TEST EXAM RIEMANN SURFACES

Exercise 1. Let *X* be a Riemann surface.

- a) Prove that a holomorphic differential is closed.
- b) Prove that for X compact, the map $\Omega^1(X) \to Rh^1(X)$ is injective.

Exercise 2. Let *X* be a compact Riemann surface of genus *g*. Recall that we have proved that there exists a meromorphic function *f* on *X* with a single pole of order $\leq g + 1$. Let $\varphi : X \to X$ be an automorphism, i.e., an invertible holomorphic map. By considering the function $f - f \circ \varphi$, show that φ can have at most 2g + 2 fixed points.

Exercise 3. Consider the compact Riemann surface *X* determined by the equation

$$w^3 = (z - \alpha_1)^2 (z - \alpha_2) \cdots (z - \alpha_k),$$

where $\alpha_1, \ldots, \alpha_k$ are distinct points in \mathbb{C} . Assume that $k = 2 \mod(3)$.

- a) *X* is a branched cover of \mathbb{P}^1 via the projection onto the *z*-coordinate. What is the degree of this covering?
- b) Determine the branch points of this covering together with their branching number.
- c) What is the genus of *X*?
- d) Find a basis for $\Omega^1(X)$.
- e) What changes if $k \neq 2 \mod(3)$?

Exercise 4. Let *X* be a compact Riemann surface.

a) Show that the sequence of sheaves

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathcal{O} \longrightarrow \mathcal{O}^* \longrightarrow 0,$$

where the third map is given by $f \mapsto e^{2\pi i f}$, is exact.

b) Use the long exact sequence in cohomology to construct the exact sequence

$$0 \longrightarrow H^1(X, \mathcal{O})/H^1(X, \mathbb{Z}) \longrightarrow H^1(X, \mathcal{O}^*) \stackrel{\delta}{\longrightarrow} H^2(X, \mathbb{Z}).$$

c) Use Serre duality to prove that

$$H^1(X, \mathcal{O})/H^1(X, \mathbb{Z}) \cong \operatorname{Jac}(X),$$

where Jac(X) is the Jacobian of X defined in the course as the quotient

$$\operatorname{Jac}(X) \cong \Omega^{1}(X) / \operatorname{Per}(\alpha_{1}, \ldots, \alpha_{g}),$$

where $\{\alpha_i\}_{i=1}^g$ is a basis of $\Omega^1(X)$ and $Per(\alpha_1, \ldots, \alpha_g)$ the associated period lattice.

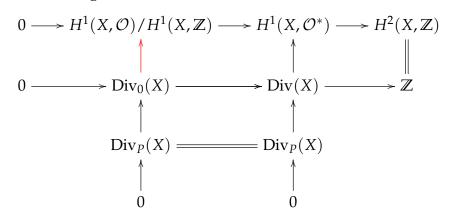
d) Show that the sequence

$$0 \longrightarrow \mathcal{O}^* \longrightarrow \mathcal{M}^* \longrightarrow \operatorname{Div} \longrightarrow 0$$

is exact. Write down the beginning of the long exact sequence in cohomology and show that this leads to the sequence

$$0 \longrightarrow \operatorname{Div}_{P}(X) \longrightarrow \operatorname{Div}(X) \stackrel{\phi}{\longrightarrow} H^{1}(X, \mathcal{O}^{*}).$$

e) We now use the following fact: $H^2(X,\mathbb{Z}) \cong \mathbb{Z}$, in such a way that $\delta \circ \phi$ equals taking the degree of a divisor. Collect all the information gathered so far in the diagram



Argue that the red arrow exists, and use this to give a proof of Abel's theorem.