M, M_r(\mathbb{C}))$ a matrix-valued one-form. ($M_r(\mathbb{C})$ denotes the $r \times r$ matrices with coefficients in $\mathbb{C}$.)

ii) For a general vector bundle, we can write $\nabla = d + A_\alpha$ in a local trivialization over U_α. On the overlap $U_\alpha \cap U_\beta$ of two local trivializations the two one forms A_α and A_β are related by (check!)

$$A_\alpha = \varphi_{\alpha\beta} A_\beta \varphi_{\alpha\beta}^{-1} + d\varphi_{\alpha\beta}\varphi_{\alpha\beta}^{-1}, \tag{5}$$

with $\varphi_{\alpha\beta} : U_{\alpha\beta} \to GL(r, \mathbb{C})$ the transition function. If we adopt the "cocycle point of view" on vector bundles, c.f. Remark **??**, we can therefore think of a connection on a vector bundle as a collection $\{A_\alpha \in \Omega^1(U_\alpha, M_r(\mathbb{C}))\}_{\alpha \in I}$ of matrix-valued 1-forms, which transform according to (5) under local gauge transformations.

iii) It can be shown by a standard partition of unity argument that a connection always exist on a vector bundle.

Remark 2.9. Connections behave well with respect to the standard constructions with vector bundles: Let E and F be vector bundles over X, with connections ∇^E, ∇^F.

i) On the direct sum, we have the obvious connection

$$\nabla^{E \oplus F} = \begin{pmatrix} \nabla^E & 0 \\ 0 & \nabla^F \end{pmatrix},$$

ii) On the tensor product we have the connection $\nabla^{E \otimes F} = \nabla^E \otimes 1 + 1 \otimes \nabla^F$, *iii*) On the dual E^*, we have the connection defined by the following equation

$$d\langle \alpha, s\rangle = \langle \nabla_{E^*}(\alpha), s\rangle + \langle \alpha, \nabla_E(s)\rangle, \quad \alpha \in \Gamma^\infty(M; E^*),\ s \in \Gamma^\infty(M; E),$$

using the dual pairing $\langle\ ,\ \rangle : \Gamma^\infty(M; E^*) \times \Gamma^\infty(M; E) \to C^\infty(M)$ between sections of E and E^*

iv) As a special case of *iii*), we obtain a connection on $\mathrm{End}(E) = E \otimes E^*$, defined by

(6) $$\nabla^{\mathrm{End}(E)}(A)(s) := \nabla_E(A(s)) - A(\nabla_E(s)) \quad \text{for } A \in \Gamma^\infty(X, \mathrm{End}(E),\ s \in \Gamma(X; E).$$

v) On the pull-back bundle f^*E for a smooth map $f : N \to M$, there is a natural pull-back connection $f^*\nabla_E$.

Finally, we shall give the precise relation between connections on principal bundles and vector bundles. For this it is important to realize that a connection A on a principal bundle gives a map $T_{\pi}(p)M \to T_pP$ for every $p \in P$, and with this we can lift vector fields X on M to vector fields $\tilde{X}$ on P: $\tilde{X}$ is the unique vector field on P satisfying

$$A(\tilde{X}) = 0, \quad T\pi(\tilde{X}) = X.$$

Proposition 2.10. *Let $P \to M$ be a principal G-bundle and let $E(V) := (P \times V)/G$ be the vector bundle associated to a representation of G. Let A be a connection on P. Then the formula*

$$(\nabla_X s) = ds(\tilde{X}),$$

where $s : P \to V$ is a G-equivariant map, defines a connection on $E(V)$.