Correction Model for the Final Exam for Topology in Physics 2018

Quickfire Questions 15 *pts*

We will start the exam off with a lightning round. This means you do **not** need to motivate your answers for these quickfire questions.

5 pt i) $\mathrm{Ind}\left(-\frac{\partial^2}{\partial x^2}-\frac{\partial^2}{\partial y^2}-\frac{\partial^2}{\partial z^2}\right)=0$

5 pt ii) Suppose $E \to M$ is a vector bundle and ∇^i for $i = 1, 2, 3$ are connections on E. Which of the following defines a connection (only one answer is correct):

- a) $-\nabla^1$,
- b) $\nabla^1 + \nabla^2$,
- c) $\nabla^1 - \nabla^2$,
- d) $\nabla^1 - \nabla^2 + \nabla^3$.

5 pt iii) Which of the following differential operators is elliptic (only one answer is correct):

- a) The Dirac operator on $\mathbb{R}^{3,1}$,
- b) The Dirac operator on $\mathbb{R}^4$,
- c) The operator $\frac{\partial}{\partial x} - \frac{\partial}{\partial y}$ on $\mathbb{R}^2$.

Problem 1: Aharonov Bohm

We parameterize $\mathbb{R}^3$ with three coordinates x, y and z. Instead of x and y, we will also use cylindrical coordinates r and θ in this exercise, where

$$x = r\cos\theta, \qquad y = r\sin\theta. \tag{1}$$

We study an infinitely long cylindrical solenoid along the z-axis, with radius R in the (x, y)-plane. For $r \leq R$, the solenoid contains a one-form gauge field

$$A = \frac{1}{2}r^2 d\theta \qquad \text{for } r \leq R \tag{2}$$

4 *pts* a. Write A in terms of x- and y-coordinates.

solution a. From $x = r\cos(\theta)$ *and* $y = r\sin(\theta)$ *we get that* $r^2 = x^2 + y^2$ *and* $\theta = \arctan\left(\frac{y}{x}\right)$.

2 pts

$$\begin{aligned} d\theta &= \frac{\partial\theta}{\partial x}\mathrm{d}x + \frac{\partial\theta}{\partial y}\mathrm{d}y \\ &= -\frac{y}{x^2+y^2}\mathrm{d}x + \frac{x}{x^2+y^2}\mathrm{d}y \end{aligned}$$

Alternatively use that

$$\begin{aligned} dx &= -r\sin\theta d\theta + \cos\theta dr &&= -yd\theta + \frac{x}{r}dr \\ dy &= r\cos\theta d\theta + \sin\theta dr &&= xd\theta + \frac{y}{r}dr \end{aligned}$$

to deduce that

$$(x^2 + y^2)d\theta = xdy - ydx.$$

2 pts

$$A = \frac{1}{2}r^2\mathrm{d}\theta = \frac{1}{2}(x^2+y^2)\left(-\frac{y}{x^2+y^2}\mathrm{d}x + \frac{x}{x^2+y^2}\mathrm{d}y\right) = \frac{1}{2}(x\mathrm{d}y - y\mathrm{d}x)$$

3 *pts* b. Show that the field strength two-form equals $F = dx \wedge dy$.

solution b. 1 pt Recall that $F = dA$.

2 pts $dA = \frac{1}{2}d(xdy - ydx) = \frac{1}{2}(dx \wedge dy - dy \wedge dx) = dx \wedge dy$.

3 *pts* c. Argue that the previous result proves that A is not an exact form.

solution c. 1 pt. If A were an exact form then there would be a 0*-form (function)* f *s.t.* $A = df$.

2 pt. This would imply:

$$0 \neq \mathrm{F} = \mathrm{dA} = \mathrm{d}(\mathrm{d}f) = \mathrm{d}^2 f = 0$$

The contradiction means that A is not exact.

For $r \geq R$, we assume that the size of the gauge field becomes constant:

$$A = \frac{1}{2}R^2 d\theta \qquad \text{for } r \geq R \tag{3}$$

3 *pts* d. For both the region inside and outside the solenoid, explain whether there are electric and/or magnetic fields present (and if so, which of the two), and whether those fields point in the x-, y- or z-direction. You don't have to worry about signs, so you don't need to mention whether a field points in the positive or negative direction.

solution d. *1 pt* *Outside the solenoid $\mathrm{F} = d\frac{1}{2}R^2 d\theta = \frac{1}{2}R^2 d^2\theta = 0$ so there is neither a magnetic nor an electric field there.*

2 pts *Inside the solenoid we have found in b) $\mathrm{F} = \mathrm{d}x \wedge \mathrm{d}y$, so there is a field. Since only the F_{xy} component of F is non-zero we recall from the shape of the field strength two-form that we have a magnetic field pointing in the z direction.*

Quantum mechanics tells us that when an electron travels around a closed loop γ in space, its wave function picks up a phase

$$\phi = \frac{e}{\hbar}\int_\gamma A \tag{4}$$

Here, e is the electron charge and $\hbar$ is Planck's constant.

4 *pts* e. Use Stokes' theorem to compute the phase shift of an electron that moves around the solenoid once and returns to its original location. Again, you do not need to worry about the sign of the answer.

solution e. *2 pts* *Let D be an area such that $\partial D = \gamma$. Using Stokes' theorem we get:*

$$\phi = \frac{e}{\hbar}\int_\gamma \mathrm{A} = \frac{e}{\hbar}\int_D \mathrm{dA}$$

2 pts *Since $dA = dx \wedge dy$ inside and 0 outside the solenoid, the integral above simply measures the area of the projection to the (x, y)-plane of part of D inside the solenoid, so*

$$\phi = \frac{e}{\hbar}\pi R^2.$$

We now want to remove the solenoid ("shrink it to zero size") while keeping a nonzero gauge field

$$A = \frac{1}{2} C d\theta, \tag{5}$$

everywhere, with C some fixed constant.

4 *pts* f. Explain why we cannot do this in $\mathbb{R}^3$, but *can* do this if we take our space to be $M = \mathbb{R}^3 \setminus \{z\text{-axis}\}$.

solution f. 2 *pts* *The gauge field* A *is not well defined along the z axis (where $x, y = 0$) since $d\theta$ is ill-defined there as shown by the result of (a). (The r^2 in a) saved us in the previous part of the exercise.)*

2 *pts* *If we remove the z-axis the form $d\theta$ is well-defined again since it only has a singularity when btoh x and y are* 0.

In the lectures, we have seen that if the second cohomology group $H^2_{dR}(M)$ vanishes (as is the case here), one has the identity

$$\Omega^2_{cl}(M) \cong \frac{\Omega^1(M)/d\Omega^0(M)}{H^1_{dR}(M)} \tag{6}$$

4 *pts* g. Using this identity, explain why the field strength F, in the setup of exercise (f), can *not* be used to describe all physically inequivalent field configurations.
Note: you do not need to compute anything; an explanation in words (and/or symbols) suffices.

solution g. *We have*

$$\Omega^2_{cl} = \frac{\Omega^1(M)/\mathrm{d}\Omega_0(M)}{\mathrm{H}^1_{\mathrm{dR}}(M)}$$

1 *pt* *The field strength is a closed two-form hence lives in the left hand side.*

1 *pt* *The gauge field A is a* 1*-form hence lives in $\Omega^1(M)$. Exercise (e) tells us that A is physically relevant up to gauge transformations, i.e. A is physically relevant up to $d\Omega^0(M)$.*

1 *pt* $H^1_{\mathrm{dR}}(M) \neq 0$.

1 *pt*

$$F \in \Omega^2_{cl} = \frac{\Omega^1(M)/\mathrm{d}\Omega_0(M)}{\mathrm{H}^1_d R} < \Omega^1(M)/\mathrm{d}\Omega_0(M) \ni A.$$

Thus the field strength can not catch all physically inequivalent field configurations.

Problem 2: BF theory

In this exercise, we study a quantum field theory in 4 Euclidean dimensions known as BF theory. Its action is

$$S_{BF} = \int_M \mathrm{Tr}\ (B \wedge F + \frac{\Lambda}{12} B \wedge B), \tag{7}$$

Here, M is a 4-dimensional manifold, which for now we assume to have no boundary. F is the field strength of a connection that in this exercise you can assume to be represented by a Lie algebra valued 1-form A, so $F = dA + A \wedge A$. The field B is a Lie algebra valued *2-form* field. Λ is a (real) numerical constant.

4 *pts* a. Find the equation of motion that results from the variation of the field B. Show that plugging the solution to this equation of motion into the action leads to

$$S_{EOM} = -\frac{3}{\Lambda} \int_M \mathrm{Tr}\ (F \wedge F) \tag{8}$$

solutiona a. 1 pt Varying the action with respect to B leads to:

$$\delta_B S_{BF} = \int_M \mathrm{Tr} \left(\delta B \wedge \mathrm{F} + \frac{\Lambda}{6} \delta B \wedge B \right) = \int_M \mathrm{Tr} \left(\delta B \wedge \left(\mathrm{F} + \frac{\Lambda}{6} B \right) \right),$$

where we used that Tr $(\delta B \wedge B + B \wedge \delta B) =$ *Tr* $(2\delta B \wedge B)$ *by the cyclic property of the trace.*

1 pt Thus the E-L equation has the solution $B = -\frac{6}{\Lambda}\mathrm{F}$

2 pt Plugging this back into the action we get:

$$S_{BF} = \int_M \mathrm{Tr} \left(-\frac{6}{\Lambda} \mathrm{F} \wedge \mathrm{F} + \frac{\Lambda}{12} \frac{36}{\Lambda^2} \mathrm{F} \wedge \mathrm{F} \right) = -\frac{3}{\Lambda} \int_M \mathrm{Tr}(\mathrm{F} \wedge \mathrm{F})$$

The path integral for BF theory can be written as

$$Z = \int DA \int DB\, e^{iS_{BF}} \tag{9}$$

4 *pts* b. Argue that after doing the B-integral, the resulting path integral equals

$$Z = \int DA\, e^{iS_{EOM}}. \tag{10}$$

That is: in this example the path integral over B can be carried out by simply inserting the solution to its equation of motion in the action. *Hint: since a path integral is not well-defined mathematically, we do **not** expect a rigorous proof here. Therefore, you are allowed to use any argument that would hold for an ordinary integral and assume without further proof that it holds for path integrals as well.*

solution b. *2 pts* *We complete the square under the trace.*

$$\frac{\Lambda}{12} Tr\,(B + \frac{6}{\Lambda}\mathrm{F})^2 = Tr\,(\frac{\Lambda}{12} B \wedge B + \frac{1}{2} B \wedge \mathrm{F} + \frac{1}{2}\mathrm{F} \wedge B + \frac{3}{\Lambda}\mathrm{F} \wedge \mathrm{F})$$
$$= Tr\,(\frac{\Lambda}{12} B \wedge B + B \wedge \mathrm{F} + \frac{3}{\Lambda}\mathrm{F} \wedge \mathrm{F})$$

where we used the cyclicity of the trace. Thus we find $L_{BF} = \frac{\Lambda}{12}(B + \frac{6}{\Lambda}\mathrm{F})^2 - \frac{3}{\Lambda}\mathrm{F}^2$.

2 pts *The first term that is squared gives a Gaussian integral term proportional to* π *while the last term is exactly the term one given.*

3 *pts* c. Describe over which space the resulting A-integral should be performed.

solution c. *1 pt* *The integral is performed over all gauge fields* A *which take values in the space of connections on a suitable principal bundle* P *over* M*, we will denote it* $C(P)$.

1 pt *Since* Z *should be gauge invariant we need to get rid of the gauge group* $\mathcal{G}(P)$ *of* P.

1 pt *The gauge group acts on* $C(P)$ *and thus the integral should be performed over* $C(P)/\mathcal{G}(P)$.

4 *pts* d. Show that the expression Tr $(F \wedge F)$ appearing in the action is a total derivative, and explain why this implies that on a manifold M without boundary, BF theory is not a very interesting theory.

solution d. *2 pt* *First of all*

$$F \wedge F = (dA + A \wedge A) \wedge (dA + A \wedge A)$$
$$= dA \wedge dA + dA \wedge A \wedge A + A \wedge A \wedge dA + A \wedge A \wedge A \wedge A.$$

The last term vanishes inside the trace since it can be written as a commutator $\frac{1}{8}\,[[A, A], [A, A]] = A \wedge A \wedge A \wedge A$. *The second and third terms are equal in the trace so we find*

$$Tr\,(F \wedge F) = Tr\,(dA \wedge dA + 2dA \wedge A \wedge A).$$

1 pt

$$d(A \wedge dA) = dA \wedge dA$$

and

$$d(A \wedge A \wedge A) = dA \wedge A \wedge A - A \wedge dA \wedge A + A \wedge A \wedge dA.$$

So that in the trace

$$d\,Tr\,(\frac{2}{3} A \wedge A \wedge A) = Tr\,(dA \wedge A \wedge A)$$

and we find

$$d\,Tr\,(A \wedge dA + \frac{2}{3} A \wedge A \wedge A) = Tr\,(F \wedge F).$$

1 pt *Since* $Tr\,(F \wedge F) = dC$ *fro some* 3*-form* C *we have by Stokes' theorem that* $\int_M Tr\,(F \wedge F) = \int_{\partial M} C$*. Thus in the case that* $\partial M = \emptyset$ *the action for BF theory only tells us that* $B = -\frac{6}{\Lambda} F$*.*

5 *pts* e. Using characteristic classes, explain how you could also have arrived at the conclusion of part (d) without a lengthy computation.

solution e. *1 pt* *Characteristic classes are given by invariant polynomials of the curvature of a connection*

1 pt *Characteristic classes do not depend on the choice of connection*

1 pt *The connection d has no curvature*

1 pt *This means that the cohomolgy class of* $Tr\,(F \wedge F) = (-4\pi^2)(c_2 - c_1^2)$ *is* 0*, i.e. the form is exact.*

1 pt *In fact* $Tr\,(F \wedge F) = dL(d, d + A)$ *where* $L(d, d + A)$ *is the transgression form corresponding to the invariant polynomial* $Tr\,(F \wedge F)$*.*

Problem 3: The Euler Class

Recall the definition of the Hodge star operator $\star\colon \Omega^p(M) \to \Omega^{n-p}(M)$ on a Riemannian manifold (M,g) of dimension $n = 2m$. It is given by

$$\star\,(dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_p}) = \frac{\sqrt{|g|}}{(n-p)!}\epsilon^{\mu_1\cdots\mu_p}{}_{\nu_1\cdots\nu_{n-p}} dx^{\nu_1} \wedge \cdots \wedge dx^{\nu_{n-p}}$$

on the basis p-forms. Recall in particular that $\star^2\alpha = (-1)^{np+p}\alpha$ for a p-form α. Then set

$$d^*\alpha = (-1)^{np+n+1} \star d \star \alpha$$

and recall that this is the **adjoint** of the exterior derivative d for the usual non-degenerate (positive definite) bilinear pairing induced on $\Omega^\bullet(M)$ by g. In this problem we are considering $\Omega^\bullet(M)$ as sections of the exterior algebra of the complexified cotangent bundle, i.e. $\Omega^\bullet(M) = \Gamma(M; \bigwedge^\bullet T^*M_{\mathbb{C}})$. In other words we are considering complex-valued differential forms.

3 *pts* a. Show that $\text{Ker}\ d + d^* = \text{Ker}\ d \cap \text{Ker}\ d^*$ if we view all these operators as acting on $\Omega^\bullet(M)$.
HINT: you do not need the explicit formula for the Hodge star.

solution a. *1 pt* *We show that* $\text{Ker}\ d \cap \text{Ker}\ d^* \subset \text{Ker}\ d + d^*$. *So suppose* $\alpha \in \Omega^\bullet(M)$ *such that* $d\alpha = d^*\alpha = 0$. *Then* $(d+d^*)\alpha = d\alpha + d^*\alpha = 0 + 0 = 0$.

2 pt *We show that* $\text{Ker}\ d + d^* \subset \text{Ker}\ d \cap \text{Ker}\ d^*$. *We use the facts that* $d + d^*$ *is self-adjoint and* $d^2 = 0$ *which implies* $(d^*)^2 = (d^2)^* = 0$. *Then suppose* $\alpha \in \Omega^\bullet(M)$ *such that* $d\alpha + d^*\alpha = 0$ *then*

$$0 = \langle (d+d^*)\alpha, (d+d^*)\alpha \rangle = \langle (d+d^*)^2\alpha, \alpha \rangle =$$

$$\langle dd^*\alpha + d^*d\alpha, \alpha \rangle = \langle d^*\alpha, d^*\alpha \rangle + \langle d\alpha, d\alpha \rangle.$$

Now since the pairing is positive definite this implies that $d^*\alpha = d\alpha = 0$.

It can be shown that the map $f\colon \text{Ker}\ d + d^* \to \bigoplus_{p=0}^n H^p_{\text{dR}}(M)$ that maps a form $\alpha \in \text{Ker}\ d + d^*$ to its cohomology class is an **isomorphism**.

Recall the definition of the Euler characteristic

$$\chi(M) := \sum_{i=0}^{n} (-1)^i \beta_i(M)$$

where $\beta_i(M) = \text{Dim}\ H^i_{\text{dR}}(M)$ are called the *Betti numbers* of M. From now on we set $X\colon E \to F$ to be the operator $d + d^*$ acting between the vector bundles

$$E = {\textstyle\bigwedge}^{even} T^*M_{\mathbb{C}} = \bigoplus_{i=0}^{m} {\textstyle\bigwedge}^{2i} T^*M_{\mathbb{C}} \quad \text{and} \quad F = {\textstyle\bigwedge}^{odd} T^*M_{\mathbb{C}} = \bigoplus_{i=0}^{m-1} {\textstyle\bigwedge}^{2i+1} T^*M_{\mathbb{C}}.$$

3 *pts* b. Show that Ind $X = \chi(M)$.

solution b. *1 pt* *First we note that* $\text{Ind } X = \text{Dim Ker } X - \text{Dim Coker } X = \text{Dim Ker } X - \text{Dim Ker } X^*$.

1 pt *Now* $X = d + d^* \colon E \to F$ *and* $X^* = d + d^* \colon F \to E$.

1 pt *The isomorphism* $\text{Ker } d+d^* \simeq \bigoplus_{k=0}^n H^k_{\text{dR}}(M)$ *implies that* $\text{Ker } X \simeq H^{even}_{\text{dR}}(M)$ *and* $\text{Ker } X^* \simeq H^{odd}_{\text{dR}}(M)$ *and thus*

$$\text{Ind } X = \text{Dim } H^{even}_{\text{dR}}(M) - \text{Dim } H^{odd}_{\text{dR}}(M) = \chi(M)$$

.

We will now use the result from (b) and the Atiyah–Singer index theorem to express $\chi(M)$ as the integral of a characteristic class. Recall the Atiyah–Singer index theorem for X

$$\text{Ind } X = (-1)^{\frac{n(n+1)}{2}} \int_M \text{Ch } (E - F) \frac{\text{Td } (TM_{\mathbb{C}})}{\text{e } (TM)}.$$

Recall that the classes in the integrand can be given in terms of their corresponding invariant polynomials of the curvature F of a connection on the vector bundle V:

$$\text{Ch } (V) = \text{Tr } e^{\frac{iF}{2\pi}} \quad \text{and} \quad \text{Td } (V) = \text{Det } \frac{iF}{2\pi(1 - e^{-\frac{iF}{2\pi}})}.$$

To determine the Euler class we first recall the splitting principle which essentially says that we may consider $V = L_1 \oplus L_2 \oplus \ldots \oplus L_k$ a sum of line bundles L_i. For the complexified tangent bundle we find in particular $TM_{\mathbb{C}} = L_1 \oplus \overline{L_1} \oplus \ldots \oplus L_m \oplus \overline{L_m}$ with first chern classes $x_i = c_1(L_i) = -c_1(\overline{L_i})$; we have

$$\text{e } (TM) = \prod_{i=1}^m x_i.$$

For the following subproblem it will also be useful to consider the facts that for vector bundles L and L' we have

- $c_1(L) = \text{Tr } \frac{iF}{2\pi}$ for F the curvature of a connection on L. Note in particular that this means that $c_1(L) = \frac{iF}{2\pi}$ if L is a line bundle.
- $c_1(L) = -c_1(L^*)$,
- $\text{Ch } (L \otimes L') = \text{Ch } (L) \text{Ch } (L')$,
- $\text{Ch } (L \oplus L') = \text{Ch } (L) + \text{Ch } (L')$ and
- $\bigwedge^\bullet (L \oplus L') = (\bigwedge^\bullet L) \otimes (\bigwedge^\bullet L')$.

5 *pts* c. Show that

$$\mathrm{Ind}\, X = \int_M \mathrm{e}\,(TM).$$

solution c. Since $\frac{2m(2m+1)}{2} = m \bmod 2$ *we need to show that*

$$(-1)^m \mathrm{Ch}\,(E-F)\frac{\mathrm{Td}\,(TM_{\mathbb{C}})}{\mathrm{e}\,(TM)} = \mathrm{e}\,(TM).$$

2 pt Since, using the splitting principle, we may assume

$$TM_{\mathbb{C}} = L_1 \oplus \overline{L_1} \oplus L_2 \oplus \overline{L_2} \oplus \ldots \oplus L_m \oplus \overline{L_m}.$$

and we know that $c_1(L) = \mathrm{Tr}(\frac{iF_L}{2\pi})$ *for* F_L *the curvature of a connection on* L *we find the form* $\frac{iF}{2\pi} = \mathrm{Diag}(x_1, -x_1, x_2, -x_2, \ldots, x_m, -x_m)$ *for a connection* F *on* $TM_{\mathbb{C}}$. *Thus*

$$\mathrm{Td}\,(TM_{\mathbb{C}}) = \mathrm{Det}\,\frac{iF}{2\pi(1-e^{-\frac{iF}{2\pi}})} = \prod_{i=1}^{m} \frac{(-1)^m x_i^2}{(1-e^{-x_i})(1-e^{x_i})}.$$

3 pt The decomposition of $TM_{\mathbb{C}}$ *implies the decomposition*

$$T^*M_{\mathbb{C}} = L_1^* \oplus \overline{L_1}^* \oplus L_2^* \oplus \overline{L_2}^* \oplus \ldots \oplus L_m^* \oplus \overline{L_m}^*$$

This implies by the identity for the wedge product above that

$$\sum_{i=0}^{n} (-1)^i \bigwedge^i T^*M_{\mathbb{C}} = \left(\bigotimes_{i=1}^{m} (\mathbb{C}_M - L_i^*)\right) \otimes \left(\bigotimes_{i=1}^{m} (\mathbb{C}_M - \overline{L_i}^*)\right)$$

where $\mathbb{C}_M$ *denotes the trivial line bundle on* M. *Thus we find by the multiplicativity of* Ch *that*

$$\mathrm{Ch}\,(E-F) = \mathrm{Ch}\,(\sum_{i=0}^{n} (-1)^i \bigwedge^i T^*M_{\mathbb{C}}) = \prod_{i=1}^{m} \mathrm{Ch}\,(\mathbb{C}_M - L_i^*)\mathrm{Ch}\,(\mathbb{C}_M - \overline{L_i}^*)$$

Thus using the additivity of Ch *and the facts that* $\mathrm{Ch}\,(\mathbb{C}_M) = e^0 = 1$, $\mathrm{Ch}\,(L_i^*) = e^{-x_i}$ *and* $\mathrm{Ch}\,(\overline{L_i}^*) = e^{x_i}$ *we find that*

$$\mathrm{Ch}\,(E-F) = \prod_{i=1}^{m} \mathrm{Ch}\,(\mathbb{C}_M - L_i^*)\mathrm{Ch}\,(\mathbb{C}_M - \overline{L_i}^*) = \prod_{i=1}^{m} (1-e^{-x_i})(1-e^{x_i})$$

Putting this together with the fact that $\mathrm{e}\,(TM) = \prod_{i=1}^{m} x_i$ *yields*

$$(-1)^m \mathrm{Ch}\,(E-F)\frac{\mathrm{Td}\,(TM_{\mathbb{C}})}{\mathrm{e}\,(TM)} = \prod_{i=1}^{m} \frac{(-1)^m(1-e^{-x_i})(1-e^{x_i})(-1)^m x_i^2}{(1-e^{-x_i})(1-e^{x_i})x_i}$$

$$= \prod_{i=1}^{m} x_i = \mathrm{e}\,(TM).$$

Consider the torus $\mathbb{T}^2$ given by $S^1 \times S^1$ with coordinates (θ, ϕ) ranging from 0 to 2π. We equip it with the metric g obtained by restricting the Euclidean metric to the embedding $\iota\colon \mathbb{T}^2 \hookrightarrow \mathbb{R}^3$ given by

$$\iota(\theta,\phi) = ((2+\cos\theta)\cos\phi, (2+\cos\theta)\sin\phi, \sin\theta).$$

Note that this differs greatly from the metric induced by viewing $\mathbb{T}^2$ as the unit square with opposite sides identified, even though the topology remains unchanged.

Recall that the Christoffel symbols Γ^k_{ij} corresponding to a local frame $\{e_1, e_2\}$ are defined by

$$\nabla_{e_i} e_j = \sum_{k=1}^{2} \Gamma^k_{ij} e_k$$

where ∇ is the Levi-Civita connection on $T\mathbb{T}^2$. In the following you may use that the non-zero Christoffel symbols with respect to the frame $\{\frac{\partial}{\partial\theta}, \frac{\partial}{\partial\phi}\}$ are given by

$$\Gamma^\phi_{\theta\phi} = \Gamma^\phi_{\phi\theta} = \frac{-\sin\theta}{2+\cos\theta} \quad \text{and} \quad \Gamma^\theta_{\phi\phi} = \sin\theta(2+\cos\theta).$$

4 *pts* d. Determine the function $F(\theta,\phi)$ such that

$$\mathrm{e}\,(TM) = F(\theta,\phi) d\theta \wedge d\phi,$$

when we use the curvature of the Levi-Civita connection to find $\mathrm{e}\,(TM)$. *HINT: to obtain the correct normal form of the curvature you will have to change to the orthonormal frame* $\{\frac{\partial}{\partial\theta}, \frac{1}{2+\cos\theta}\frac{\partial}{\partial\phi}\}$.

solution d. *2 pt* *To compute* $\mathrm{e}\,(TM)$ *we need to find the matrix* Γ *such that* $\nabla = d + \Gamma$, *in which case* $F_\nabla = d\Gamma + \Gamma\wedge\Gamma$.

1 pt *Since* $d\partial_\theta = d\partial_\phi = 0$ *we have*

$$\Gamma = \begin{pmatrix} \Gamma^\theta_{\theta\theta} d\theta + \Gamma^\theta_{\phi\theta} d\phi & \Gamma^\theta_{\theta\phi} d\theta + \Gamma^\theta_{\phi\phi} d\phi \\ \Gamma^\phi_{\theta\theta} d\theta + \Gamma^\phi_{\phi\theta} d\phi & \Gamma^\phi_{\theta\phi} d\theta + \Gamma^\phi_{\phi\phi} d\phi \end{pmatrix} =$$

$$= \begin{pmatrix} 0 & \sin\theta(2+\cos\theta) d\phi \\ \frac{-\sin\theta}{2+\cos\theta} d\phi & \frac{-\sin\theta}{2+\cos\theta} d\theta \end{pmatrix}.$$

So we find

$$F_\nabla = \begin{pmatrix} 0 & \cos\theta(2+\cos\theta) \\ \frac{-\cos\theta}{2+\cos\theta} & 0 \end{pmatrix} d\theta \wedge d\phi.$$

1 pt *To get it in the normal form for which we may read of the value of the first Chern class (and thus euler class in this case) we need to express it in an orthonormal basis which yields*

$$\begin{pmatrix} 0 & \cos\theta \\ -\cos\theta & 0 \end{pmatrix} d\theta \wedge d\phi.$$

Thus we find that $F(\theta,\phi) = \frac{i\cos\theta}{2\pi}$.

6 *pts* e. How could you have determined the value of

$$\int_0^{2\pi}\int_0^{2\pi} F(\theta,\phi)d\theta d\phi$$

without the computation at (d)?

solution e. *1 pt* *The value of the integral is of course* 0 *since* $\int_0^{2\pi}\cos\theta d\theta = 0$.

3 pt *The Euler class is a characteristic class and thus a topological invariant, this mean we may use any connection to compute it. Additionally the Euler characteristic is obviously a topological invariant.*

2 pt *The model of the 2-torus given by identifying opposite sides of a square is obviously flat. This means the curvature vanishes identically and thus* $\mathrm{e}\,(TM) = 0$.

4 *pts* f. Use arguments and the previous results rather than complicated computations to determine the Betti numbers of the 2-torus.

solution f. *1 pt* *The torus is connected and thus* $\beta_0(\mathbb{T}^2) = 1$.

1 pt *The torus is oriented, connected and compact and thus* $\beta_2(\mathbb{T}^2) = 1$

2 pt *By the above* $\chi(\mathbb{T}^2) = 0$ *and thus* $\beta_1(\mathbb{T}^2) = \beta_0(\mathbb{T}^2) + \beta_0(\mathbb{T}^2) = 2$.

$\star$ Problem 4: The Clifford algebra and the rotation Lie algebra 15 *pts*

We consider the Clifford algebra Cliff_3 generated by $\psi_i,\ i=1,2,3$ satisfying

$$\psi_i\psi_j = -\psi_j\psi_i, \quad i\neq j,$$
$$\psi_i^2 = -1.$$

4 *pts* a. Show that the elements $J_i \in \mathrm{Cliff}_3(\mathbb{R}),\ i=1,2,3$, defined as

$$J_1 := \frac{1}{2}\psi_2\psi_3, \quad \text{and cyclic permutations,}$$

satisfy the commutation relations of the Lie algebra $\mathfrak{so}(3)$:

$$[J_1, J_2] = J_3,, \quad \text{and cyclic permutations.}$$

solution a.

$$[J_1, J_2] = \frac{1}{4}(\psi_2\psi_3\psi_3\psi_1 - \psi_3\psi_1\psi_2\psi_3) =$$

$$\frac{1}{4}(-\psi_2\psi_1 - (-1)^2\psi_1\psi_3\psi_3\psi_2) = \frac{1}{4}(\psi_1\psi_2 + \psi_1\psi_2) = J_3$$

cyclic permutations will not change this calculation since the J_i are also defined as cyclic permutations of each other.

5 *pts* b. For any element $c \in \mathrm{Cliff}_3(\mathbb{R})$ define its exponential by the formal power series

$$\exp(c) := \sum_{k=0}^{\infty} \frac{c^k}{k!} \in \mathrm{Cliff}_3(\mathbb{R}).$$

Show that

$$\exp(\theta\psi_i\psi_j) = \cos(\theta) + \sin(\theta)\psi_i\psi_j, \quad i\neq j.$$

for all $i,j = 1,2,3,\ i\neq j$ and $\theta\in\mathbb{R}$.

solution b. *1 pt Note that* $\psi_i\psi_j\psi_i\psi_j = (-1)^3 = -1$.

1 pt So that we get that $(\psi_i\psi_j)^{2k} = ((\psi_i\psi_j)^2)^k = (-1)^k$

1 pt For the odd case we get that $(\psi_i\psi_j)^{2k+1} = (\psi_i\psi_j)^{2k}\psi_i\psi_j = (-1)^k\psi_i\psi_j$.

1 pt Filling in the exponential we get

$$\exp(\theta\psi_i\psi_j) = \sum_{k=0}^{\infty}\frac{1}{k!}\theta^k(\psi_i\psi_j)^k = \sum_{k=0}^{\infty}\frac{(-1)^k}{2k!}\theta^{2k} + \sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)!}\theta^{2k+1}\psi_i\psi_j.$$

1 pt Thus we recognize that since $\exp(i\theta) = \cos\theta + i\sin\theta$ *we have*

$$\exp(\theta\psi_i\psi_j) = \cos(\theta) + \sin(\theta)\psi_i\psi_j.$$

As usual we use $\psi(v)$ to denote $x\psi_1 + y\psi_2 + z\psi_3$ for $(x, y, z) = v \in \mathbb{R}^3$. Recall the definition of the group Spin(3):

$$\text{Spin }(3) := \{\psi(v_1)\psi(v_2)\ldots\psi(v_{2k}) \mid \|v_i\| = 1, k \in \mathbb{N}\},$$

where the $\|\cdot\|$ denotes the Euclidean norm.

3 pts c. Show that $\exp(\theta\psi_i\psi_j) \in \text{Spin }(3)$ for all $i, j = 1, 2, 3$ and $\theta \in \mathbb{R}$.

solution c. *We will denote the ith standard basis vector by* e_i.

1 pt Note that by (b) $\exp(\theta\psi_i\psi_j) = \cos(\theta) + \sin(\theta)\psi_i\psi_j$.

1 pt Note that $\cos(\theta) + \sin(\theta)\psi_i\psi_j = \psi_i(-\cos(\theta)\psi_i + \sin(\theta)\psi_j)$

1 pt Note that $\|e_i\| = 1$ *and* $\|-cos(\theta)e_i + \sin(\theta)e_j\| = \cos^2(\theta) + \sin^2(\theta) = 1$ *while we just showed that* $\exp(\theta\psi_i\psi_j = \psi(e_i)\psi(-\cos(\theta)e_i + \sin(\theta)e_j)$. *So indeed these are elements of* Spin (3).

3 pts d. Denote by G the Lie group given by the exponents of elements of the Lie algebra in (a), i.e. the Lie algebra generated by the J_i. Consider the group homomorphism $\rho\colon G \to SO(3)$ given on the generators by

$$\rho(\exp(\theta J_1)) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(\theta) & -\sin(\theta) \\ 0 & \sin(\theta) & \cos(\theta) \end{pmatrix}, \quad \rho(\exp(\theta J_2)) = \begin{pmatrix} \cos(\theta) & 0 & \sin(\theta) \\ 0 & 1 & 0 \\ -\sin(\theta) & 0 & \cos(\theta) \end{pmatrix}$$

and

$$\rho(\exp(\theta J_3)) = \begin{pmatrix} \cos(\theta) & -\sin(\theta) & 0 \\ \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Argue that this map is $2:1$ (two to one).

solution d. *1 pt Note first that* $J_i = \frac{1}{2}\psi_j\psi_k$ *for* i, j, k *some cyclic permutation of* $1, 2, 3$ *and thus* $\exp(\theta J_i) = \cos(\frac{\theta}{2}) + \sin(\frac{\theta}{2})\psi_j\psi_k$.

1 pt So we see that $\rho(\exp(2\pi J_i)) = \rho(\exp(0J_i))$, *but*

$$\exp(2\pi J_i) = \cos(\pi) + \sin(\pi)\psi_j\psi_k = -1 \neq 1 = \cos(0) + \sin(0)\psi_j\psi_k = \exp(0J_i).$$

1 pt Since the element -1 *we find in the kernel is the same for each* J_i *we conclude that the kernel of* ρ *must equal* $\{-1, 1\}$ *and thus the map is* $2:1$.