TEST EXAM RIEMANN SURFACES

Exercise 1. Let X be a Riemann surface.

a) Prove that a holomorphic differential is closed.
b) Prove that for X compact, the map $\Omega^1(X) \to Rh^1(X)$ is injective.

Exercise 2. Let X be a compact Riemann surface of genus g. Recall that we have proved that there exists a meromorphic function f on X with a single pole of order $\leq g+1$. Let $\varphi : X \to X$ be an automorphism, i.e., an invertible holomorphic map. By considering the function $f - f \circ \varphi$, show that φ can have at most $2g+2$ fixed points.

Exercise 3. Consider the compact Riemann surface X determined by the equation

$$w^3 = (z - \alpha_1)^2 (z - \alpha_2) \cdots (z - \alpha_k),$$

where $\alpha_1, \ldots, \alpha_k$ are distinct points in $\mathbb{C}$. Assume that $k = 2 \bmod(3)$.

a) X is a branched cover of $\mathbb{P}^1$ via the projection onto the z-coordinate. What is the degree of this covering?
b) Determine the branch points of this covering together with their branching number.
c) What is the genus of X?
d) Find a basis for $\Omega^1(X)$.
e) What changes if $k \neq 2 \bmod(3)$?

Exercise 4. Let X be a compact Riemann surface.

a) Show that the sequence of sheaves

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathcal{O} \longrightarrow \mathcal{O}^* \longrightarrow 0,$$

where the third map is given by $f \mapsto e^{2\pi i f}$, is exact.

b) Use the long exact sequence in cohomology to construct the exact sequence

$$0 \longrightarrow H^1(X, \mathcal{O})/H^1(X, \mathbb{Z}) \longrightarrow H^1(X, \mathcal{O}^*) \xrightarrow{\delta} H^2(X, \mathbb{Z}).$$

c) Use Serre duality to prove that

$$H^1(X, \mathcal{O})/H^1(X, \mathbb{Z}) \cong \mathrm{Jac}(X),$$

where $\mathrm{Jac}(X)$ is the Jacobian of X defined in the course as the quotient

$$\mathrm{Jac}(X) \cong \Omega^1(X)/\mathrm{Per}(\alpha_1, \ldots, \alpha_g),$$

where $\{\alpha_i\}_{i=1}^{g}$ is a basis of $\Omega^1(X)$ and $\mathrm{Per}(\alpha_1, \ldots, \alpha_g)$ the associated period lattice.

d) Show that the sequence

$$0 \longrightarrow \mathcal{O}^* \longrightarrow \mathcal{M}^* \longrightarrow \mathrm{Div} \longrightarrow 0$$

is exact. Write down the beginning of the long exact sequence in cohomology and show that this leads to the sequence

$$0 \longrightarrow \mathrm{Div}_P(X) \longrightarrow \mathrm{Div}(X) \xrightarrow{\phi} H^1(X, \mathcal{O}^*).$$

e) We now use the following fact: $H^2(X, \mathbb{Z}) \cong \mathbb{Z}$, *in such a way that* $\delta \circ \phi$ *equals taking the degree of a divisor.* Collect all the information gathered so far in the diagram

$$\begin{array}{ccccccc}
0 & \longrightarrow & H^1(X,\mathcal{O})/H^1(X,\mathbb{Z}) & \longrightarrow & H^1(X,\mathcal{O}^*) & \longrightarrow & H^2(X,\mathbb{Z}) \\
 & & \uparrow & & \uparrow & & \| \\
0 & \longrightarrow & \mathrm{Div}_0(X) & \longrightarrow & \mathrm{Div}(X) & \longrightarrow & \mathbb{Z} \\
 & & \uparrow & & \uparrow & & \\
 & & \mathrm{Div}_P(X) & = & \mathrm{Div}_P(X) & & \\
 & & \uparrow & & \uparrow & & \\
 & & 0 & & 0 & &
\end{array}$$

Argue that the red arrow exists, and use this to give a proof of Abel's theorem.