Lecture notes on Noncommutative Geometry

Hessel Posthuma

Version: June 24, 2015

CHAPTER 1

C^*-algebras and the Gelfand–Naimark theorem

1. Definition and basics

1.1. Definition and examples. Recall that an *involution* of an algebra over $\mathbb{C}$ is an anti-linear map $a \mapsto a^*$, $a \in A$ satisfying

$$(ab)^* = b^*a^*, \quad \text{for all } a, b \in A,$$

and $(a^*)^* = a$.

DEFINITION 1.1. A C^*-algebra is an algebra A over $\mathbb{C}$ equipped with an involution $a \mapsto a^*$ together with a norm $\| \; \| : A \to \mathbb{R}_{\geq 0}$ satisfying the properties:

i) A is complete with respect to the norm $\| \; \|$,

ii)
$$\|ab\| \leq \|a\| \, \|b\| \,, \quad \text{for all } a, b \in A,$$

iii)
$$\|a^*a\| = \|a\|^2 \,, \quad \text{for all } a \in A,$$

REMARK 1.2.

- The identity iii) above involving the $*$ is called the *C^*-identity*: without it we have defined a *Banach algebra* (with or without an involution.)
- It follows easy from the axioms (exercise!) that the $*$-involution is isometric:
$$\|a^*\| = \|a\| \,, \quad \text{for all } a \in A.$$
- When A has a unit, we call A a unital C^*-algebra. Clearly we have that $\|1\| = 1$.

REMARK 1.3. Property ii) above implies that left and right multiplication with a fixed element in a Banach algebra is bounded, hence continuous. The following observation will be used at several places: *in a Banach algebra, the multiplication is jointly continuous.* Indeed, if $a_n \to a$ and $b_n \to b$ for $n \to \infty$, we want to show that $a_n b_n \to ab$. Suppose that n is large enough so that $\|b - b_n\| \leq 1$. Then we find that

$$\begin{aligned} \|a_n b_n - ab\| &\leq \|a_n - a\| \, \|b_n\| - \|a\| \, \|b - b_n\| \\ &\leq \|a_n - a\| \left(\|b\| + 1 \right) - \|a\| \, \|b - b_n\| \,, \quad \to 0, \; n \to \infty \end{aligned}$$

EXAMPLE 1.4. The following two examples are the most important to keep in mind:

i) Let X be a compact Hausdorff topological space. Denote by $C(X)$ the commutative algebra of continuous $\mathbb{C}$-values functions. We introduce the following norm:

$$\|f\|_\infty := \sup_{x \in X} |f(x)|.$$

With this norm, and the involution given by $f^*(x) = \overline{f(x)}$, $C(X)$ is a C^*-algebra.

ii) Let $\mathcal{H}$ be a separable Hilbert space. Denote by $B(\mathcal{H})$ the algebra of bounded operators on $\mathcal{H}$, equippped with the adjoint as a $*$-operator. With the operator norm

$$\|A\| = \sup_{\substack{v \in \mathcal{H} \\ ||v||=1}} ||A(v)||,$$

this is a C^* algebra. Notice that this algebra is noncommutative unless $\mathcal{H}$ is one-dimensional.

EXAMPLE 1.5. To give an interesting example of the type of algebras that NCG is really about, consider the following: Let Γ be a discrete group, and equip it with the counting measure to form the Hilbert space $\mathcal{H} := L^2(\Gamma)$. Each element $\gamma \in \Gamma$ defines a unitary operator U_γ on $L^2(\Gamma)$ by means of the formula

$$(U_\gamma f)(\gamma') := f(\gamma'\gamma^{-1}), \quad f \in L^2(\Gamma).$$

These operators generate a subalgebra in $B(\mathcal{H})$ isomorphic to the group algebra $\mathbb{C}[\Gamma]$. We now that the norm closure in $B(\mathcal{H})$ of this algebra to define the so-called *reduced group C^*-algebra*, denoted $C^*_r(\Gamma)$. This algebra plays an important role in the harmonic analysis of the group.

DEFINITION 1.6. A morphism of C^* algebras is a morphism of algebras $\varphi : A \to B$ that preserves the $*$: $\varphi(a^*) = \varphi(a)^*$ for all $a \in A$.

1.2. The spectrum. An important notion in the theory of Banach and C^*-algebras is that of the spectrum of an element:

DEFINITION 1.7. Let A be a unital Banach algebra. The *spectrum* of $a \in A$ is defined as

$$\mathrm{sp}(a) := \{\lambda \in \mathbb{C},\ a - \lambda 1 \text{ is not invertible}\}.$$

EXERCISE 1.8. Determine the spectrum of a matrix $A \in M_n(\mathbb{C})$ and a function $f \in C(X)$, where X is a compact Hausdorff topological space.

THEOREM 1.9 (Gelfand). *For any $a \in A$, $\mathrm{sp}(a) \subset \mathbb{C}$ is:*

i) *contained in the closed disc of radius $\|a\|$,*
ii) *compact,*
iii) *nonemtpy.*

PROOF. First we have:

LEMMA 1.10. *For $a \in A$ with $\|a\| < 1$,*

$$\lim_{N\to\infty} \sum_{k=0}^{N} a^k = (1-a)^{-1}$$

PROOF OF LEMMA. Show that the sequence of partial sums is Cauchy. By completeness of A, the sum will converge, and the identity is clear. □

When $|\lambda| > \|a\|$, by the Lemma we have

$$(\lambda - a)^{-1} = \lambda^{-1}(1 - \lambda^{-1}a)^{-1} = \sum_{n=0}^{\infty} \lambda^{-n-1}a^n,$$

and this proves i).

Next, consider the *resolvent* $R_a : \mathbb{C}\backslash \mathrm{sp}(a) \to A$ defined as

$$R_a(\lambda) = (\lambda - a)^{-1}.$$

Let $\lambda \notin$ sp(a), and consider $|\zeta| < \left\|R(\lambda)^{-1}\right\|$. Because of the identity $(\lambda 1 - \zeta 1 - a) = (\lambda 1 - a)(1 - \zeta R(\lambda))$, we see that

$$R(\lambda - \zeta) = (\lambda 1 - \zeta 1 - a)^{-1} = \sum_{n=0}^{\infty} R_a(\lambda)^{n+1}\zeta^n. \tag{1}$$

It follows that the complement of $\mathrm{sp}(a) \subset \mathbb{C}$ is open, so that $\mathrm{sp}(a)$ is closed.

Finally, we turn to iii). Let $|\lambda| > 2\|a\|$. Then we have $1 - \left\|\frac{a}{\lambda}\right\| \geq \frac{1}{2}$, and therefore

$$\left\|(1-\lambda^{-1}a)^{-1} - 1\right\| = \left\|\sum_{k=1}^{\infty}\left(\frac{a}{\lambda}\right)^k\right\| \leq \sum_{k=1}^{\infty}\left\|\frac{a}{\lambda}\right\|^k \leq \frac{\|\lambda^{-1}a\|}{1 - \|\lambda^{-1}a\|} < 1.$$

It follows that $\left\|(1-\lambda^{-1}a)^{-1}\right\| < 2$, for $|\lambda| > 2\|a\|$.

DEFINITION 1.11. The dual Banach space A^* is the space of linear maps $\rho : A \to \mathbb{C}$ with

$$\|\rho\| := \sup_{\substack{a\in A \\ \|a\|=1}} |\rho(a)| < \infty.$$

It is a Banach space in the norm $\|\ \|$.

To prove iii), assume $\mathrm{sp}(a) = \emptyset$. Then, for any $\rho \in A^*$,

$$\lambda \mapsto \rho(R_a(\lambda)), \tag{$\star$}$$

defines a complex function from $\mathbb{C}$ to $\mathbb{C}$. Equation (1) shows that it has local power series expansions, hence must be holomorphic. By the preceding calculation, this function is bounded, and therefore, by Liouville's theorem, it must be constant. To evaluate this constant, we consider the values at $\lambda = 0$ and 1: $\rho(a^{-1}) = \rho((a-1)^{-1})$. By the Hahn–Banach theorem, A^* separates points in A, so $a^{-1} = (a-1)^{-1}$ which implies $0 = 1$. Hence we have a contradiction and iii) is proved. □

COROLLARY 1.12 (Gelfand–Mazur). *Let A be a unital C^*-algebra in which every nonzero element is invertible. Then $A \cong \mathbb{C}$.*

PROOF. Let $a \in A$. By the previous theorem, the spectrum of a is nonempty, i.e., there exists a $\lambda \in \mathbb{C}$ such that $a - \lambda$ is not invertible. By the assumption, we must have $a - \lambda = 0$. The map $a \mapsto \lambda$ now defines the desired isomorphism. □

Next, introduce the *spectral radius* $r(a)$ of $a \in A$ by

$$r(a) := \sup_{\lambda \in \mathrm{sp}(a)} |\lambda|.$$

By the Theorem above, we have $r(a) \leq \|a\|$, but we can do better:

THEOREM 1.13 (Beurling's formula).

$$r(a) = \lim_{n \to \infty} \sqrt[n]{\|a^n\|}.$$

PROOF. We have already seen that $r(a) \leq \|a\|$. Furthermore, if $\lambda \in \mathrm{sp}(a)$, we have that $\lambda^n \in \mathrm{sp}(a^n)$ by the holomorphic functional calculus, see below. It follows from this that $\lambda^n \leq \|a^n\|$ and therefore

$$r(a) \leq \inf_n \sqrt[n]{\|a^n\|}.$$

We now fix $\rho \in A^*$ and consider the function $f(\lambda)$ defined in $(\star)$ in the proof of Theorem 1.9. This is a holomorphic function for $|\lambda| > r(a)$, and by Lemma 1.10 it has a power series expansion

$$f(\lambda) = \sum_{n=0}^{\infty} \lambda^{-n-1} \rho(a^n).$$

We now write $\lambda = re^{i\theta}$ and we integrate $\lambda^{n+1} f(\lambda)$ over the circle with radius r:

$$\begin{aligned}\int_0^{2\pi} r^{n+1} e^{i(n+1)\theta} f(re^{i\theta}) d\theta &= \sum_{k=0}^{\infty} \int_0^{2\pi} r^{n-k} e^{i(n-k)\theta} \rho(a^k) d\theta \\ &= 2\pi \rho(a^n).\end{aligned}$$

Next, observe that $|f(\lambda)| = |\rho(R_a(\lambda))| \leq \|\rho\| \, \|R_a(\lambda)\|$. Therefore, if we set $M(r) := \sup_\theta \|R_a(\lambda)\|$, we get the estimate

$$|\rho(a^n)| \leq r^{n+1} M(r) \, \|\rho\| \, .$$

It follows from the Hahn–Banach theorem that there always exists a $\rho \in A^*$ with $\|\rho\| = 1$ and $|\rho(a^n)| = \|a^n\|$, so we see that $\|a^n\| \leq r^{n+1} M(r)$. From this inequality we now get

$$\lim_{n \to \infty} \sqrt[n]{\|a^n\|} \leq \inf_r = r(a),$$

and with the previous inequality for $r(a)$ this proves the theorem. □

DEFINITION 1.14. An element $a \in A$ is said to be selfadjoint if $a = a^*$.

COROLLARY 1.15. *For $a \in A$ selfadjoint. $\|a\| = r(a)$.*

PROOF. By the C^*-identity: $\|a^2\| = \|a^*a\| = \|a\|^2$. □

REMARK 1.16. For any $a \in A$, we now have by the C^*-identity that $\|a\| = \sqrt{r(a^*a)}$, so the norm of an element a in a unital C^*-algebra is determined by the algebraic structure (since $r(a)$ only depends on this), and is therefore unique. Another consequence is the following:

COROLLARY 1.17. *A morphism $\varphi : A \to B$ of C^*-algebra is automatically continuous and satisfies $\|\varphi\| \leq 1$.*

PROOF. Let $a \in A$. Clearly $\mathrm{sp}(\varphi(a)) \subset \mathrm{sp}(a)$. Now we have

$$\|\varphi(a)\|^2 = \|\varphi(a^*a)\| = r(\varphi(a^*a)) \leq r(a^*a) = \|a\|^2 .$$

Therefore, φ is bounded with norm $\|\varphi\| \leq 1$, hence continuous. □

1.3. The holomorphic functional calculus in Banach algebras.

THEOREM 1.18. *Let A be a unital Banach algebra. For $a \in A$ and a holomorphic function $f(z)$ in a region in $\mathbb{C}$ that contains $\mathrm{sp}(a)$, we can define $f(a) \in A$ and this association satisfies:*

i) $\mathrm{sp}(f(a)) = f(\mathrm{sp}(a))$,

ii) $(fg)(a) = f(a)g(a)$, *for two holomorphic functions f, g in region containing* $\mathrm{sp}(a)$.

PROOF. Let us first assume that f is defined on a ball $B(0, R)$ around the origin of radius R large enough to contain $\mathrm{sp}(a)$. Then f has a Taylor expansion at 0: $f(z) = \sum_{n=0}^{\infty} \alpha_n z^n$, and the series converges absolutely inside $B(0, R)$. Therefore, since $\mathrm{sp}(a)$ is contained in a disk of radius $\|a\|$, the series $\sum_n \alpha_n a^n$ converges in norm to an element $f(a) \in A$. Next, we see that with $\lambda \in \mathrm{sp}(a)$,

$$\begin{aligned} f(\lambda)1 - f(a) &= \sum_{n=1}^{\infty} \alpha_n(\lambda^n 1 - a^n) \\ &= (\lambda 1 - a) \sum_{n=1}^{\infty} \alpha_n P_n(\lambda, a), \end{aligned}$$

where $P_n(\lambda, a) = \sum_{k=0}^{n-1} \lambda^k a^{n-k-1}$. Since $\|P_n(\lambda, a)\| \leq nR^{n-1}$, the series $\sum_{n=1}^{\infty} \alpha_n P_n(\lambda, a)$ converges to an element $b \in A$, and this element commutes with a. Therefore, if $f(\lambda)1 - f(a)$ is invertible with inverse c, then bc is an inverse for $(\lambda 1 - a)$ contradicting the fact that $\lambda \in \mathrm{sp}(a)$: we now see that $f(\lambda) \in \mathrm{sp}(f(a))$.

We will not spel out the details of the general case, but here is a sketch: for f a holomorphic function on an open neighborhood of $\mathrm{sp}(a)$, the idea is to make sense of the formula

$$f(a) := \frac{1}{2\pi i} \oint_{\gamma} f(\lambda)(\lambda 1 - a)^{-1} d\lambda, \tag{2}$$

where γ is a closed contour going once around $\mathrm{sp}(a)$. For this we must treat the theory of holomorphic functions from $\mathbb{C}$ to A together with their path integrals. This can be done using the dual A^*, for this it is important that A^* separates points in A. For example: $g : \mathbb{C} \to A$ is holomorphic if $\rho \circ g$ is holomorphic for all $\rho \in A^*$. What emerges with this definition, is a beautiful theory in which almost all properties for holomorphic functions on $\mathbb{C}$ hold true for those with values in A. The fact that definition (2) coincides with our previous definition in terms of the power series expansion of f then follows by applying Cauchy's theorem. □

LEMMA 1.19. *Let A be a unital C^*-algebra.*

i) for $u \in A$ is unitary*: $u^* = u^{-1}$, we have* $\mathrm{sp}(u) \subset \mathbb{T}$*,*

ii) for $a \in A$ selfadjoint*, we have* $\mathrm{sp}(a) \subset \mathbb{R}$*.*

PROOF. One easily shows that for any invertible element $a \in A$, and $\lambda \in \mathrm{sp}(a)$,

$$\lambda^{-1} \in \mathrm{sp}(a^{-1}), \quad \bar{\lambda} \in \mathrm{sp}(a^*).$$

For a unitary $u \in A$, we have $\|u\| = 1$, so that we have $|\lambda| \leq 1$ and $\bar{\lambda}^{-1} \leq 1$ for $\lambda \in \mathrm{sp}(u)$ from the above. This proves the first claim.

For the second: for a selfadjoint $a = a^* \in A$, we apply the holomorphic functional calculus to form the element $\exp(ia) \in A$. It follows from Theorem 1.18 that $\exp(ia)^* = \exp(-ia) = \exp(ia)^{-1}$, so $\exp(ia)$ is unitary. Therefore the second claim follows from Theorem 1.18 *i*). □

2. The Gelfand–Naimark theorem

We now turn to *commutative* C^*-algebras. The fundamental theorem of Gelfand–Naimark, proved in 1943, shows that a commutative C^*-algebra is always of the form $C_0(X)$, where X is a locally compact Hausdorff space.

2.1. The Gelfand spectrum of a commutative C^*-algebra. Let A be a unital commutative C^*-algebra. We define the *Gelfand spectrum* of A as follows:

$$\mathrm{Spec}(A) := \{\mu : A \to \mathbb{C} \text{ homomorphism}\}.$$

LEMMA 2.1. *For all $\mu \in \mathrm{Spec}(A)$;*

i) $\mu(1) = 1$,

ii) $\|\mu\| = 1$.

PROOF. i) For all $a \in A$, we have the following equality:

$$\mu(a) = \mu(1 \cdot a) = \mu(1)\mu(a) \Longrightarrow \mu(1) = 1.$$

ii) We already know that for $|z| > \|a\|$, the element $a - z \in A$ is invertible. Therefore $\mu(a - z) = \mu(a) - z \neq 0$, and we see $\mu(a) \neq |z|$ for $|z| > \|a\|$. It follows that $|\mu(a)| \leq \|a\|$, and equality is attained e.g. by the unit element $1 \in A$ by i). □

Of course we have $\mathrm{Spec}(A) \subset A^*$, and we can equip $\mathrm{Spec}(A)$ with the restriction of the weak $*$-topology on A^* in which $\rho_n \to \rho$ if $\rho_n(a) \to \rho(a)$ for all $a \in A$.

PROPOSITION 2.2. Spec(A) *is compact in the weak* $*$*-topology.*

PROOF. Suppose that $\mu_n \to \mu$ with $\mu_n \in \mathrm{Spec}(A)$. Let $a, b \in A$. Then we have

$$|\mu(ab) - \mu(a)\mu(b)| \leq |\mu(ab) - \mu_n(ab)| + |\mu_n(a)\mu_n(b) - \mu(a)\mu(b)|$$

For the second term, we have

$$\begin{aligned} |\mu_n(a)\mu_n(b) - \mu(a)\mu(b)| &= |(\mu_n(a) - \mu(a))\mu_n(b) + \mu(a)(\mu_n(b) - \mu(b))| \\ &\leq \|b\| \, |\mu_n(a) - \mu(a)| + \|a\| \, |\mu_n(b) - \mu(b)| \\ &\longrightarrow 0, \; n \to \infty. \end{aligned}$$

Therefore, $\mu(ab) = \mu(a)\mu(b)$ for all $a, b \in A$ and $\mu \in \mathrm{Spec}(A)$, so $\mathrm{Spec}(A)$ is a closed subset of the unit ball in A^*. By the Banach–Alaoglu Theorem, the unit ball in A^* is compact and the statement follows. □

DEFINITION 2.3. The *Gelfand transform* of a commutative unital C^*-algebra A is the map

$$\Gamma : A \to C(\mathrm{Spec}(A)), \quad a \mapsto \{\mu \mapsto \mu(a)\}.$$

(The weak $*$-topology is the weakest topology on A^* that makes these functions continuous.)

2.2. Maximal ideals. The Gelfand spectrum of A can be identified with the set of maximal ideals in A. This identification plays a role in the proof of the Gelfand–Naimark theorem, and it also brings the spectrum of a commutative C^*-algebra closer to the notion of spectrum as used in algebraic geometry.

DEFINITION 2.4. An ideal in a C^*-algebra is a closed linear subspace $I \subset A$ with the property

$$a \in I \Longrightarrow ab, \; ba \in A, \quad \text{for all } b \in A.$$

Remark that, besides the usual algebraic condition for an ideal, there is a topological condition in the axiom that I needs to be closed. As usual, we define an ideal to be maximal if the only ideals containing it are either I itself or A.

THEOREM 2.5. *Let A be a commutative unital C^*-algebra. There is a bijective correspondence*

$$\mathrm{Spec}(A) \leftrightarrow \{I \subset A, \text{ maximal ideal}\}.$$

PROOF. For $\mu \in \mathrm{Spec}(A)$ we define $I_\mu = \ker \mu$. Since $\dim_{\mathbb{C}}(A/I_\mu) = 1$, this ideal is clearly maximal.

Conversely, Let $I \subset A$ be a maximal ideal. Let $b \in A$ with $b \notin I$. Consider now $J := \{ba + i, \; a \in A, \; i \in I\}$. Since A is commutative, J is an ideal that strictly contains I, and therefore $J = A$. Therefore $1 \in J$ and there exists an $a \in A$ such that $1 = ba + I$. In other words: the elements $\bar{b} \in A/I$ is invertible. By Corollary 1.12, we conclude that

$A/I \cong \mathbb{C}$. Define $\mu : A \to \mathbb{C}$ to be the composition of the projection to A/I together with this isomorphism to $\mathbb{C}$. Then μ clearly is a character and $I = \ker \mu$. □

COROLLARY 2.6. *Let A be a commutative unital C^*-algebra. Then for any $a \in A$ we have*

$$\mathrm{sp}(a) = \{\mu(a),\ \mu \in \mathrm{Spec}(A)\}.$$

PROOF. Let $\lambda \in \mathrm{sp}(a)$. Examining the proof of Theorem 2.5, we conclude that there exists a $\mu \in \mathrm{Spec}(A)$ with $\mu(\lambda 1 - a) = 0$, in other words $\mu(a) = \lambda$. Conversely, if $\mu(a) = \lambda \in \mathbb{C}$, we have $\lambda 1 - a \in \ker \mu$, and we conclude that $\lambda 1 - a$ is not invertible. Hence $\lambda \in \mathrm{sp}(a)$. □

As known from elementary algebra, for an ideal $I \subset A$, the quotient A/I has a canonical algebra structure. For C^*-algebras we can even equip this quotient with a C^*-algebra structure with norm given by

$$\|a + I\| := \inf_{i \in I} \|a + i\|.$$

2.3. The Gelfand–Naimark theorem. We now come to the actual proof of the Gelfand–Naimark theorem:

THEOREM 2.7. *Let A be a commutative, unital C^*-algebra. Then the Gelfand transform in def. 2.3 is an isomorphism of C^*-algebras*

$$A \cong C(\mathrm{Spec}(A)).$$

PROOF. We start with the following basic properties of the Gelfand transform:

$$\Gamma(ab) = \Gamma(a)\Gamma(b), \quad \Gamma(a^*) = \Gamma(a)^*, \quad \text{for all } a, b \in A.$$

The first property follows easily from the fact that $\mu \in \mathrm{Spec}(A)$ is a homomorphism, the second follows from the property $\mu(a^*) = \overline{\mu(a)}$, which is more difficult to prove. We omit this argument; we may as well add this property to the definition of $\mathrm{Spec}(A)$.

Next we claim that $\mathrm{sp}(a) = \mathrm{sp}(\Gamma(a)$: Indeed we have

$$a - \lambda I \text{ not invertible} \Longrightarrow J := \{(a - \lambda I)b,\ b \in A\} \subset A \text{ ideal not containing } 1.$$

By Zorn's Lemma, J is contained in a maximal ideal, so by Theorem 2.5, there exists a $\mu \in \mathrm{Spec}(A)$ such that $\mu(a) = \lambda$. Conversely,

$$(a - \lambda I) \text{ invertible} \Longrightarrow \mu(a - \lambda I)\mu((a - \lambda I)^{-1}) = 1,\ \forall \mu \in \mathrm{Spec}(A),$$

and therefore $\mu(a - \lambda I) \neq 0$ for all $\mu \in \mathrm{Spec}(A)$.

It follows therefore that $r(a) = \|\Gamma(a)\|_\infty = \|a\|$ when a is selfadjoint. For general $a \in A$ we have

$$\|a\|^2 = \|a^* a\| = \|\Gamma(a^* a)\|_\infty = \|\Gamma(a)^* \Gamma(a)\|_\infty = \|\Gamma(a)\|_\infty^2,$$

so Γ is an isometry, and hence injective. We conclude by showing that it is surjective: $\Gamma(A) \subset C(\mathrm{Spec}(A))$ is a closed subalgebra that separates points of $\mathrm{Spec}(A)$. The Stone–Weierstrass theorem now implies that $\Gamma(A) = C(\mathrm{Spec}(A))$. ☐

2.4. Noncompact spaces. When X is locally compact, but not compact, we can define a C^*-algebra as follows:

$$C_0(X) := \{f : X \to \mathbb{C}\text{continuous},\ f \text{ vanishes at } \infty\}.$$

Here, "vanishes at ∞" means that

$$\forall \epsilon > 0,\ \exists K \subset X \text{ compact, such that } |f(x)| < \epsilon,\ \forall x \in K.$$

With the sup-norm, we again get a commutative C^*-algebra, but without unit. For general nonunital C^*-algebras A, there is a way to get a unital C^*-algebra, called it *unitization* $\tilde{A}$: As a vector space we define $\tilde{A} := A \oplus \mathbb{C}$, and define the multiplication by

$$(a, z) \cdot (b, w) = (ab + wa + zb, zw), \quad a, b \in A,\ z, w \in \mathbb{C}.$$

So far, so good. What is more difficult, is to find a good norm so that $\tilde{A}$ is a C^*-algebra. There are two ways of doing this: first, one can use the GNS constructions (see exercises) to realize A as a norm closed subalgebra of the algebra of bounded operators on a Hilbert space. Then $\tilde{A}$ is the smallest C^*-subalgebra (see §3) generated by A and 1. Another, more direct way is to define

$$\|(a, z)\| := \sup_{b \in A,\ \|b\| \leq 1} \|ab + zb\|.$$

EXERCISE 2.8. Check that this norm satisfies the C^*-identity.

In any case: whatever the construction, the unitization $\tilde{A}$ is unique up to isomorphism. (Recall that the norm in a C^*-algebra is unique.) By construction, $\tilde{A}$ fits into a short exact sequence of C^*-algebras

$$0 \to A \to \tilde{A} \to \mathbb{C} \to 0,$$

which is canonically split by the canonical map $\mathbb{C} \to \tilde{A}$.

2.5. Interlude: some category theory. The Gelfand–Naimark theorem is best expressed in the language of category theory. Let us therefore briefly introduce a few basic concepts.

DEFINITION 2.9. A category $\mathscr{C}$ consists of:

i) a class of objects,
ii) a set of morphisms $\mathrm{Hom}_{\mathscr{C}}(A, B)$ for any pair of objects A, B.

These data should satisfy:

a) (Composition) There is a composition of morphisms

$$\circ : \mathrm{Hom}_{\mathscr{C}}(A,B) \times \mathrm{Hom}_{\mathscr{C}}(B,C) \to \mathrm{Hom}_{\mathscr{C}}(A,C),$$

which is associative in the obvious way,

b) (Existence of units) For each object A there exists a unit morphisms $1_A \in \mathrm{Hom}_{\mathscr{C}}(A,A)$.

EXAMPLE 2.10. Here are some examples relevant for us:

i) The category $\underline{Top}$ consists of compact topologycal Hausdorff spaces with morphisms given by continuous maps.

ii) The category $\underline{AbC^*}$ consists of commutative C^*-algebras with unit together with morphisms of C^*-algebras.

DEFINITION 2.11. A *functor* $F : \mathscr{C} \to \mathscr{D}$ between two categories $\mathscr{C}$ and $\mathscr{D}$ maps objects to objects and morphisms to morphisms such that

- $F(1_A) = 1_{F(A)}$, for all objects A in $\mathscr{C}$,
- $F(f \circ g) = F(f) \circ F(g)$ for any pair of composable morphisms f and g in $\mathscr{C}$

What is more important than the definition of a functor is that of a natural transformation:

DEFINITION 2.12. Let $F, G : \mathscr{C} \to \mathscr{D}$ be two functors between the same categories. A *natural transformation* $\tau : F \to G$ is an assignment to each object A of $\mathscr{C}$ of an arrow $\tau(A) \in \mathrm{Hom}_{\mathscr{D}}(F(A), G(A))$ such that for any arrow $f : A \to B$ in $\mathscr{C}$ the following diagram commutes:

$$\begin{array}{ccc} F(A) & \xrightarrow{\tau(A)} & G(A) \\ {\scriptstyle F(f)}\downarrow & & \downarrow{\scriptstyle G(f)} \\ F(B) & \xrightarrow{\tau(B)} & G(B) \end{array}$$

When there exists a natural transformation between F and D, we write $F \simeq G$.

With this notion we can formulate when two categories are "the same":

DEFINITION 2.13. Two categories $\mathscr{C}$ and $\mathscr{D}$ are *equivalent* if there exist functors

$$F : \mathscr{C} \to \mathscr{D}, \quad G : \mathscr{D} \to \mathscr{C},$$

such that

$$F \circ G \simeq \mathrm{id}_{\mathscr{D}}, \quad G \circ F \simeq \mathrm{id}_{\mathscr{C}}.$$

Notice that this is weaker than the notion of a *strict isomorphism* between $\mathscr{C}$ and $\mathscr{D}$: this is a functor $F : \mathscr{C} \to \mathscr{D}$ which is a bijection on morphisms and objects.

2.6. Gelfand–Naimark: the categorical version. Now that we have some categorical background, we can state the most precise version of the Gelfand–Naimark theorem. We consider the two categories described in Example 2.10. There are obvious functors

$$\underline{Top} \xrightarrow{F} \underline{AbC^*}, \quad X \mapsto C_0(X),$$

as well as

$$\underline{AbC^*} \xrightarrow{G} \underline{Top}, \quad A \mapsto \mathrm{Spec}(A).$$

Clearly, $G \circ F \simeq \mathrm{id}_{\underline{Top}}$ because every $x \in X$ gives rise to a evaluation homomorphism on $C_0(X)$ by $ev_x(f) = f(x)$.

THEOREM 2.14 (Gelfand–Naimark, categorical version). *The Gelfand transform defines a natural isomorphism*

$$F \circ G \xrightarrow{\Gamma} \mathrm{id}_{\underline{AbC^*}}.$$

Therefore the categories $\underline{Top}$ and $\underline{AbC^*}$ are weakly equivalent.

3. The continuous functional calculus

Let A be a C^*-algebra. For any subset $F \subset A$, the *subalgebra generated by F*, written $C^*(F)$, is the smallest C^*-subalgebra containing F. It can be constructed concretely as follows: let

$$W_n := \{a_1 \cdots a_n,\ a_i \in F \cup F^*,\ i = 1, \ldots n\}.$$

Then we define $W := \bigcup_n W_n$ and finally $C^*(F)$ is the norm closure of W: the algebra operations on W extend to $C^*(F)$ because they are continuous in norm, cf. Remark 1.3.

DEFINITION 3.1. An element $a \in A$ is called *normal* if $aa^* = a^*a$, i.e., a commutes with its adjoint.

We now consider the subset $F := \{a\}$ of A. By the construction above, this generates a C^*-algebra $C^*(a)$. When a is normal, this algebra is in fact commutative. We now assume this is the case.

PROPOSITION 3.2. *There is a canonical homeomorphism* $\mathrm{Spec}(C^*(a)) \cong \mathrm{sp}(a)$.

PROOF. For $\mu \in \mathrm{Spec}(C^*(a))$, we have that $\mu(a) \in \mathrm{sp}(a)$ since $a - \mu(a)1 \in \ker \mu$. This defines the desired homeomorphism. □

By the Gelfand–Naimark theorem, it follows that $C^*(a) \cong C(\mathrm{sp}(a))$. Therefore, we conclude that for each continuous function $f \in C(\mathrm{sp}(a))$, there exists a unique $f(a) \in A$. This construction is called the *continuous functional calculus* for the C^*-algebra A. Applied to $A = B(\mathcal{H})$, this reduces to the usual continuous functional calculus for bounded operators on a Hilbert space.

THEOREM 3.3 (The continuous functional calculus). *Let $a \in A$ be a normal element of a C^*-algebra with unit. For each $f \in C(\mathrm{sp}(a))$, there exists a unique element $f(a) \in A$. This assignment $f \mapsto f(a)$ satisfies the following properties:*

i) $(fg)(a) = f(a)g(a)$ *for all* $f, g \in C(\mathrm{sp}(a))$,
ii) $f(a)^* = \bar{f}(a)$.
iii) $\|f(a)\| = \|f\|_\infty$
iv) $\mathrm{sp}(f(a)) = f(\mathrm{sp}(a))$,
v) $(h \circ f)(a) = h(f(a))$ *for all* $f \in C(\mathrm{sp}(a))$, *and* $h \in C(\mathrm{sp}(f(a)))$.

The first three properties are just a reformulation of the statement that the (inverse) Gelfand transform $f \mapsto f(a)$ defines an isometric $*$-isomorfism $C(\mathrm{sp}(a)) \cong C^*(a)$. The last two properties are not difficult to prove.

CHAPTER 2

K-theory

1. Topological K-theory: an overview

Here we give a brief review of K-theory for topological spaces. Classic references for this are [**At, K**].

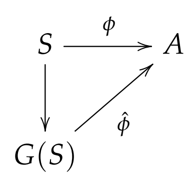
1.1. Generalized cohomology theories. K-theory is an example of a so-called *generalised cohomology theory*. To motivate its definiton, consider the ordinary (singular) cohomology of a topological space.

1.2. The Grothendieck group of an abelian semigroup. Recall that an abelian semigroup is a set S equipped with an operation

$$+ : S \times S \to S,$$

which is associative and satisfies $a + b = b + a$ for all $a, b \in S$. The difference with an abelian group is that there need not be a unit and inverses are not required to exist. When it exists, we speak of a *semigroup with unit*. A morphism of semigroups is defined in the obvious way.

DEFINITION 1.1. Let S be an abelian semigroup with unit. The *Grothendieck group* $G(S)$ of S is the abelian group satisfying the following universal property: There exists a morphism $S \to G(S)$ of semigroups and for all other morphisms $\phi : S \to A$ to abelian groups, there exists a unique homomorphism $\hat{\phi} : G(S) \to A$ of groups making the following diagram commute:



It is easy to prove that the Grothendieck group is unique up to isomorphism if it exists. Let us therefore provide an explicit construction: on $S \times S$ introduce the equivalence relation

$$(x_1, y_1) \sim (x_2, y_2) \iff \exists z \in S \text{ such that } x_1 + y_2 + z = x_2 + y_1 + z.$$

You should check that this indeed is an equivalence relation. Remark that because of the lack of inverses, we may not have the cancellation property that $x + z = y + z$ for some $z \in S$ implies that $x = y$. With the equivalence relation we can define

$$G(S) := (S \times S)/\sim .$$

On this set, we define the addition component wise:

$$[x_1, y_1] + [x_2, y_2] = [x_1 + x_2, y_1 + y_2].$$

EXERCISE 1.2. Check that is indeed well defined, and that $G(S)$ is an abelian group with this operation: what is the unit, inverse? Construct the morphism $S \to G(S)$. Show that each element in $G(S)$ can be written as

$$[x] - [y] \in G(S), \quad , x, y \in S.$$

EXAMPLE 1.3. It is easy to check that $G(\mathbb{N}) = \mathbb{Z}$.

1.3. Vector bundles. Let X be a paracompact Hausdorff space.

DEFINITION 1.4. A vector bundle is a surjective continuous map $p : E \to X$ such that $p^{-1}(x)$ carries the structure of a vector space. Furthermore, it is required that E is locally trivial: each $x \in X$ has a open neighborhood $U \subset X$ such that there exists a homeomorphism $\varphi : p^{-1}(U) \to U \times \mathbb{C}^r$, linear on each fiber, which is compatible with the projection p in the following way: the diagram

$$\begin{array}{ccc} \pi^{-1}(U) & \xrightarrow{\varphi} & U \times \mathbb{C}^n \\ {\scriptstyle p}\downarrow & \swarrow {\scriptstyle pr_1} & \\ U & & \end{array}$$

where pr_1 is the projection onto the first component.

We often simply write E for a vector bundle, taking the projection for granted. The map φ as above is called a *local trivialization* of E. It is only required to exist locally around each point, there may not exist a *global trivialization*. Finally, a *morphism of vector bundles* $\phi : E \to F$ is a continuous map, compatible with the projections, which is linear on each fiber.

There is a very convenient local description of vector bundles as follows: fix an open cover $\mathcal{U} = \{U_i\}_{i \in I}$ such that over each U_i there exists a local trivialization $\varphi_i : E|_{U_i} \to U_i \times \mathbb{C}^n$. On the overlap $U_i \cap U_j \neq \emptyset$, $i, j \in I$ the two local trivializations φ_i, φ_j are related by a unique continuous map $\psi_{ij} : U_{ij} \to GL(n, \mathbb{C})$ as follows:

$$\varphi_i \circ \varphi_j^{-1} = (id_{U_{ij}}, \psi_{ij}).$$

On triple overlaps $U_i \cap U_j \cap U_k \neq \emptyset$, the ψ's satsify the *cocycle* relation

$$\psi_{ij} \cdot \psi_{jk} = \psi_{ik}. \tag{3}$$

Conversely, given a covering $\mathcal{U} = \{U_i\}_{i\in I}$ of X together with functions $\psi_{ij} : U_{ij} \to GL(n,\mathbb{C})$ satisfying the cocycle identity (3), we can construct a vector bundle

$$E := \left(\coprod_{i\in I} U_i \times \mathbb{C}^n \right) \Big/ \sim,$$

where

$$(x, v_i) \sim (y, v_j) \Longleftrightarrow x = y \in U_{ij} \text{ and } \psi_{ij}(v_j) = v_i.$$

These two constructions are inverses of each other (up to isomorphism).

There are several constructions with vector bundles:

- For any continuous map $f : X \to Y$ and vector bundle $E \to Y$, we can construct the *pull-back bundle* f^*E defined fiber wise as
$$(f^*E)_x = E_{f(x)}, \quad \text{for all } x \in X.$$
- Given two vector bundles E and F over X, we can form their direct sum $E \oplus F$ by taking locally the direct sum of the vector spaces over the fibers.
- Likewise, we can define the tensor product $E \otimes F$.

1.4. $K^0(X)$**.** Assume now that X is compact. Denote by $\underline{\mathscr{V}ect}(X)$ the category of vector bundles over X: its objects are vector bundles, and morphisms are given by morphisms of vector bundles as defined above. Out of this we can construct the set of isomorphism classes of objects:

$$Vect(X) := Ob(\underline{\mathscr{V}ect}(X))/isomorphism$$

We write $[E] \in Vect(X)$ for the isomorphism class of a vector bundle E. It is easy to check that the direct sum of vector bundles induces on $Vect(X)$ the structure of an abelian semigroup. It even has a unit given by the zero vector bundle. This allows us to define:

DEFINITION 1.5. For X compact, $K^0(X)$ is defined as the Grothendieck group of the semigroup $Vect(X)$.

2. The Serre–Swan theorem

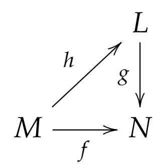
We have already seen that the category of locally compact Hausdorff spaces is equivalent to the category of commutative C^*-algebras by the Gelfand–Naimark theorem. To generalize K-theory to the category of general (possibly noncommutative) C^*-algebras, we have to first translate its construction in algebraic terms. Key in this is the so-called Serre–Swan theorem. This theorem gives a correspondence between vector bundles over locally compact spaces and certain modules over $C_0(X)$. Let us first recall some algebraic definitions:

DEFINITION 2.1. Let A be an algebra, and M a (left) module over A.

i) M is *finitely generated* if there are elements $m_1, \ldots, m_k$ such that

$$M = \{a_1 m_1 + \ldots + a_k m_k,\ a_1, \ldots, a_k \in A\}.$$

ii) M is *projective* if it has the following universal lifting property: for every A-modules N and L, and module maps $f : M \to N$ and $g : L \to N$ with g surjective, there exists a $h : M \to L$ making the following diagram commutative:



EXERCISE 2.2. Proof that M is projective if and only if it is a direct summand of a free module. With this, show that a finitely generated projective module is the same as an idempotent $e \in M_n(A)$, for some n. (Idempotent means that $e^2 = e$.)

Let X now be a compact topological Hausdorff space, and consider a vector bundle E over X. The global sections of E are defined by:

$$\Gamma(E) := \{s : X \to E \text{ continuous}, p \circ s = id_X\}.$$

Clearly, $\Gamma(E)$ is a vector space over $\mathbb{C}$, because of the fiber wise vector space structure on E. It is even a module over $C(X)$, by the formula

$$(f \cdot s)(x) := f(x)s(x), \quad x \in X.$$

The key idea to the noncommutative generalization of K-theory to general C^*-algebras is the following classical theorem:

THEOREM 2.3 (Serre–Swan). *Let X be a compact space. Then the global sections functor defines an equivalence of categories*

$$\Gamma : \underline{Vect}(X) \to \mathscr{M}od_{fgp}(C(X))$$

PROOF. Let us first verify that the modules obtained as the image of the functor Γ are indeed finitely generated and projective. For this fact, we use the following:

THEOREM 2.4. *For any vector bundle E, there exists a vector bundle F such that $E \oplus F$ is trivial.*

PROOF OF THEOREM 2.4. Recall the following

LEMMA 2.5. *Over a paracompact space X, any short exact sequence of vector bundles*

$$0 \longrightarrow E \longrightarrow E' \xrightarrow{\pi} E'' \longrightarrow 0,$$

splits, that is: there exists a morphism of vector bundles $\sigma : E'' \to E'$ satisfying $p \circ \sigma = id_{E''}$.

Such a σ as above is called a *splitting* of the sequence. With this, one easily constructs an isomorphism $E' \cong E \oplus E''$. Remark that in general the kernel of a morphism $F : E' \to E''$ in general need not be a vector bundle: a very easy example is the map $F : [0,1] \times \mathbb{C} \to [0,1] \times \mathbb{C}$ defined by $F(t,z) = tz$. Clearly, this defines a morphism of the trivial line bundle over $[0,1]$ to itself. Its kernel is $\{0\}$ over $(0,1]$ but $\ker F = \mathbb{C}$ at 0. We see in this way that the function $x \mapsto \operatorname{rank} F_x$ is not locally constant in general. If it is, $\ker F$ and $\operatorname{coker} F$ do form vector bundles over X.

PROOF OF LEMMA 2.5. □

Let us proceed to the proof of Theorem 2.4.

Let us fix a finite cover $\mathcal{U} = \{U_i\}$ $i = 1, \ldots, m$ with the property that over each U_i there exists a trivialization $E \cong U_i \times \mathbb{C}^r$. In other words: over each U_i we can find sections $s_{i1}, \ldots, s_{ir}$ with the property that they are linear independent in each fiber (i.e., form a basis). We also fix a partitition of unity $\{\psi_i\}$ subordinate to $\mathcal{U}$. With this we define $\sigma_{ik} : X \to E$ by $\sigma_{ik} := \psi_i s_{ik}$. We put $n = rm$, and construct $\beta : X \times \mathbb{C}^n \to E$ by

$$\beta(x,t) := \sum_{i,k} t_{ik}\sigma_{ik}(x).$$

By construction, the map β is a surjective morphism of vector bundles. Its rank is therefore constant, and $\ker \beta$ forms a vector bundle. The short exact sequence

$$0 \to \ker \beta \to X \times \mathbb{C}^n \to E \to 0,$$

splits by the previous Lemma, and we see that $F := \ker \beta$ has the desired properties. □

To continue the proof of Theorem 2.3, we now make some easy observations about the functor Γ: First of all, it maps a trivial vector bundle to a finitely generated *free* module over $C(X)$: Clearly, $\Gamma(X \times \mathbb{C}^n) = C(X, \mathbb{C}^n) = C(X)^{\oplus n}$. Conversely, any free, finitely generated module over $C(X)$ is isomorphic to a module of this kind: we see that Γ defines an equivalence of categories

$$\underline{\{\text{Trivial vector bundles over } X\}} \xrightarrow{\cong} \underline{\{\text{Free finitely generated modules over } C(X)\}}$$

DEFINITION 2.6. Let $\mathcal{C}$ be a category. Its *Karoubi envelope* $\tilde{\mathcal{C}}$ is given by the category whose objects are pairs (X, e) where $X \in \mathrm{Ob}(\mathcal{C})$ and $e \in \mathrm{Hom}_{\mathcal{C}}(X, X)$ is idempotent: $e^2 = e$. A morphism from (X, e) to (Y, e') in $\tilde{\mathcal{C}}$ is given by a morphism $F \in \mathrm{Hom}_{\mathcal{C}}(X, Y)$ with $e \circ F = F = F \circ e'$.

PROPOSITION 2.7. *The Karoubi envelope of the category of trivial vector bundles over X is equivalent to the category* $\underline{\mathscr{V}ect}(X)$*of vector bundles*

PROOF. A morphism of vector bundles from the trivial bundle $X \times \mathbb{C}^n$ to itself is determined by a matrix valued function $\varphi \in C(X, M_n(\mathbb{C}))$. Being idempotent in the category of vector bundles means that the matrix value $\varphi(x)$ is an idempotent in $M_n(\mathbb{C})$ for each $x \in X$.

CLAIM 2.8. *The kernel* $x \mapsto \ker \varphi(x)$ *forms a vector bundle over* X.

PROOF. We have to show local triviality of the vibration $\ker\varphi \to X$. For this we fix $x_0 \in X$ and define $F \in C(X, M_n(\mathbb{C}))$ by $F(x) := 1 - \varphi(x) - \varphi(x_0) + 2\varphi(x_0)\varphi(x)$. By construction, we have that $\varphi(x_0)F(x) = F(x)\varphi(x)$, so that F fits into a commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & \ker\varphi & \longrightarrow & X\times\mathbb{C}^n & \xrightarrow{\varphi} & X\times\mathbb{C}^n \\ & & \downarrow & & \downarrow{\scriptstyle F} & & \downarrow{\scriptstyle F} \\ 0 & \longrightarrow & X\times\ker\varphi(x_0) & \longrightarrow & X\times\mathbb{C}^n & \xrightarrow{\varphi(x_0)} & X\times\mathbb{C}^n \end{array}.$$

Since $F(x_0) = id_{\mathbb{C}^n}$, there is a neighborhood of U of x_0 over which F is invertible, since $GL(n,\mathbb{C})$ is open in $M_n(\mathbb{C})$. Therefore, over U, F induces a homeomorphism between $\ker\varphi|_U$ and $U \times \ker\varphi(x_0)$, proving the claim. □

This claim defines a functor from the Karoubi envelope of the category of trivial vector bundles to the category of vector bundles over X. To see that this defines an equivalence, we use Swan's theorem to embed a vector bundle E into a trivial bundle $X \times \mathbb{C}^n$. This gives us an idempotent $\varphi \in C(X, M_n(\mathbb{C}))$ by taking, for each $x \in X$ the projection onto $E_x \subset \mathbb{C}^n$. This proves the claim □

PROPOSITION 2.9. *The Karoubi envelope of the category of finitely generated free modules over an algebra* A *is equivalent to the category* $\mathscr{M}_{fgp}(A)$ *of finitely generated projecttive modules over* A.

□

3. K-theory for C^*-algebras

Let A be a unital C^*-algebra. Denote by $V(A)$ the set of isomorphism classes of finitely generated projective modules. This set has an abelian semigroup structure induced by taking the direct sum of modules, and with this we can define

$$K_0(A) := G(V(A)).$$

By the Serre–Swan theorem, this definition extends the definition of K-theory (in degree zero) to the category of C^*-algebras. Below we are going to given a more explicit description of this group.

3.1. Equivalence of projectors in C^*-algebras. Let A be a unital C^*-algebra. A *projector* in A is an element $p \in A$ satsifying $p^2 = p^* = p$.

EXAMPLE 3.1. The name comes from the example given by $A = B(\mathcal{H})$ for some Hilbert space $\mathcal{H}$. Given a closed subspace $K \subset \mathcal{H}$, define

$$p_K := \begin{cases} id_K & \text{on } K \\ 0 & \text{on } K^\perp. \end{cases}$$

EXERCISE 3.2. Show that $\mathrm{sp}(p) \subset \{0,1\}$ for a projector p in any C^*-algebra A.

We denote by $\mathrm{Proj}(A)$ the set of projectors in A. There are three different types of equivalence on projectors:

DEFINITION 3.3. Let $p, q \in A$ be projectors. Define the following equivalence relations:

(homotopy) $\quad p \sim_h q \Longleftrightarrow \exists e \in C^0([0,1], \mathrm{Proj}(A))$ with $e(0) = p, e(1) = q$,

(Unitary) $\quad p \sim_u q \Longleftrightarrow \exists u \in A$ unitary, $q = upu^*$,

(Murray–Von Neumann) $\quad p \sim q \Longleftrightarrow \exists v \in A$such that $v^*v = p,\ vv^* = q$.

EXERCISE 3.4. Prove that these are indeed equivalence relations.

PROPOSITION 3.5. *Let* $p, q \in \mathrm{Proj}(A)$. *Then*

$$p \sim_h q \Longrightarrow p \sim_u q \Longrightarrow p \sim q.$$

PROOF. Assume that $p \sim_h q$ and that $\|p - q\| < 1/2$. Define

$$z := pq + (1-p)(1-q).$$

One easily checks that $pz = pq = zq$. Furthermore,

$$\|(z-1)\| = \|p(1-q) + (1-p)((1-q) - (1-p))\| \leq 2\,\|(p-q)\|\,| < 1.$$

With this, the series $\sum_k (z-1)^k$ converges to z^{-1} and z is invertible. This shows that $q = z^{-1}pz$. To get unitary equivalence, we use the following version of the *polar decomposition*: Because z is invertible, we see that z^*z is a selfadjoint strictly positive element with $\mathrm{sp}(z^*z) \subset (0, \infty)$. Using the functional calculus of §3, we construct the following elements in A:

$$|z| := \sqrt{z^*z}, \quad u := \frac{z}{|z|}.$$

We see from the construction of §3 that the element $|z^*z|$ lies in the sub C^*-algebra $C^*(z^*z)$ generated by z^*z, and because $z^*zq = z^*pz = qz^*z$, it follows that $|z|$ commutes with q, and therefore $q = upu^*$.

This proves the first implication under the condition that $\|p - q\| < 1/2$. To prove it in general, we can (by compactness of $[0,1]$ and continuity) divide the unit interval into a finite number of pieces $[t_i - t_{i+1}]$ such that for the homotopy $e(t)$ between p and q, we have that $\|(e(t_i) - e(t_{i+1})\| < 1/2$. With the transitivity property of $\sim_u$, this proves the implication.

To prove the second implication, suppose that $p = uqu^*$ for some unitary $u \in A$. Define $v = uq \in A$. Then clearly $v^*v = q$ and $vv^* = p$, proving the second implication. □

There are examples that show that the converse of these implications are not true in general. To arrange this, we need to stabilize the algebra as follows.

DEFINITION 3.6. Let A be a C^*-algebra. The matrix algebra $M_n(A)$ consists of $n \times n$ matrices with entries in A, equipped with the obvious multiplication combining the multiplication in A with those of matrices. We equip this algebra with a $*$-operation by

$$\begin{pmatrix} a_{11} & \dots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \dots & a_{nn} \end{pmatrix}^* = \begin{pmatrix} a_{11}^* & \dots & a_{n1}^* \\ \vdots & & \vdots \\ a_{1n}^* & \dots & a_{nn}^* \end{pmatrix}.$$

Finally, to get a C^*-norm, fix an injective homomorphism $\rho : A \to B(\mathcal{H})$ for some Hilbert space $\mathcal{H}$. (This is possible by the GNS construction). This defines an obvious injection $\rho_n : M_n(A) \to B(\mathcal{H}^n)$, where $\mathcal{H}^n = \mathcal{H} \oplus \dots \mathcal{H}$ (n copies). The norm is then defined by the restriction of the norm on $B(\mathcal{H}^n)$.

PROPOSITION 3.7. *Let $p, q \in \mathrm{Proj}(A)$. We have:*

(i))
$$p \sim q \Longrightarrow \begin{pmatrix} p & 0 \\ 0 & 0 \end{pmatrix} \sim_u \begin{pmatrix} q & 0 \\ 0 & 0 \end{pmatrix}$$

(i))
$$p \sim_u q \Longrightarrow \begin{pmatrix} p & 0 \\ 0 & 0 \end{pmatrix} \sim_h \begin{pmatrix} q & 0 \\ 0 & 0 \end{pmatrix}$$

PROOF. Assume that $p = v^*v$ and $q = vv^*$. Then the unitary element

$$u = \begin{pmatrix} v & 1 - vv^* \\ v^*v - 1 & v^* \end{pmatrix}$$

gives the first implication. For the second we assume that $q = upu^*$ and construct the following one-parameter family of unitary elements in $M_2(A)$:

$$U(t) := \begin{pmatrix} u \cos \pi t/2 & \sin \pi t/2 \\ -\sin \pi t/2 & u^* \cos \pi t/2 \end{pmatrix} \tag{4}$$

Since $U(t)$ is unitary, the conjugation

$$e(t) = U(t) \begin{pmatrix} p & 0 \\ 0 & 0 \end{pmatrix} U(t)^*$$

defines a one-parameter family of projections in $M_2(A)$. This defines the homotopy proving the second implication. □

We now consider the inclusion $M_n(A) \hookrightarrow M_{n+1}(A)$ given by

$$A \mapsto \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix}.$$

Clearly, this map is an isometry, so we get a natural norm on the direct limit

$$M_\infty(A) := \varinjlim_n M_n(A) \tag{5}$$

of the algebra of matrices of arbitrary size. The algebra $M_\infty(A)$ is *not* complete in this norm, but this is not needed in the following. Combining the previous two propositions, we now have:

THEOREM 3.8. *The equivalence relations $\sim_h$, $\sim_u$ and $\sim$ coincide on* $\mathrm{Proj}(M_\infty(A))$.

3.2. K_0 of a C^*-algebra. Let A be a unital C^*-algebra. Given the fact that the three equivalence relations on $\mathrm{Proj}(M_\infty(A))$ coincide, we now define

$$\mathcal{P}(A) := \pi_0(\mathrm{Proj}(M_\infty(A)) \cong \mathrm{Proj}(M_\infty(A))/\sim,$$

where $\sim$ can be any of the equivalence relations. We can equip $\mathcal{P}(A)$ with a semigroup structure by means of

$$[p] + [q] = \left[\begin{pmatrix} p & 0 \\ 0 & q \end{pmatrix}\right]$$

DEFINITION 3.9. The K_0-group of a unital C^*-algebra A is defined as the Grothendieck group of the semigroup $\mathcal{P}(A)$; $K_0(A) := G(\mathcal{P}(A))$.

To connect to topological K-theory, we need one final result:

PROPOSITION 3.10. *The set of idempotents in a C^*-algebra is homotopy equivalent to the set of projections.*

PROOF. Let e be an idempotent in a C^*-algebra A, i.e., $e^2 = e$. Define

$$z := 1 + (e - e^*)(e^* - e).$$

This is an invertible element in A, which allows us to define $p := ee^*z^{-1}$. Since z^{-1} commutes with e and e^*, one checks that $p = p^*$ and $p^2 = p$, i.e., p is a projection. With this,

$$e_t := (1 - tp - 1e)e(1 - te - tp)$$

is a continuous family of idempotents (remark that $(1 - tp - 1e)$ is the inverse of $(1 - te - tp)$ since $ep = p$ and $pe = e$) with $e_0 = e$ and $e_1 = p$. This proves the proposition. ∎

EXAMPLE 3.11. Consider the case of the trivial C^*-algebra $\mathbb{C}$. Since this is the commutative C^*-algebra of continuous functions on a point, we find $K_0(\mathbb{C}) \cong \mathbb{Z}$. Slightly more difficult is the case of the noncommutative algebra $M_n(\mathbb{C})$. But since

$$M_\infty(M_N(\mathbb{C})) \cong M_\infty(\mathbb{C}),$$

we see that also $K_0(M_N(\mathbb{C})) \cong \mathbb{Z}$. (This is in fact a baby-example of Morita invariance.)

3.3. Nonunital algebras. When A is not unital, we proceed analogous to the topological K-theory of noncompact spaces: denote by $\tilde{A}$ the unitization (this is algebraic version of the one-point compactification of a locally compact topological space). The third map in the short exact sequence

$$0 \to A \to \tilde{A} \to \mathbb{C} \to 0, \tag{6}$$

induces a map on the level of K-theory

$$K_0(\tilde{A}) \to \mathbb{Z}.$$

DEFINITION 3.12. For A any C^*-algebra, we define

$$K_0(A) := \ker(K_0(\tilde{A}) \to \mathbb{Z}).$$

The reader should check that this definition agrees with the previous one if A happened to have a unit. The main reason for this definition in the nonunital case is the property of half-exactness (see below). For the moment, we record a first property of the K_0-group of C^*-algebras:

PROPOSITION 3.13. *The assignment $A \rightsquigarrow K_0(A)$ defines a covariant functor from the category of C^*-algebras (with homomorphisms of C^*-algebras as arrows) to the category of abelian groups.*

PROOF. The proof is left as an easy exercise □

We write $\varphi_* : K_0(A) \to K_0(B)$ for the map induced by a morphism $\varphi : A \to B$.

REMARK 3.14. The short exact sequence (6) admits a canonical splitting by the morphism $\lambda : \mathbb{C} \to \tilde{A}$ defined by $\lambda(z) := (0, z)$. Pre-composing with π, we get the idempotent $s : \tilde{A} \to \tilde{A}$, $s(a, z) = z$. Consider now an element $x = [p] - [q]$ in $K_0(A)$ represented by two projects in $M_N(\tilde{A})$, for some N large enough. By definition we have $[p] - [q] \in \ker(\pi_*)$, so $s_*([p]) = s_*([q])$. With this property, we have, by definition of the Grothendieck group:

$$\begin{aligned}[p]-[q] &= \left[\begin{pmatrix} p & 0 \\ 0 & 0 \end{pmatrix}\right] - \left[\begin{pmatrix} 0 & 0 \\ 0 & q \end{pmatrix}\right] \\ &= \left[\begin{pmatrix} p & 0 \\ 0 & 1_n - q \end{pmatrix}\right] - \left[\begin{pmatrix} 0 & 0 \\ 0 & 1_n \end{pmatrix}\right] \\ &= \left[\begin{pmatrix} p & 0 \\ 0 & 1_n - q \end{pmatrix}\right] - s_* \left[\begin{pmatrix} p & 0 \\ 0 & 1_n - q \end{pmatrix}\right].\end{aligned}$$

So we see that any element in $K_0(A)$ can be represented as $[p] - s_*[p]$ for some projector $p \in \mathrm{Proj}(M_\infty(\tilde{A}))$. This is also called the *standard picture* of $K_0(A)$.

EXAMPLE 3.15. As an example, we consider $C_0(\mathbb{R}^2)$. We can identify $\widetilde{C_0(\mathbb{R}^2)} \cong C(S^2)$, and let us therefore consider the classification of vector bundles over S^2 [1]. It can be proved that any complex vector bundle over S^2 splits as a direct sum of line bundles (rank one), c.f. **[BT]**. We therefore restrict to the case of line bundles. There are two generators (under taking tensor products!) corresponding to the trivial line bundle and the so-called tautological bundle by identifying $S^2 \cong \mathbb{CP}^1$. The trivial line bundle is represented by the function $1_{\mathbb{CP}^1}$ on $\mathbb{CP}^1$, whereas the tautological line bundle can be represented by the projector $p \in C(\mathbb{CP}^1, M_2(\mathbb{C}))$ defined as

$$p(z) = \frac{1}{1+z^2}\begin{pmatrix} 1 & \bar{z} \\ z & |z|^2 \end{pmatrix}, \; z \in \mathbb{C}, \quad p(\infty) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

The map $K^0(\mathbb{CP}^1) \to \mathbb{Z}$ sends $[1_{\mathbb{CP}^1}]$ and $[p]$ to 1. Therefore we get a class

$$\beta := [p] - [1_{\mathbb{CP}^1}] \in K_0(C_0(\mathbb{R}^2)),$$

called the *Bott element*.

4. First functorial properties

4.1. Homotopy invariance. We now come to the first functorial properties satisfied by the functor K_0. We start by defining the relevant notion of homotopy:

DEFINITION 4.1. Let A and B be C^*-algebras.

- Two $*$-morphisms $\varphi, \psi : A \to B$ are said to be homotopic, written $\varphi \sim_h \psi$ if there exists a family of $*$-morphisms $\{\Psi_t\}_{t\in[0,1]}$ with $\Psi_0 = \varphi$ and $\Psi_1 = \psi$ such that for each $a \in A$, the function $t \mapsto \Psi_t(a) \in B$ is continuous.
- Two C^*-algebras are homotopic if there exist $*$-morphism $\varphi : A \to B$ and $\psi : B \to A$ with $\varphi \circ \psi \sim_h id_B$ and $\psi \circ \varphi \sim_h id_A$.

We can now state:

[1] It is known in general that $Vect_k(S^q) \cong \pi_{q-1}(U(k))$, c.f. **[BT**, §23**]**

PROPOSITION 4.2. *If two $*$-homomorphisms $\varphi, \psi : A \to B$ are homotopic,*

$$\varphi_* = \psi_* : K_0(A) \to K_0(B).$$

PROOF. Left as an exercise. □

4.2. Half-exactness. Half exactness is of crucial importance to extend the K_0-functor to a full-blown homology theory on the category of C^*-algebras. Let us first recall that a *short exact sequence* of C^*-algebras is given by a sequence of $*$-morphisms

$$0 \longrightarrow I \xrightarrow{i} A \xrightarrow{\pi} B \longrightarrow 0,$$

where i is injective, π surjective and we have $\mathrm{Im}(i) = \ker(\pi)$. It then follows that $i(I) \subset A$ is an ideal (since it equals $\ker(\pi)$), and $B \cong A/i(I)$. So we may equivalently think of an ideal I inside A giving rise to the short exact sequence. We say that the sequence is *split* if there exists a splitting morphism $\sigma : B \to A$ such that $\pi \circ \sigma = id_B$.

PROPOSITION 4.3. *The functor K_0 is half-exact: a short exact sequence*

$$0 \longrightarrow I \xrightarrow{i} A \xrightarrow{\pi} B \longrightarrow 0$$

of C^*-algebras is mapped to an exact sequence

$$K_0(I) \xrightarrow{i_*} K_0(A) \xrightarrow{\pi_*} K_0(B).$$

PROOF. See the exercises. □

4.3. Stability. We shall be very brief about this one:

PROPOSITION 4.4 (Stability). *Let A be a unital C^*-algebra, and suppose there exists an increasing chain*

$$A_1 \subset A_2 \subset \ldots \subset A$$

of unital C^*-subalgebras, whose union is dense in A. Then the induced map

$$\varinjlim K_0(A_i) \to K_0(A)$$

is an isomorphism.

PROOF. Omitted. □

EXAMPLE 4.5. We have already see the chain

$$\mathbb{C} \subset M_2(\mathbb{C}) \subset M_3(\mathbb{C}) \subset \ldots,$$

whose union is $M_\infty(\mathbb{C})$. This algebra has an induced norm, and its completion is the $K(\mathcal{H})$ of compact operators on a separable Hilbert space. Abstractly, a compact operator is characterized by the property that it sends bounded subsets of $\mathcal{H}$ into relatively compact subsets. By example 3.11, we have $K_0(M_n(\mathbb{C})) \cong \mathbb{Z}$ and the maps induced by the inclusion are given by the identity. Therefore $K_0(K(\mathcal{H})) \cong \mathbb{Z}$ by stability of K-theory.

5. Higher K-groups

We are now going to define the higher K-theory groups and proof the fundamental long exact sequence associated to a short exact sequence of C^*-algebras. First we need some general constructions with C^*-algebras. First, given a C^*-algebra A, its *cone* CA, and suspension SA are defined as

$$CA := C_0((0,1], A), \quad SA := C_0((0,1), A),$$

equipped with the point wise multiplication and $*$, and the norm is given by

$$||f|| := \sup_{t\in(0,1]} ||f(t)||.$$

Remark that element in CA are just continuous functions f on the interval $[0,1]$ with values in A and $f(0) = 0$. In SA we have the additional requirement that also $f(1) = 0$. These are again C^*-algebras and they fit into an exact sequence

$$0 \to SA \to CA \to A \to 0.$$

We have:

LEMMA 5.1. *The cone of a C^*-algebra is contractible.*

PROOF. Let A be a C^*-algebra. Consider the family of $*$-morphisms $\Psi_t : CA \to CA$ defined by $\Psi_t(f)(s) = f(st)$. This defines a homotopy between the identity morphism on CA and the zero map $CA \to 0$. □

EXERCISE 5.2. Prove that taking the suspension of a C^*-algebra defines a covariant functor from the category of C^*-algebras to itself, which is exact as well as preserves homotopy equivalences

Let us now define the higher K-groups as follows:

DEFINITION 5.3. Let A be a C^*-algebra. We define

$$K_i(A) := K_0(S^i A).$$

Next, given two C^*-algebras A and B, their *direct sum* $A \oplus B$ is defined using the component wise product and equipped with the norm

$$||(a,b)|| = \max\{||a||, ||b||\}.$$

Given a $*$-morphism $\varphi : A \to B$, define the *mapping cone* $C_\varphi(A,B)$ as

$$C_\varphi(A,B) := \{(a,f) \in A \oplus CB,\ \varphi(a) = f(1)\}.$$

Remark that $CA = C_{id}(A,A)$. The cone of a morphism fits into an exact sequence

$$0 \longrightarrow SB \longrightarrow C_\varphi(A,B) \longrightarrow A \longrightarrow 0. \tag{7}$$

We now start with the following Lemma:

LEMMA 5.4. *Let*

$$0 \longrightarrow I \xrightarrow{i} A \xrightarrow{\pi} B \longrightarrow 0$$

be a short exact sequence of C^*-algebras with B contractible, then $i_* : K_0(I) \to K_0(A)$ is an isomorphism.

PROOF. By homotopy invariance we have that $K_0(B) \cong 0$, and therefore half-exactnes implies that i_* is surjective. To show injectivity, we factor i as

$$I \longrightarrow D_i \xrightarrow{\phi} C_\pi \xrightarrow{\pi_1} A,$$

with

$$D_i := \{f \in C([0,1], A),\ f(1) \in i(I)\},$$

and $\phi(f) = (f(0), \pi \circ g)$ with $g(t) = f(1-t)$, and π_1 comes from the short exact sequence (7). We now claim that all three maps induce injections on K_0: the first inclusion $I \hookrightarrow D_i$ is in fact a homotopy equivalence by the family $\Psi_t(f)(s) = f(s + t - st)$. The map ϕ fits into a short exact sequence

$$0 \longrightarrow CI \longrightarrow D_i \xrightarrow{\phi} C_\pi \longrightarrow 0,$$

so half-exactness yields injectivity of ϕ_*. Finally, the short exact sequence

$$0 \longrightarrow SB \longrightarrow C_\pi \xrightarrow{\pi_1} A \longrightarrow 0$$

together with half-exactness and the fact that SB is contractible implies that $(\pi_1)_*$ is injective. □

Now, given any short exact sequence

$$0 \longrightarrow I \xrightarrow{i} A \xrightarrow{\pi_0} B \longrightarrow 0,$$

we can now associate another short exact sequence given by

$$0 \longrightarrow I \xrightarrow{e} C_{\pi_0}(A, B) \xrightarrow{q} C(B) \longrightarrow 0,$$

with $e(a) = (e, 0)$ and $q(a, f) = f$. By Lemma 5.1, we can apply the previous Lemma to obtain an isomorphism

$$K_0(I) \cong K_0(C_{\pi_0}(A, B)).$$

As explained above, the mapping cone $C_{\pi_0}(A, B)$ fits into the exact sequence

$$0 \longrightarrow SB \longrightarrow C_{\pi_0}(A, B) \xrightarrow{\pi_1} A \longrightarrow 0. \tag{8}$$

Applying Proposition 4.3 yields, using Definition 5.3, a map

$$\partial : K_1(A/I) \to K_0(I).$$

DEFINITION 5.5. The map ∂ is called the *connecting* (or *boundary*) *map* of the short exact sequence.

With the connecting map, we have extended our exact sequence one step to the left:

$$K_1(A/I) \xrightarrow{\partial} K_0(I) \longrightarrow K_0(A) \longrightarrow K_0(B).$$

We already knew it was exact at $K_0(A)$, and now we can conclude exactness at $K_0(I)$. Next, we iterate this argument by applying it to π_1 in (8) to obtain $K_0(SB) \cong K_0(C_{\pi_1})$ together with the exact sequence

$$0 \longrightarrow SA \longrightarrow C_{\pi_1} \xrightarrow{\pi_2} C_{\pi_0} \longrightarrow 0,$$

which induces a map

$$K_1(A) \to K_1(A/I)$$

on the level of K-theory.

Continuing like this we get the following exact sequences:

$$0 \longrightarrow I \xrightarrow{i} A \xrightarrow{\pi_0} B \longrightarrow 0, \tag{9}$$

$$0 \longrightarrow SB \longrightarrow C_{\pi_0} \xrightarrow{\pi_1} A \longrightarrow 0, \tag{10}$$

$$0 \longrightarrow SA \longrightarrow C_{\pi_1} \xrightarrow{\pi_2} C_{\pi_0} \longrightarrow 0, \tag{11}$$

$$0 \longrightarrow SC_{\pi_0} \longrightarrow C_{\pi_2} \xrightarrow{\pi_3} C_{\pi_1} \longrightarrow 0, \tag{12}$$

$$0 \longrightarrow SC_{\pi_1} \longrightarrow C_{\pi_3} \xrightarrow{\pi_4} C_{\pi_2} \longrightarrow 0, \tag{13}$$

$$0 \longrightarrow SC_{\pi_2} \longrightarrow C_{\pi_4} \xrightarrow{\pi_5} C_{\pi_3} \longrightarrow 0, \tag{14}$$

$$\ldots, \tag{15}$$

together with isomorphisms

$$K_0(C_{\pi_0}) \cong K_0(I), \tag{16}$$

$$K_0(C_{\pi_1}) \cong K_0(SB) = K_1(B), \tag{17}$$

$$K_0(C_{\pi_2}) \cong K_0(SA) = K_1(A), \tag{18}$$

$$K_0(C_{\pi_3}) \cong K_0(SC_{\pi_0}) \cong K_0(SI) = K_1(I), \tag{19}$$

$$K_0(C_{\pi_4}) \cong K_0(SC_{\pi_1}) \cong K_0(S^2B) \cong K_2(B), \tag{20}$$

$$\ldots \tag{21}$$

We can now prove:

THEOREM 5.6. *The short exact sequence of C^*-algebras*

$$0 \longrightarrow I \xrightarrow{i} A \xrightarrow{\pi_0} B \longrightarrow 0$$

induces a long exact sequence on the level of K-theory:

$$\ldots \longrightarrow K_2(B) \xrightarrow{\partial} K_1(I) \longrightarrow K_1(A) \longrightarrow K_1(B) \xrightarrow{\partial} K_0(I) \longrightarrow K_0(A) \longrightarrow K_0(B).$$

PROOF. Should now be clear. □

REMARK 5.7. We have to work a bit if we want to have an explicit description of the connecting map $\partial : K_1(A/I) \to K_0(I)$. First remark that for B a unital C^*-algebra, SB is not unital and one has

$$\widetilde{SB} \cong \{f \in C(I,B),\ f(0) = f(1) \in \mathbb{C}1_B\}. \tag{22}$$

The canonical map to $\mathbb{C}$ is given by evaluation at 0 (or 1), and therefore elements in $K_0(SB) = K_1(B)$ are given by formal differences $[p] - [q]$ of projection valued continuous maps $p, q : I \to M_\infty(B)$ with

$$p(0) = p(1) = q(0) = q(1) \in M_\infty(\mathbb{C}).$$

Now given an element in $K_1(A/I)$ represented by such projectors $[p] - [q]$, p can be lifted to a continuous map P from I to $\mathrm{Proj}(M_\infty(A))$. However, we may not have $P(0) = P(1)$ anymore. This is measured by the formal difference

$$\mathrm{Twist}(p) := [P(0)] - [P(1)] \in K_0(A).$$

This element is independent of the choice of lift P of p, and maps to zero under the canonical map to $K_0(A/I)$. Therefore, by Proposition 4.3, we have a canonical element

$$\mathrm{Twist}(p) \in K_0(I).$$

With this definition, the connecting map is given by $\partial([p] - [q]) = \mathrm{Twist}(p) - \mathrm{Twist}(q)$.

COROLLARY 5.8. *Suppose that the short exact sequence*

$$0 \to I \to A \to A/I \to 0,$$

is split by a $*$-homomorphism from A/I to A. The the induced sequences

$$0 \to K_p(I) \to K_p(A) \to K_p(A/I) \to 0$$

are exact.

6. A concrete description of K_1

The previous definition of the higher K-groups is very good for their cohomological properties. In practise it is convenient to have a more concrete description of these groups, in terms of explicit cycles. Here we give such a description of K_1. By Bott periodicity, this suffices for the general K-groups.

DEFINITION 6.1. Let A be a unital C^*-algebra. The abelian group $K_1'(A)$ is generated by elements $[u]$ with $u \in M_N(A)$ unitary, for some $N \in \mathbb{N}$, subject to the relations:

- $[u] = [v]$ if u and v are homotopic in $U(M_N(A))$,
- $[1]$ is the unit of $K_1'(A)$,
- $[u] + [v] = [u \oplus v]$.

In other words: $K_1'(A) = \pi_0(U_\infty(A))$ if $U_\infty(A)$ denotes the group of unitary elements in $M_\infty(A)$. That this forms an abelian group is not entirely obvious, but given unitaries $u, v \in U_\infty(A)$, one shows that

$$\begin{pmatrix} u & 0 \\ 0 & 1 \end{pmatrix} \sim_h \begin{pmatrix} 1 & 0 \\ 0 & u \end{pmatrix}$$
$$\begin{pmatrix} u & 0 \\ 0 & v \end{pmatrix} \sim_h \begin{pmatrix} uv & 0 \\ 0 & 1 \end{pmatrix} \sim_h \begin{pmatrix} vu & 0 \\ 0 & 1 \end{pmatrix}$$
$$\begin{pmatrix} u & 0 \\ 0 & u^* \end{pmatrix} \sim_h \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

PROPOSITION 6.2. *There exists a natural isomorphism $K_1(A) \cong K_1'(A)$.*

PROOF. Let us first construct a map from $K_0(SA)$ to $K_1'(A)$. From the description in Remark 5.7 we see that classs in $K_0(SA)$ are represented by formal differences $[p] - [q]$ of projection valued continuous maps from $[0,1]$ to $\mathrm{Proj}(M_\infty(A))$ with $p(0) = p(1) = q(0) = q(1) \in M_\infty(\mathbb{C})$. Since

$$[p] - [q] = ([p] - [p(0)]) - ([q] - [q(0)]),$$

and the two terms separately define elements in $K_0(SA)$, it suffices to define the map on $[p] - [p(0)]$. Clearly, the projection $p(t)$ in $M_\infty(A)$ is homotopic to $p(1)$, so by Proposition 3.7, there exists $u(t) \in U_\infty(A)$ with $p(t) = u(t)p(1)u(t)^*$. Since $p(0) = p(1)$, we have $[p(0), u(0)] = 0$. Since any projection in $M_n(\mathbb{C})$ is equivalent to $\mathrm{diag}(1_m, 0_{n-m})$, we have

$$u(0) = \begin{pmatrix} v & 0 \\ 0 & w \end{pmatrix}, \quad p(0) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

With this, we define the map $[p] - [p(0)] \mapsto [v] \in K_1'(A)$. Conversely, given $v \in U_\infty(A)$, let $u(t)$ be a homotopy between

$$u(0) = \begin{pmatrix} v & 0 \\ 0 & v^* \end{pmatrix} \quad \text{and} \quad u(1) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

The family $p(t) = u(t)\mathrm{diag}(1,0)u(t)^*$ defines an element in $K_0(SA) = K_1(A)$. This defines the inverse. □

EXAMPLE 6.3. Recall that a bounded operator $T \in B(\mathcal{H})$ is *Fredholm* if its kernel and cokernel are finite dimensional vector spaces. In that case, the integer

$$\mathrm{index}(T) := \dim\ker(T) - \dim\mathrm{coker}(T)$$

is called the *index* of T.

We have already encountered the algebra $K(\mathcal{H})$ of compact operators. It is in fact an ideal in $B(\mathcal{H})$, so we have a short exact sequence,

$$0 \longrightarrow K(\mathcal{H}) \longrightarrow B(\mathcal{H}) \xrightarrow{\pi} C(\mathcal{H}) \to 0, \tag{23}$$

where $C(\mathcal{H}) := B(\mathcal{H})/K(\mathcal{H})$ is called the *Calkin algebra*. The relation between the Calkin algebra and Fredholm operators is given by the following classical result:

THEOREM 6.4 (Atkinson's theorem). *$T \in B(\mathcal{H})$ is Fredholm if and only if its image in the Calkin algebra is invertible.*

Let us now consider an *essentially unitary* operator $T \in B(\mathcal{H})$, i.e., satisfying

$$T^*T - 1,\ TT^* - 1 \in K(\mathcal{H}).$$

By Atkinson's theorem, such an operator is Fredholm, but we also see that it induces a class $[\pi(T)] \in K_1(C(\mathcal{H}))$. The connecting map of the short exact sequence (23) yields a map

$$\partial : K_1(C(\mathcal{H})) \to K_0(K(\mathcal{H})) \cong \mathbb{Z},$$

where the isomorphism follows from stability as in Example 4.5. Unravelling the definitions, we now see that

$$\partial([\pi(T)]) = \mathrm{index}(T).$$

EXAMPLE 6.5 (The Toeplitz extension). Let $L^2(S^1)$ be the Hilbert space of square integrable functions on the unit circle S^1 in the complex plane. The Hardy subspace $\mathcal{H}(S^1)$ is given by the boundary values of holomorphic functions on the interior disk. If we fix a basis $\{z^n\}_{n\in\mathbb{Z}}$ of $L^2(S^1)$, the Hardy space is the closure of the span of the positive Fourier modes z^n, $n \geq 0$. Denote by π the projector in $L^2(S^1)$ onto $\mathcal{H}(S^1)$. Given a continuous function on S^1, the associated *Toeplitz operator* $T_f \in B(\mathcal{H}(S^1))$ is defined as

$$T_f(g) = \pi(fg).$$

LEMMA 6.6. *When f is non-vanishing, T_f is Fredholm*

PROOF. Let us start with the following: denote by M_f the bounded operator on $L^2(S^1)$ given by multiplying with f. The continuous functions f for which

$$[\pi, M_f] \in K,$$

forms a C^*-subalgebra of $C(S^1)$. Since $f(z) = z$ is in this subalgebra (the commutator is a rank one projection), we conclude that this subalgebra must be all of $C(S^1)$ since z generates this algebra by the Stone–Weierstrass theorem. Therefore, $[\pi, M_f]$ is compact for all $f \in C(S^1)$.

For the second, first remark that

$$\begin{aligned} T_{f_1}T_{f_2} &= \pi M_{f_1}\pi M_{f_2} \\ &= \pi M_{f_1 f_2} + \pi[M_{f_1}, \pi]M_{f_2} \\ &= T_{f_1 f_2} + \text{compact operator.} \end{aligned}$$

Therefore, if f is non vanishing, $T_{1/f}$ is an inverse modulo compact operators, and hence T_f is Fredholm by Atkinson's theorem. □

LEMMA 6.7. *The map $\alpha : C(S^1) \to C(\mathcal{H}(S^1))$ given by the image of $f \mapsto T_f$, is an injective $*$-homomorphism of C^*-algebras*

PROOF. The proof of the previous Lemma shows that α is a homomorphism. Clearly $T_f^* = T_{\bar{f}}$, so α is also a $*$-homomorphism. The kernel is therefore an ideal in $C(S^1)$ and corresponds to a closed subset $J \subset S^1$. Since α commutes with the canonical rotation action on $C(S^1)$ and $C(\mathcal{H}(S^1))$, J is either empty or the entire S^1. The first case is clearly impossible, so the conclusion follows. □

Now we let $\mathcal{T}$ be the C^*-algebra generated by the operators T_f with f continuous, together with all the compact operators K on $\mathcal{H}$. By the previous Lemmata, we have a short exact sequence

$$0 \to K \to \mathcal{T} \to C(S^1) \to 0, \tag{24}$$

called the *Toeplitz extension*. A nonvanishing function $f : S^1 \to \mathbb{C}/\{0\}$ induces a map from $\mathbb{Z}$ to $\mathbb{Z}$ on fundamenal groups. This is given by multiplication with an integer called the *winding number* of f.

THEOREM 6.8 (Toeplitz index theorem). *For f nonvanishing,*

$$\text{index}(T_f) = -\text{winding}(f).$$

REMARK 6.9. When f is C^1, we can write the right hand side as

$$-\int_{S^1} \frac{df}{f}.$$

7. Bott periodicity

The famous Bott periodicity theorem is the following statement:

THEOREM 7.1 (Bott periodicity). *For any C^*-algebra A, there are natural isomorphisms*

$$K_{i+2}(A) \cong K_i(A).$$

Here, natural means that for any $*$-morphism $\varphi : A \to B$, we have a commutative diagram

$$\begin{array}{ccc} K_{i+2}(A) & \xrightarrow{\varphi_*} & K_{i+2}(B) \\ \cong\downarrow & & \downarrow\cong \\ K_i(A) & \xrightarrow{\varphi_*} & K_i(B) \end{array}$$

COROLLARY 7.2. *A short exact sequence of C^*-algebras*

$$0 \longrightarrow I \xrightarrow{i} A \xrightarrow{\pi_0} B \longrightarrow 0$$

induces a six-term periodic exact sequence on the level of K-theory:

$$\begin{array}{ccccc} K_0(I) & \longrightarrow & K_0(A) & \longrightarrow & K_0(A/I) \\ {\scriptstyle\partial_1}\uparrow & & & & \downarrow{\scriptstyle\partial_0} \\ K_1(A/I) & \longleftarrow & K_1(A) & \longleftarrow & K_1(I) \end{array}$$

The map ∂_0 is defined as the composition of $K_0(A/I) \cong K_2(A/I) \stackrel{\partial_2}{\to} K_1(I)$.

In the following, we give a sketch of the proof of Bott periodicity. It consists of several steps:

Step 1. *(The minimal tensor product of C^*-algebras.)* Taking tensor products of C^*-algebras is a subtle matter. Clearly, the algebraic tensor product is too small: when A and B are infinite dimensional C^*-algebras, their tensor product $A \otimes_{\mathbb{C}} B$ is not complete. To get another feeling of the problem, convince yourself that $C_0(\mathbb{R}) \otimes C_0(\mathbb{R})$ is not isomorphic to $C_0(\mathbb{R}^2)$, a property we would reasonably require.

The solution is to find a completion of the algebraic tensor product. There are several possibilities for this. One is given by the so-called *minimal (or spatial) completion*: Consider faithful embeddings ρ_1, ρ_2 of A and B into the bounded operators on Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$. (These exist by the GNS-construction.) For these two Hilbert spaces, we consider their tensor product $\mathcal{H}_1 \otimes \mathcal{H}_2$ (of Hilbert spaces!) Two bounded operators $S \in B(\mathcal{H}_1)$ and $T \in B(\mathcal{H}_2)$, define a bounded operator $S \otimes T$ on $\mathcal{H}_1 \otimes \mathcal{H}_2$. This extends to define an embedding of the algebraic tensor product $A \otimes_{\mathbb{C}} B$ in $B(\mathcal{H}_1 \otimes \mathcal{H}_2)$. We then define the minimal tensor product $A \hat{\otimes} B$ to be the C^*-completion of $A \otimes_{\mathbb{C}} B$ in $B(\mathcal{H}_1 \otimes \mathcal{H}_2)$. One can prove that this completion is independent of the choices of ρ_1 and ρ_2. Furthermore, it has the advantage of being functorial under $*$-morphisms of C^*-algebras.

LEMMA 7.3. *Let X be Hausdorff topological space. For any C^*-algebra B, there is a canonical isomorphism*

$$C_0(X) \hat{\otimes} B \cong C_0(X; B).$$

PROOF. □

REMARK 7.4. With the minimal tensor product, it is not difficult to prove that completion of the infinite matrix algebra (5) in the norm is equal to $K \hat{\otimes} A$. Together with stability of K-theory, this proves that

$$K_p(K \hat{\otimes} A) \cong K_p(A).$$

Step 2 *(The external product in K-theory)* For general noncommutative C^*-algebras, K-theory has no (graded) ring structure. However, there does exist an *external* product: for C^*-algebras A, B this is a map

$$\times : K_p(A) \times K_q(B) \to K_{p+q}(A \hat{\otimes} B). \tag{25}$$

The basic idea is very simple: Let A and B be unital C^*-algebras, and suppose that p is a projector in A and q a projector in B. Their tensor product $p \otimes q$ defines a projector in $A\hat{\otimes}B$. This extends easily to matrix algebras: if $p \in M_m(A)$ and $q \in Mn(B)$ are projectors, then $p \otimes q$ defines a projector in $M_{mn}(A\hat{\otimes}B)$, using the canonical isomorphism $M_m(A)\hat{\otimes}M_n(B) \cong M_{mn}(A\hat{\otimes}B)$. This gives us the product

$$\times : K_0(A) \times K_0(B) \to K_0(A\hat{\otimes}B),$$

for A and B unital.

In the nonunital case, we use the restriction of the product map

$$\times : K_0(\tilde{A}) \times K_0(\tilde{B}) \to K_0(\tilde{A}\hat{\otimes}\tilde{B})$$

to $K_0(A) \times K_0(B)$. We have split exact sequences

$$0 \to A\hat{\otimes}B \to A\hat{\otimes}\tilde{B} \to A \to 0,$$
$$0 \to A\hat{\otimes}B \to \tilde{A}\hat{\otimes}B \to B \to 0.$$

Using Corollary (5.8), we see that

$$K_0(A\hat{\otimes}B) = \ker\left(\pi_* : K_0(\tilde{A}\hat{\otimes}\tilde{B}) \to K_0(\tilde{A}) \times K_0(\tilde{B})\right),$$

with $\pi := (1 \otimes \pi_2, \pi_1 \otimes 1)$. This is exactly the image of the map above, so this defines the external product on K_0 for nonunital algebras. For the general product (25), one uses the canonical isomorphism

$$S^{p+q}(A\hat{\otimes}B) \cong S^p(A)\hat{\otimes}S^q(B).$$

Step 3 *(The Bott map.)* Consider now the Bott element $b \in K_0(C_0(\mathbb{R}^2))$ constructed in Example 3.15. Using the isomorphism $S^2A \cong C_0(\mathbb{R}^2; A)$, we get a map

$$\beta_A : K_0(A) \xrightarrow{b\times -} K_0(C_0(\mathbb{R}^2)\hat{\otimes}A) \overset{\text{Lemma 7.3}}{\cong} K_0(C_0(\mathbb{R}^2; A)) \overset{\text{Def.}}{\cong} K_2(A),$$

called the *Bott map*.

Step 4 *(The inverse to the Bott map)* Given a short exact sequence of C^*-algebras

$$0 \to I \to A \to A/I \to 0,$$

it is not true in general that the sequence

$$I\hat{\otimes}B \to A\hat{\otimes}B \to (A/I)\hat{\otimes}B \tag{26}$$

is exact for a generic C^*-algebra B. (However, when B is so-called *nuclear*, this is true.)

Consider now the Toeplitz extension (24). It is true that the sequence

$$0 \to K\hat{\otimes}A \to \mathcal{T}\hat{\otimes}A \to C(S^1)\hat{\otimes}A \to 0,$$

is exact for a general C^*-algebra. (This can be proved by a technical argument using the properties of section $f \mapsto T_f$ of the Toeplitz short exact sequence (24).) The connecting map in K-theory of this sequence is given by

$$\partial : K_1(C(S^1; A)) \to K_0(K \hat{\otimes} A) \cong K_0(A).$$

Using the inclusion $C_0(\mathbb{R}; A) \subset C(S^1; A)$, this gives a map

$$\alpha_A : K_2(A) \cong K_1(SA) \to K_0(A).$$

Step 5 *(Proof of Bott periodicity)* We conclude with the proof of the fact that α_A and β_A are inverses of each other.

LEMMA 7.5. *The map α_A defined above has the following properties:*

i) For $A = \mathbb{C}$, we have $\alpha_{\mathbb{C}}(b) = 1$, where b is the Bott generator.
ii) For all A and B, the following diagram commutes:

$$\begin{array}{ccc} K_2(A) \times K_0(B) & \xrightarrow{\times} & K_2(A \hat{\otimes} B) \\ \downarrow{\scriptstyle \alpha_A \otimes 1} & & \downarrow{\scriptstyle \alpha_{A \hat{\otimes} B}} \\ K_0(A) \times K_0(B) & \xrightarrow{\times} & K_0(A \hat{\otimes} B) \end{array}$$

PROOF. We have already seen that the connecting map in K-theory applied to the Toeplitz extension is given by taking the index of the Fredholm Toeplitz operator, so property $i)$ follows from Theorem 6.8. For this we have to identify the class $b' \in K_1(C_0(\mathbb{R}))$ that corresponds to the Bott element. This turns out to be the class induced by the $U(1)$-valued map $z \mapsto \bar{z}$ on S^1. Property $ii)$ follows from the fact that

$$\partial(x \times y) = \partial(x) \times y,$$

for the connecting map in K-theory for the sequence (26) incase this happens to be exact, where $x \in K_p(A/I)$ and $y \in K_q(B)$. We have already mentioned that this holds true for the Toeplitz extension. □

With this Lemma, we easily see that α_A is a left inverse of β_A:

$$\alpha_A(\beta_A(x)) = \alpha_A(b \times x) = \alpha_{\mathbb{C}}(b) \times x = 1 \times x = x.$$

This shows that the Bott map is injective, so it remains to show surjectivity: for this we consider the following rotation in $SO(4, \mathbb{R})$:

$$\tau := \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}.$$

Its determinant is 1, so it is in the connected component of the identity, and the map induced on $K^0(\mathbb{R}^4)$ is the identity by homotopy invariance. But written out algebraically, τ interchanges the two factors in

$$C_0(\mathbb{R}^4) \cong C_0(\mathbb{R}^2)\hat{\otimes}C_0(\mathbb{R}^2).$$

Therefore, for $y \in K_0(A\hat{\otimes}C_0(\mathbb{R}^2))$ we see that

$$b \times y = \sigma_*(y) \times b,$$

where $\sigma : A\hat{\otimes}C_0(\mathbb{R}^2) \to C_0(\mathbb{R}^2)\hat{\otimes}A$ is the "flip". It follows that

$$y = \alpha_{A\hat{\otimes}C_0(\mathbb{R}^2)}(b \times y) = \alpha_{A\hat{\otimes}C_0(\mathbb{R}^2)}(\sigma_*(y) \times b) = \alpha_A(\sigma_*(y)) \times b.$$

This concludes the proof of Bott periodicity.

REMARK 7.6. Unravelling all the definitions and isomorphisms, one finds the following explicit description of the map $\partial_0 : K_0(A/I) \to K_1(I)$ in Corollary 7.2: Given a projector $p \in M_n(A/I)$ we consider a lift $x \in M_n(A)$ with $x^* = x$. (So x need not be a projector anymore.) Then

$$e^{2\pi i x} \in U_n(\tilde{I}),$$

since $\pi(e^{2\pi i x}) = e^{2\pi i p} = 1$, by Exercise 3.2. Its class in $K_1(I)$ is independent of the choice of lift and this defines $\partial_0([p])$.

REMARK 7.7. Bott periodicity gives us the following natural isomorphisms:

$$K_0(A) \cong K_2(A) = K_1(SA) = \pi_0(U_\infty(\widetilde{SA})).$$

Using the description (22), we have a natural isomorphism

$$U_\infty(\widetilde{SA}) \cong \{f \in C(S^1, U_\infty(A)),\ f(0) = 1\},$$

which is nothing but the based loop group of $U_\infty(A)$! Its group of connected components is exactly $\pi_1 U_\infty(A)$. We therefore find the following picture of K-theory, complementing that of Section 6:

$$K_0(A) = \pi_1(U_\infty(A))$$
$$K_1(A) = \pi_0(U_\infty(A)).$$

CHAPTER 3

Cyclic theory

Cyclic theory refers to both Hochschild (co)homology and cyclic (co)homology of algebras. We will see in this chapter that indeed these naturally come together in a whole "package" . Cyclic homology plays the role of de Rham cohomology of manifolds in noncommutative geometry. It is also the natural recipient of a noncommutative generalization of the Chern character.

Contrary to K-theory, cyclic theory can be set up completely algebraically, and therefore from now on we will work over a fixed field $\mathbb{K}$, and all tensor products $\otimes$ are algebraic over $\mathbb{K}$.

1. Hochschild theory

1.1. bimodules. Let A be an algebra over $\mathbb{K}$. We have already discussed left modules over A, and we now denote the category of left A modules by $\underline{A\text{-Mod}}$. There is also a notion of right modules: this is a $\mathbb{K}$-vector space M equipped with a right action $m \mapsto ma$, $m \in M$, $a \in A$ satisfying the obvious axioms. The category of right A modules is denoted by $\underline{\text{Mod-}A}$. A *bimodule* over A is a $\mathbb{K}$-vector space M equipped with both a left and right A-module structure that commute with each other. We write the left module structure as $m \mapsto am$ and the right module structure as $m \mapsto ma$ so that the compatibility condition is spelled out as

$$(a_1 m)a_2 = a_1(ma_2).$$

We denote by A^{op} the opposite algebra of A: this has the same underlying vector space, but the multiplication is defined as

$$a_1 \cdot_{op} a_2 = a_2 a_1, \quad \text{for all } a_1, a_2 \in A.$$

This defines another associative algebra (check!) that will only coincide with A when A is commutative. With this, it is easy to see that a right A-module is nothing but a left A^{op}-module. (check again!)

We can now consider the *enveloping algebra*

$$A^e := A \otimes A^{op}.$$

EXERCISE 1.1. Show that an A-bimodule is the same thing as either a left A^e-module or a right A^e-module.

Because of this, we can define the category of $A-A$ bimodules as the category of left A^e-modules.

EXAMPLE 1.2. We always have the bimodule $M = A$ equipped with the obvious left and right module structure.

1.2. Hochschild homology. Let A be a unital algebra and M be a bimodule. The space of *Hochschild chains* of degree k is defined as

$$C_k(A,M) := A^{\otimes k} \otimes M$$

The Hochschild boundary map $b : C_k(A,M) \to C_{k-1}(A,M)$ is defined as

$$\begin{aligned} b(m \otimes a_1 \otimes \ldots \otimes a_k) =& ma_1 \otimes a_2 \otimes \ldots \otimes a_k \\ &+ \sum_{i=1}^{k-1} (-1)^i m \otimes a_1 \otimes \ldots \otimes a_i a_{i+1} \otimes \ldots \otimes a_k \\ &+ (-1)^k a_k m \otimes a_1 \otimes \ldots \otimes a_{k-1}. \end{aligned} \tag{27}$$

EXERCISE 1.3. Check that $b \circ b = 0$.

DEFINITION 1.4. The *Hochschild homology with coefficients in* M, written $H_\bullet(A,M)$ is defined to be the homology of the chain complex $(C_\bullet(A,M), b)$. When $M = A$, we simply write $HH_\bullet(A)$.

1.3. Hochschild cohomology. The dual theory is called Hochschild cohomology. For this we define the *Hochschild cochains* as

$$C^k(A,M) := \mathrm{Hom}_{\mathbb{K}}(A^{\otimes k}, M).$$

The boundary operator dualizes to the *Hochschild coboundary operator* $b : C^k(A,M) \to C^{k+1}(A,M)$:

$$\begin{aligned} b\varphi(a_0 \otimes \ldots \otimes a_k) =& a_0 \varphi(a_1 \otimes a_2 \otimes \ldots \otimes a_k) \\ &+ \sum_{i=0}^{k-1} (-1)^{i+1} \varphi(a_0 \otimes \ldots \otimes a_i a_{i+1} \otimes \ldots \otimes a_k) \\ &+ (-1)^{k+1} \varphi(a_1 \otimes \ldots \otimes a_{k-1}) a_k, \end{aligned} \tag{28}$$

where $\varphi \in C^k(A,M)$. The cohomology of this complex defines the Hochschild cohomology $H^\bullet(A,M)$.

1.4. Low degrees. In low degrees, the Hochschild (co)homology groups have a clear interpretation, showing why they are important. For degree zero, we have for example:

$$H^0(A,M) = M^A = \{m \in M, am = ma, \ \forall a \in A\} \tag{29}$$

$$H_0(A,M) = M_A = M/\{am - ma, a \in A, m \in M\}. \tag{30}$$

(One could call these spaces the *invariants* resp. the *coinvariants* with respect to the action of A.) Remark that for $M = A$,

$$HH_0(A) = A/[A, A],$$

is the dual space of traces on A.

In degree one, we notice that the kernel of b in $C^1(A, M)$ is given by maps $f : A \to M$ satisfying

$$f(a_1 a_2) = a_1 f(a_2) + f(a_1) a_2, \quad \text{for all } a_1, a_2 \in A.$$

Such maps are called *derivations* and the space of all derivations is written as $\mathrm{Der}(A, M)$. The image of b in $C^1(A, M)$ is given by derivations of the form $f_m(a) = am - ma$ for $m \in M$. Such derivations are called *inner*, and we find:

$$H^1(A, M) = \mathrm{Der}(A, M)/\mathrm{Inn}(A, M).$$

REMARK 1.5. When A is commutative, we can go a both further. The *Käbler differentials* Ω^1_A of A is the A-module with the following presentation: it is generated by elements da for all $a \in A$, with $dx = 0$ for $x \in \mathbb{K}$, and subject to the relations

$$d(a_1 + a_2) = da_1 + da_2, \quad d(a_1 a_2) = a_1 da_2 + a_2 da_1, \quad \text{for all } a_1, a_2 \in A.$$

Recall that any left A-module can be automatically given a bimodule structure (these bimodules are called *symmetric*). We therefore see that the $d : A \to \Omega^1_A$ is a derivation, and the point of this construction is that this derivation is *universal*:

PROPOSITION 1.6. *For any A-module, there is an isomorphism*

$$\mathrm{Der}(A, M) \cong \mathrm{Hom}_A(\Omega^1_A, M).$$

In fact, one can also prove that

$$HH_1(A) \cong \Omega^1_A. \tag{31}$$

We now continue with A being a general (i.e., not necessarily commutative) algebra, and move to degree 2. For $H^2(A, M)$ we should consider so-called *Hochschild extensions* of A by M, i.e., short exact sequences of the form:

$$0 \longrightarrow M \longrightarrow E \xrightarrow{\epsilon} A \longrightarrow 0,$$

subject to the following conditions: E is an algebra and the morphism $\epsilon : E \to A$ is such that the kernel $\ker(\epsilon)$ is an *ideal of square zero*. This implies that $\ker(\epsilon)$ can be given the structure of an A-bimodule and we assume that this bimodule is isomorphic to M. Finally, we assume that over $\mathbb{K}$ there exists a splitting of the sequence above.

If we choose such a splitting $\sigma : A \to E$ (not an algebra homomorphism!), we get a vector space isomorphism $E \cong A \oplus M$. The multiplication will then look like

$$(a_1, m_1) \cdot (a_2, m_2) = (a_1 a_2, a_1 m_2 + m_2 a_1 + f(a_1, a_2)),$$

for some map $f : A \otimes A \to M$.

EXERCISE 1.7. Show that associativity of the product implies that f is a cocycle: $bf = 0$. Furthermore, show that choosing a different splitting $\sigma' : A \to E$ merely changes f by a coboundary, and conclude that a Hochschild extension determines a unique class in $H^2(A, M)$. Show that $H^2(A, M)$ classifies all Hochschild extensions up to isomorphism.

2. Some homological algebra

The above definition of Hochschild theory uses explicit chain complexes. Although this is a completely straightforward definition, it has the disadvantage that it leads in practice to horrible computations, even for the simplest algebras. To make the theory more accessible for computations, we need some tools from homological algebra.

Let R be a ring, and consider the abelian category $\underline{\text{Mod-}R}$ of right R-modules. Given a *left* R-module B, the tensor product over R defines a functor

$$\otimes_R B : \underline{\text{Mod-}R} \to \underline{Ab}, \quad M \mapsto M \otimes_R B,$$

with

$$M \otimes_R B := M \otimes B / \{mr \otimes b - m \otimes rb,\ m \in M, r \in R, b \in B\}.$$

This functor turns out to be *right-exact*: any short exact sequence

$$0 \to M \to N \to O \to 0$$

in $\underline{\text{Mod-}R}$, is send to an exact sequence

$$M \otimes_R B \to N \otimes_R B \to O \otimes_R B \to 0.$$

The non-exactness of the first arrow is measured by the abelian group $\mathrm{Tor}_1(O, B)$. In fact one can define groups $\mathrm{Tor}_n(M, B)$ for any $n \in \mathbb{N}$ and object $M \in \underline{R\text{-Mod}}$, so that we get a long exact sequence

$$\ldots \to \mathrm{Tor}_{n+1}(O, B) \to \mathrm{Tor}_n(M, B) \to \mathrm{Tor}_n(N, B) \to \mathrm{Tor}_n(O, B) \to \ldots$$
$$\ldots \to \mathrm{Tor}_1(O, B) \to \mathrm{Tor}_0(M, B) \to \mathrm{Tor}_0(N, B) \to \mathrm{Tor}_0(O, B) \to 0,$$

with $\mathrm{Tor}_0(M, B) = M \otimes_R B$. Let us recall its construction: let M be an object of $\underline{\text{Mod-}R}$. A *projective resolution* is a complex of R-modules

$$\ldots \to P_2 \to P_1 \to P_0 \to M \to 0,$$

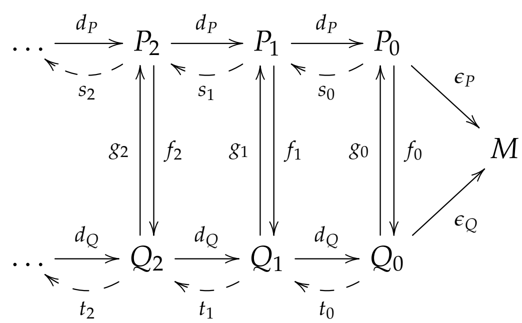
which is exact (i.e., has no homology) and each P_i is projective, c.f. Definition 2.1. It can be shown that such a resolution exists. To define the Tor-groups, we choose one and consider the resulting complex by tensoring over R:

$$\ldots \to P_2 \otimes_R B \to P_1 \otimes_R B \to P_0 \otimes_R B \to 0$$

Then $\mathrm{Tor}_n(M, B)$ is the n-th homologgy group of this complex. This definition is independent, up to natural isomorphism, of the resolution chosen. This follows from the following

LEMMA 2.1. *Any two projective resolutions of M in the category R-Mod are chain homotopic to each other.*

PROOF. Let $(P_\bullet, d_P)$ and $(Q_\bullet, d_Q)$ be two projective resolutions of M. Using projectively, we shall construct maps as in the following diagram:



These maps will satisfy

$$g_n \circ f_n - 1_{P_n} = d_P \circ s_n + s_{n-1} \circ d_P, \quad f_n \circ g_n - 1_{Q_n} = d_Q \circ t_n + t_{n-1} \circ d_Q,$$

therefore proving the Lemma.

To construct the first one, f_0, is easy: since ϵ_Q is surjective, this follows by Definition 2.1 of P_0 being projective. The next one, f_1 is a little bit more difficult because we need a surjective map to lift: for this we use the fact that $0 = \epsilon_P \circ d_P = \epsilon_Q \circ f_0 \circ d_P$, so $f_0 \circ d_P$ maps P_1 onto $\ker(\epsilon_Q) = \mathrm{im}(d_Q)$, a submodule of Q_0. Projectivity of P_1 then guarantees the existence of f_1. In this way we proceed inductively to construct the higher f_k's. The maps g_k are constructed in the same way using projectivity of $Q_\bullet$.

Now we turn to the construction of the maps s_n. Again, the first one, s_0 is easy: we have

$$\epsilon_P \circ g_0 \circ f_0 = \epsilon_Q \circ f_0 = \epsilon_P,$$

so $g_0 \circ f_0 - 1_{P_0}$ maps P_0 onto $\ker(\epsilon_P) = \mathrm{im}(d_P)$. Projectivity of P_0 now gives the existence of s_0. Next we have

$$d_P \circ (g_1 \circ f_1 - s_0 \circ d_P) = d_P \circ g_1 \circ f_1 - g_0 \circ f_0 \circ d_P + d_P = d_P.$$

Therefore $g_1 \circ f_1 - 1_{P_1} - s_0 \circ d_P$ maps P_1 onto $\ker(d_P) = \mathrm{im}(d_P)$. Projectivity of P_1 now gives s_1, and we proceed analogously to define the higher s_n's. The construction of the maps t_n is done in the same way using projectivity of Q_n. □

Let us now return to Hochschild homology $HH_\bullet(A)$. We have already remarked that an A-bimodule is simply a left A^e-module. We see from (30) that

$$H_0(A, M) \cong M/[A, M] \cong A \otimes_{A^e} M.$$

This suggests to look at $\mathrm{Tor}_n^{A^e}(M,A)$. For this we need a projective resolution of A in the category of A-bimodules. Consider now $B_n(A) := A^{\otimes(n+2)}$. These fit into a complex

$$\ldots \xrightarrow{b'} A^{\otimes(n+2)} \xrightarrow{b'} A^{\otimes(n+1)} \xrightarrow{b'} \ldots \xrightarrow{b'} A\otimes A \xrightarrow{m} A, \tag{32}$$

where m is the multiplication and

$$b'(a_0\otimes\ldots\otimes a_{n+1}) = \sum_{i=0}^{n}(-1)^i a_0\otimes\ldots\otimes a_i a_{i+1}\otimes\ldots\otimes a_{n+1}.$$

Indeed these are maps of A-bimodules if we fix the bimodule structure to be given by multiplication from the left on the first copy of A and from the right on the last copy of A. With these maps, this complex forms a resolution, called the *Bar-resolution*:

THEOREM 2.2. *When A is unital, the Bar-resolution is a projective resolution of A in the category of A-bimodules and we have*

$$HH_\bullet(A)\cong \mathrm{Tor}_\bullet^{A^e}(A,A).$$

PROOF. To check that (32) is a complex is straightforward. One easily checks that the map

$$s(a_0\otimes\ldots\otimes a_n) = 1\otimes a_0\otimes\ldots\otimes a_n$$

satisfies $b's + sb' = id$, so provides a contracting homotopy and shows that the complex is exact. Each object in the complex can be written as $A^e\otimes V_n$, with V_n some vector space, so it is a free A^e-module, hence projective: We have shown that we can use this complex to compute the Tor-groups.

For this, we tensor (32) over A^e with A. Using the isomorphism

$$B_n(A)\otimes_{A^e} A\cong A^{\otimes(n+1)},\quad (a_0\otimes\ldots\otimes a_{n+1})\otimes a\mapsto a_{n+1}aa_0\otimes a_1\otimes\ldots\otimes a_n,$$

we find exactly the Hochschild complex (27). □

3. Cyclic homology

We still consider a unital algebra A over $\mathbb{K}$, and take as our module $M = A$: in that case the space of Hochschild chains $C_k(A) = A^{\otimes(k+1)}$ has a "cyclic symmetry" by permuting the factors in the tensor products. This observation lies at the heart of the definition of cyclic homology as we now explain.

3.1. The B-operator. On the space of Hochschild chains $C_k(A) = A^{\otimes(k+1)}$ (with values in $M = A$), there exists a degree *in*creasing operator $B: C_k(A)\to C_{k+1}(A)$ defined by

$$\begin{aligned} B(a_0\otimes\ldots\otimes a_k) = \sum_{i=0}^{k}(-1)^{ki}\big(&1\otimes a_i\otimes\ldots\otimes a_k\otimes a_0\otimes\ldots\otimes a_{i-1}\\ &- a_i\otimes 1\otimes a_{i+1}\otimes\ldots a_k\otimes a_0\otimes\ldots\otimes a_{i-1}\big). \end{aligned}\tag{33}$$

LEMMA 3.1. *The operator B has the following properties:*

(i) $$B^2 = 0$$

(ii) $$Bb + bB = 0.$$

PROOF. Direct computation. □

DEFINITION 3.2. A *mixed complex* (M, b, B) is a $\mathbb{Z}$-graded vector space $M = \bigoplus_{n \geq 0} M_n$, equipped with two operators $b : M_n \to M_{n-1}$, $B : M_n \to M_{n+1}$ satisfying

$$b^2 = B^2 = bB + Bb = 0.$$

We therefore see that $(C_\bullet(A), b, B)$ is a mixed complex.

3.2. Connes' double complex. We can construct a double complex $\mathcal{B}$ out of a mixed complex (M, b, B) as follows: we define $\mathcal{B}_{p,q} := M_{q-p}$ for $p \geq 0$. With this strange re-indexing, we can get the b and B-operator to be the vertical and horizontal chain operators as follows:

(34)
$$\begin{array}{ccccccc}
\cdots & & \cdots & & \cdots & & \cdots \\
\downarrow{\scriptstyle b} & & \downarrow{\scriptstyle b} & & \downarrow{\scriptstyle b} & & \downarrow{\scriptstyle b} \\
M_3 & \xleftarrow[B]{} & M_2 & \xleftarrow[B]{} & M_1 & \xleftarrow[B]{} & M_0 \\
\downarrow{\scriptstyle b} & & \downarrow{\scriptstyle b} & & \downarrow{\scriptstyle b} & & \\
M_2 & \xleftarrow[B]{} & M_1 & \xleftarrow[B]{} & M_0 & & \\
\downarrow{\scriptstyle b} & & \downarrow{\scriptstyle b} & & & & \\
M_1 & \xleftarrow[B]{} & M_0 & & & & \\
\downarrow{\scriptstyle b} & & & & & & \\
M_0 & & & & & &
\end{array}$$

As with any double complex, we can define the *total complex*

$$\mathrm{Tot}_\bullet(\mathcal{B}) := \bigoplus_{p+q=\bullet} \mathcal{B}_{p,q},$$

equipped with the differential $b + B$.

DEFINITION 3.3. Let (M, b, B) be a mixed complex.

i) The *Hochschild homology* $HH_\bullet(M)$ is the homology of the complex $(M_\bullet, b)$
ii) The *cyclic homology* $HC_\bullet(M)$ is the homology of the complex $(\mathrm{Tot}_\bullet(\mathcal{B}), b + B)$

Applying this to the mixed complex $(C_\bullet(A), b, B)$ defines the *cyclic homology of A*.

3.3. The SBI-sequence. Hochschild and cyclic homology of a mixed complex are related by the so-called SBI-sequence. To derive this, first recall the shift-functor: if $(C_\bullet, \partial)$ is a chain complex, its *p-shift* $C[p]$ is defined as $C[p]_k := C_{p+k}$ equipped with the same differential. With this notation, there is a a short exact sequence of complexes as follows:

$$0 \longrightarrow (M, b) \longrightarrow \mathrm{Tot}(\mathcal{B}) \xrightarrow{S} \mathrm{Tot}(\mathcal{B})[-2] \to 0.$$

Here, the second map is the inclusion of the Hochschild chain complex as the first column in (34), and the third map is the projection onto the whole double complex, minus the first column. Since this part of the double complex has exactly the same shape, we can write this as $\mathrm{Tot}(\mathcal{B})[-2]$. Applying the long exact sequence in homology associated to this short exact sequence, we obtain:

THEOREM 3.4. *Let (M, b, B) be a mixed complex. Its Hochschild and cyclic homology are related by the following exact sequence:*

$$\ldots \longrightarrow HC_{k-1}(M) \xrightarrow{B} HH_k(M) \xrightarrow{I} HC_k(M) \xrightarrow{S} HC_{k-2}(M) \longrightarrow HH_{k-1}(M) \longrightarrow \ldots$$

3.4. A lemma on mixed complexes. We shall now prove a Lemma that is very useful in practical computations. Recall that a morphism of chain complex is said to be a *quasi-isomorphism* if it induces an isomorphism on homology groups. A morphism $\varphi : (M, b, B) \to (N, b, B)$ of mixed complexes is a map of $\mathbb{Z}$-graded vector spaces which commutes with the two operators b and B. In particular, it is a morphism of Hochschild chain complexes.

LEMMA 3.5. *Let $\varphi : (M, b, B) \to (N, b, B)$ be a morphism of mixed complexes. When φ is a quasi-isomorphism for the Hochschild chain complexes, it induces an isomorphism on the level of cyclic homology.*

PROOF. Since the map φ commutes with b and B, it induces a commutative diagram relating the two SBI-sequences:

$$\begin{array}{ccccccccccc} \ldots & \longrightarrow & HH_k(M) & \xrightarrow{I} & HC_k(M) & \xrightarrow{S} & HC_{k-2}(M) & \xrightarrow{B} & HH_{k-1}(M) & \longrightarrow & \ldots \\ & & \downarrow \varphi & & \downarrow \varphi & & \downarrow \varphi & & \downarrow \varphi & & \\ \ldots & \longrightarrow & HH_k(N) & \xrightarrow{I} & HC_k(N) & \xrightarrow{S} & HC_{k-2}(N) & \xrightarrow{B} & HH_{k-1}(N) & \longrightarrow & \ldots \end{array}$$

In low degrees, the SBI-sequence looks like

$$\ldots \to HC_1 \to HH_2 \to HC_2 \to HC_0 \to HH_1 \to HC_1 \to 0 \to HH_0 \to HC_0 \to 0.$$

Therefore $HH_0 = HC_0$ so φ induces an isomorphism on the level of HC_0. By the five-Lemma, we conclude that is gives an isomorphism for HC_1 as well. Proceeding inductively, we now get the result. □

3.5. Periodic cyclic homology. There is a variant of cyclic homology that is usually better behaved. It arises by stabilizing the bicomplex $\mathcal{B}_{\bullet,\bullet}$ with respect to the periodicity operator S:

$$\mathcal{B}^{per}_{\bullet,\bullet} = \lim_{\substack{\leftarrow \\ S}} \mathcal{B}_{\bullet,\bullet}$$

In a diagram, the complex looks like:

$$\begin{array}{ccccccccc}
 & & & & & & \dots & & \dots \\
 & & & & & & \downarrow & & \downarrow \\
 & & & & \dots & \longleftarrow & M_1 & \longleftarrow & M_0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & \dots & \longleftarrow & M_1 & \longleftarrow & M_0 & & \\
 & & & & \downarrow & & & & \\
\dots & \longleftarrow & M_1 & \longleftarrow & M_0 & & & & \\
 & & \downarrow & & & & & & \\
\dots & \longleftarrow & M_0 & & & & & &
\end{array}$$

The cohomology of the associated total complex $\mathrm{Tot}(\mathcal{B}^{per})$ is called the *periodic cyclic homology*, written $HC^{per}_\bullet(M)$. Because now the double complex is stable under the shift, we have an isomorphism of complexes:

$$S : \mathrm{Tot}(\mathcal{B}^{per}) \to \mathrm{Tot}(\mathcal{B}^{per})[2].$$

Therefore we have $HC^{per}_k(M) \cong HC^{per}_{k+2}(M)$ for all k, i.e., $HC^{per}_\bullet(M)$ is $\mathbb{Z}/2$ periodic!

3.6. A strategy for computations. In practice, for a given algebra, one actually wants to compute its cyclic theory. A fundamental problem with the definitions of Hochschild and cyclic homology given here is that they use chain complexes that are "very big", making such computations very difficult even for very small algebras. Lemma 3.5, together with the discussion in §2, suggests the following more practical strategy:

- Find another projective resolution $(P_\bullet, d_P)$ of your algebra A (besides the bar-resolution) in the category of A-bimodules. As explained in §2, such a resolution can be used to compute the Hochschild homology by taking $P_\bullet \otimes_{A^e} A$. The general rule for the choice of such a resolution is: the smaller the better!
- We know by the general theory that $(P_\bullet, d_P)$ and the bar-resolution are equivalent, but now we want to construct explicit chain homotopies inducing this equivalence. With these maps, we can try to construct the analogue of the B-operator.
- With the new differential we form the associated mixed complex, and claim this computes the cyclic homology: this follows from Lemma 3.5.

In the next section, we will apply this strategy to compute the cyclic homology of the algebra of smooth functions on a manifold.

4. The Hochschild–Kostant–Rosenberg theorem

The Hochschild–Kostant–Rosenberg theorem computes the Hochschild homology of a smooth commutative algebra A in terms of its algebraic differential forms $\Omega_A^\bullet$. This computation can be extended to include cyclic homology (this theory was not yet developed at the time of the HKR-theorem), and there are many variants of this. Here we are eventually interested in the case that $A = C^\infty(M)$ and we bring topology into play. We shall therefore first briefly recall the algebraic HKR-theorem, after which we prove its smooth version for $A = C^\infty(M)$. (This was first done by Connes in **[Co82]**.)

4.1. HKR: the algebraic version. Let A be a unital algebra over $\mathbb{K}$. We have already encountered the Kähler differentials Ω_A^1 of A in Remark 1.5. We now consider the exterior algebra of Ω_A^1:

$$\Omega_A^\bullet := \bigwedge_A^\bullet \Omega_A^1 = A \oplus \Omega_A^1 \oplus \Omega_A^2 \oplus \dots$$

This a graded-commutative A-algebra, freely generated by Ω_A^1. Elements in Ω_A^k look like

$$a_0 da_1 \wedge \dots da_k, \quad \text{for } a_0, \dots, a_k \in A.$$

There is a "de Rham differential" $d : \Omega_A^k \to \Omega_A^{k+1}$ defined as

$$d(a_0 da_1 \wedge \dots da_k) = da_0 \wedge \dots \wedge da_k.$$

It obviously squares to zero.

As before, let $C_\bullet(A)$ be the Hochschild chain complex of A and consider the map $\Psi : C_\bullet(A) \to \Omega_A^\bullet$ defined by

$$\Psi(a_0 \otimes \dots \otimes a_k) := \frac{1}{k!} a_0 da_1 \wedge \dots \wedge da_k. \tag{35}$$

LEMMA 4.1. *Ψ defines a morphism of mixed complexes:*

$$\Psi : (C_\bullet(A), b, B) \to (\Omega^\bullet, 0, d).$$

PROOF. Simply check by explicit computation that

$$\Psi \circ b = 0$$
$$\Psi \circ B = d \circ \Psi.$$

We omit the details. □

THEOREM 4.2 (Hochschild–Kostant–Rosenberg). *When A is a smooth*[1]*, commutative, Noetherian algebra, the map Ψ induces a isomorphism on the level of Hochschild homology:*

$$HH_\bullet(A) \cong \Omega^\bullet_A.$$

We will only prove this theorem for the algebra of polynomials on a finite dimensional vector space $V \cong \mathbb{C}^n$. This algebra can be identified as $S(V^*)$, the symmetric tensor algebra. The idea is to use a different resolution, not just the Bar-resolution. For this we choose the so-called *Koszul resolution* $K_\bullet(V)$:

$$\begin{aligned} 0 \to \bigwedge^n V \otimes S(V^*) \otimes S(V^*) \xrightarrow{\partial} \bigwedge^{n-1} V^* \otimes S(V^*) \otimes S(V^*) \xrightarrow{\partial} \dots \\ \dots \xrightarrow{\partial} V \otimes S(V^*) \otimes S(V^*) \xrightarrow{\partial} S(V^*) \otimes S(V^*) \xrightarrow{m} 0. \end{aligned} \tag{36}$$

with

$$\partial(a_0 \otimes a_1 \otimes dy_{i_1} \wedge \dots \wedge dy_{i_k}) = \sum_{j=1}^{k} (-1)^j (y_{i_j} a_0 \otimes a_1 - a_0 \otimes y_{i_j} a_1) \otimes dy_{i_1} \wedge \dots \wedge \widehat{dy_{i_j}} \wedge \dots \wedge dy_{i_k}$$

We can view the Koszul resolution as a graded commutative differential algebra by means of the product $(a_0 \otimes a_1 \otimes \omega) \cdot (a_0' \otimes a_1' \otimes \omega') := a_0 a_0' \otimes a_1 a_1' \otimes \omega \wedge \omega'$. One can easily prove that $K_\bullet(V) \otimes K_\bullet(W) \cong K_\bullet(V \oplus W)$, for two vector spaces V and W. For a one-dimensional vector space L, it is easy to see that $K_\bullet(L)$ is a resolution, i.e., the sequence

$$0 \longrightarrow S(L^*) \otimes S(L^*) \otimes L^* \xrightarrow{\partial} S(L^*) \otimes S(L^*) \xrightarrow{m} S(L^*) \longrightarrow 0$$

is exact. With this, one proves that $K_\bullet(V)$ is a resolution of $S(V^*)$ for any vector space V.

With the Koszul resolution one easily computes the Hochschild homology: take the tensor product $- \otimes_{S(V^*)^e} S(V^*)$ to get the chain complex

$$\dots \xrightarrow{0} \bigwedge^n V^* \otimes S(V^*) \xrightarrow{0} \bigwedge^{n-1} V^* \otimes S(V^*) \dots \xrightarrow{0} S(V^*) \longrightarrow 0.$$

This proves Theorem 4.2 for the algebra $S(V^*)$.

4.2. A remark about topologies and tensor products. We now want to compute the cyclic homology of the commutative algebra $C^\infty(M)$, where M is a compact smooth manifold. As remarked before, it would be naive to use the algebraic tensor product, since then we would not have the desirable property that

$$C^\infty(M) \otimes C^\infty(M) \cong C^\infty(M \times M). \tag{37}$$

[1]This means that for any $\mathbb{K}$-algebra C and a square zero ideal $I \subset C$, the canonical map $\mathrm{Hom}(A, C) \to \mathrm{Hom}(A, C/I)$ is surjective. Coordinate rings of nonsingular varieties are examples of smooth algebras.

However, $C^\infty(M)$ is more than just a commutative algebra: it also has a topology in which $f_n \to f$ if

$$\lim_{n\to\infty} \sup_{x\in M} |Df_n(x) - Df(x)| = 0,$$

for all differential operators D on M. With this topology, $C^\infty(M)$ is a so-called *Fréchet algebra*. There is a way to complete the tensor product which has the property (37). Here, we will simply *take* (37) *as our definition of tensor product*.

4.3. A Chain map. With the definition of the tensor product as above, the space of Hochschild chains is simply given by

$$C_k(C^\infty(M)) = C^\infty(M^{\times(k+1)}),$$

with boundaries

$$\begin{aligned} bf(x_0,\ldots,x_{k-1}) &= \sum_{i=0}^{k-1}(-1)^i f(x_0,\ldots,x_i,x_i,\ldots,x_{k-1}) \\ &\qquad\qquad + (-1)^{k-1} f(x_0,x_0,\ldots,x_{k-1}), \\ Bf(x_0,\ldots,x_{k+1}) &= \sum_{i=0}^{k}(-1)^{ki} f(x_{k-i+1},\ldots,x_{k+1},x_1,\ldots,x_{k-i}). \end{aligned} \tag{38}$$

We want to define a chain map analogous to (35) in the algebraic case. For this we fix the following notation: for a vector field $X \in \mathfrak{X}(M)$ on M, we write L^i_X for the operator taking the Lie derivative of a function $f \in C^\infty(M^{\times(k+1)})$ in the i-th variable. With this, the map $\Psi^{an}_M : C^\infty(M^{\times(k+1)}) \to \Omega^k(M)$ is defined by

$$\Psi^{an}_M(f)(X_1,\ldots,X_k) = \frac{1}{k!}\sum_{\sigma\in S_k}(-1)^\sigma L^1_{X_{\sigma(1)}}\cdots L^k_{X_{\sigma(k)}} f \tag{39}$$

LEMMA 4.3. *The map* (39) *defines a morphism of mixed complexes:*

$$\Psi^{an}_M : (C_\bullet(C^\infty(M)), b, B) \to (\Omega^\bullet(M), 0, d_{dR})$$

The theorem that we want to prove is the following:

THEOREM 4.4. *The map Ψ^{an}_M induces an isomorphism on the level of Hochschild and cyclic homology:*

$$\begin{aligned} HH_k(C^\infty(M)) &\cong \Omega^k(M) \\ HC_k(C^\infty(M)) &\cong \Omega^k(M)/d\Omega^{k-1}(M) \oplus H^{k-2}_{dR}(M) \oplus H^{k-4}_{dR}(M) \oplus \ldots \end{aligned}$$

4.4. Connes' resolution. The strategy to prove the Theorem above was outlined in §3.6: we shall first find a projective resolution that is smaller than the bar-resolution. It is the analogue, in the smooth context of differentiable manifolds, of the Koszul resolution (36). Let us first start by recalling the following:

THEOREM 4.5 (Hopf 1925). *A compact manifold M admits a global nonvanishing vector field if and only if its Euler characteristic $\chi(M)$ vanishes.*

Also, remark that the Euler characteristic is multiplicative, i.e.,

$$\chi(M \times N) = \chi(M)\chi(N),$$

so we can always get zero Euler characteristic by crossing with a circle: $\chi(M \times S^1) = 0$.

EXERCISE 4.6. Let $\xi \in \mathfrak{X}(M)$ be a nonzero vector field. Show that the operation ι_ξ of contracting with ξ is a differential on $\Omega^\bullet(M)$, the space of differential forms. Prove that the homology of this complex is trivial by showing that the following map is a contracting chain homotopy:

$$h_\alpha : \Omega^k(M) \to \Omega^{k+1}(M), \quad h_\alpha \beta = \alpha \wedge \beta,$$

where $\alpha \in \Omega^1(M)$ is the one-form dual to ξ: $\langle \xi, \alpha \rangle = 1$.

We shall now work on the manifold $M \times M$ and denote by $\Delta : M \hookrightarrow M \times M$ the inclusion of the diagonal. With this, the commutative multiplication $C^\infty(M) \otimes C^\infty(M) \to C^\infty(M)$ extends to the completion $C^\infty(M) \hat{\otimes} C^\infty(M) \cong C^\infty(M \times M)$ and we can identify this map as the pull-back $\Delta^* : C^\infty(M \times M) \to C^\infty(M)$ along Δ. We define the vector space

$$\mathcal{E}_k := \Gamma^\infty \left(M \times M; \mathrm{pr}_2^* \bigwedge^k T^*_{\mathbb{C}} M \right),$$

where $\mathrm{pr}_2 : M \times M \to M$ is the projection onto the second coordinate. Connes' resolution looks as follows:

$$0 \longrightarrow \mathcal{E}_n \xrightarrow{\iota_\xi} \mathcal{E}_{n-1} \xrightarrow{\iota_\xi} \ldots \xrightarrow{\iota_\xi} C^\infty(M \times M) \xrightarrow{\Delta^*} C^\infty(M) \longrightarrow 0. \tag{40}$$

Here $\xi \in \Gamma^\infty(M \times M; \mathrm{pr}_2^* T_{\mathbb{C}} M)$ is a complex "vector field" satisfying:

i) in a neighborhood of the diagonal we have

$$\xi(x, y) = \exp_y^{-1} x \in T_y M, \tag{41}$$

where $\exp : T_y M \to M$ is the exponential map with respect to some riemannian metric,

ii) ξ is nonvanishing outside the diagonal Δ.

Remark that the first condition implies that $\xi(x, x) = 0$, for all $x \in M$.

LEMMA 4.7. *When $\chi(M) = 0$, such a ξ satisfying i) and ii) exists.*

PROOF. Let $U \subset M \times M$ be a tubular neighborhood of the diagonal Δ over which the exponential map

$$TM \to M \times M, \quad (x, V_x) \mapsto (x, \exp_x(V_x))$$

is a local diffeomorphism. On U we can define $\xi^r \in \Gamma^\infty(M; \mathrm{pr}_2^* T_{\mathbb{C}} M)$ by the equation (41). On the other hand, we can choose a nonvanishing vector field ξ^i on M, because $\chi(M) = 0$, cf. Theorem 4.5. Finally, let $\psi \in C_c^\infty(U)$ be a "bump function" for the diagonal Δ, i.e., $\psi \equiv 1$ in a neighborhood of Δ and $\psi \equiv 0$ outside a compact subset containing Δ. With these choices we can define

$$\xi(x,y) := \psi(x,y)\xi^r(x,y) + \sqrt{-1}(1-\psi(x,y))\xi^i(y).$$

By construction, this is a section of $\mathrm{pr}_2^* T_{\mathbb{C}} M$ satisfying *i*) and *ii*) above. □

With this choice of ξ we have the crucial

PROPOSITION 4.8 (Connes). *The complex* (40) *is a projective resolution of* $C^\infty(M)$ *in the category of* $C^\infty(M)$ *bimodules.*

PROOF. Clearly, the map ι_ξ satisfies $\iota_\xi^2 = 0$, and also $\Delta^* \circ \iota_\xi = 0$ since ξ vanishes on the diagonal Δ, and therefore (40) is a chain complex. Since the $\mathcal{E}_k$ consist of sections of a vector bundle over $M \times M$, the Serre–Swan theorem 2.3 implies that they are finitely generated projective modules over $C^\infty(M \times M)$. It remains to show that the homology of the complex (40) vanishes in positive degrees.

For this we define $s_k : \mathcal{E}_k \to \mathcal{E}_{k+1}$ by the formula

$$s_k(\omega) := \int_0^1 \varphi_t^* d_2(\psi\omega)\frac{dt}{t} + (1-\psi)\eta \wedge \omega,$$

where:

- $\varphi_t : U \to U$ is defined by $\varphi_t(x,y) = \exp_x(t\xi(y,x))$,
- d_2 is the exterior ("de Rham") differential with respect to the second coordinate.
- $\eta \in \mathcal{E}_1$ is a "one-form" dual to ξ on $\mathrm{supp}(1-\psi)$ (where ξ is nonvanishing): $\langle \eta, \xi \rangle = 1$.

Let us already remark that s_k is $C^\infty(M)$-linear in the first variable, so we can do a local computation by freezing the first variable $x \in M$. In U we can use *normal coordinates* around x and we we have

$$\varphi_t(x,y) = ty, \quad \xi(x,y) = -y, \tag{42}$$

this follows from (41). Suppose now that ω vanishes outside $\mathrm{supp}(\psi)$ and that $\omega(x,x) = 0$. Then we see that

$$\int_0^1 \varphi_t^* d_2(\iota_\xi \omega)\frac{dt}{t} + \iota_\xi \int_0^1 \varphi_t^* d_2(\omega)\frac{dt}{t} = \int_0^1 \varphi_t^*(L_\xi \omega)\frac{dt}{t} = \omega$$

The last equality follows, using normal coordinates, from the following small computation: suppose that $\alpha \in \Omega^k(\mathbb{R}^n)$ is a form with $\alpha(0) = 0$. Then, with $\rho_t(x) = tx$ and $R = \sum_i x^i \partial/\partial x^i$ the Euler vector field, we have

$$\alpha = \int_0^1 \frac{d}{dt}(\rho_t^* \alpha)dt = \int_0^1 t\frac{d}{dt}(\rho_t^*\alpha)\frac{dt}{t} = \int_0^1 \rho_t^*(L_R \alpha)\frac{dt}{t}.$$

We see from (42) that indeed in normal coordinates, φ_t identifies with ρ_t and ξ with R.

To complete the proof, we compute, using Exercise 4.6, for general $\omega \in \mathcal{E}_k$:

$$\begin{aligned}(s_{k-1}\iota_\xi + \iota_\xi s_k)\omega = &\int_0^1 \varphi_t^* d_2(\psi\iota_\xi\omega)\frac{dt}{t} + \iota_\xi \int_0^1 \varphi_t^* d_2(\psi\omega)\frac{dt}{t} \\ &+ (1-\psi)\eta \wedge \iota_\xi\omega + (1-\psi)\iota_\xi(\eta\wedge\omega) \\ =&\psi\omega + (1-\psi)(\iota_\xi\eta)\omega \\ =&\omega.\end{aligned}$$

This shows that the s_k define a contracting chain homotopy, and therefore the chain complex (40) is a resolution. This completes the proof of the Proposition. □

Following our general strategy for cyclic homology computations, we now use this resolution instead of the Bar-resolution to compute the Hochschild and cyclic homology of $C^\infty(M)$. But before we do that, we will compare Connes' resolution with the bar-resolution with

$$B_k(M) := C^\infty(M^{\times(k+2)})$$

differential

$$bf(x_0,\ldots,x_k) = \sum_{i=0}^{k-1}(-1)^i f(x_0,\ldots,x_i,x_i,\ldots,x_k).$$

There are maps

$$\mathcal{E}_k \underset{p_k}{\overset{i_k}{\rightleftarrows}} B_k(M)\ ,$$

with

$$i_k\omega(x,y,x_1,\ldots,x_k) := \sum_{\sigma\in S_k}(-1)^\sigma \omega(x,y)\left(\xi(x_{\sigma(1)},y),\ldots,\xi_{\sigma(k)},y)\right)$$

$$p_k(f)(x,y)(Y_1,\ldots,Y_k) := \sum_{\sigma\in S_k}(-1)^\sigma\, L_{Y_{\sigma(1)}}\cdots L_{Y_{\sigma(k)}} f\Big|_{x,y=x_1=\ldots=x_k}$$

LEMMA 4.9. *i and p are chain maps and satisfy $p\circ i = id$.*

PROOF. Omitted. □

Now we use Connes' resolution to compute the Hochschild homology. For this we take $\ldots\otimes_{C^\infty(M\times M)} C^\infty(M)$ of the resolution. To do this, we use the following

LEMMA 4.10. *Let $f : M \to N$ be a continuous map, and $E \to N$ a vector bundle over N. Then we have an isomorphism*

$$\Gamma^\infty(M; f^*E) \cong C^\infty(M)\otimes_{C^\infty(N)} \Gamma^\infty(N;E).$$

PROOF. Recall that sections of f^*E are given by maps $s : M \to E$ satisfying $s(x) \in E_{f(x)}$, for all $x \in M$. Given $g \in C^\infty(M)$ and $t \in \Gamma^\infty(N;E)$ we can define a section of f^*E by $x \mapsto g(x)t(f(x))$. This construction defines a map $C^\infty(M)\otimes\Gamma^\infty(N;E) \to \Gamma^\infty(M;f^*E)$ which clearly factors over the quotient $C^\infty(M)\otimes_{C^\infty(N)}\Gamma^\infty(N;E)$. □

We use this Lemma in the following way:

$$\mathcal{E}_k \otimes_{C^\infty(M\times M)} C^\infty(M) \cong \Gamma^\infty(M; \Delta^*\mathrm{pr}_2^* \bigwedge^k T^*_{\mathbb{C}}M) \cong \Omega^k(M). \tag{43}$$

The isomorphism is induced by the map $\omega \otimes f \mapsto f\omega|_{\Delta(M)}$. Because we are restricting to the diagonal in this map, the induced differential is zero. Furthermore, remark that the map induced by p_k is exactly the morphism (39), and therefore we have proved the Hochschild part of Theorem 4.4.

For cyclic homology, the crucial issue is to compute the B-operator under this isomorphism. For this we look at the map

$$\Omega^k(M) \xrightarrow{i'_k} C_k(C^\infty(M)) \xrightarrow{B} C_{k+1}(C^\infty(M)) \xrightarrow{p'_k} \Omega^{k+1}(M),$$

where the prime denotes the induced map under the isomorphism (43).

LEMMA 4.11. $i'_k \circ B \circ p'_{k+1} = d_{dR}$.

We can now complete our computation of cyclic homology: we have a morphism of mixed complexes Ψ^{an}_M, as in Lemma 4.3. By Proposition 4.8 and the computation above, we know that this map induces an isomorphism on Hochschild homology, and therefore, by Lemma 3.5 also on cyclic homology. With this argument, the proof of Theorem 4.4 is complete.

5. Nonunital algebras and excision

In this last section of the chapter, we discuss the extension to non-unital algebras. As it turns out, this is best done in an abstract framework, as we shall now discuss.

5.1. Cyclic objects. Recall that the *simplicial category* is the small category Δ with objects given by the sets

$$[n] := \{0 < 1 < \ldots < n\}, \quad n = 0, 1, 2, \ldots$$

A morphism $f : [n] \to [m]$ is an order preserving map. The morphisms in Δ are generated by the *faces* $\delta_i : [n-1] \to [n]$, $i = 0, \ldots n$ and the *degeneracies* $\sigma_j : [n+1] \to [n]$, $j = 0, \ldots, n$. By definition δ_i is the morphism that misses i, and σ_j maps j and $j+1$ to j. One easily verifies that these maps satisfy the following *simplicial identities*:

$$\begin{aligned}
\delta_j\delta_i &= \delta_i\delta_{j-1}, \quad i < j \\
\sigma_j\sigma_i &= \sigma_i\sigma_{j+1}, \quad i \le j \\
\sigma_j\delta_i &= \begin{cases} \delta_i\sigma_{j-1} & i < j \\ id_{[n]} & i = j \text{ or } i = j+1, \\ \delta_{i-1}\sigma_j & i > j+1. \end{cases}
\end{aligned} \tag{44}$$

DEFINITION 5.1. Let $\mathscr{C}$ be a category. A *simplicial object* in $\mathscr{C}$ is a functor $\Delta^{op} \to \mathscr{C}$. A covariant functor is called a cosimplicial object.

By the description of Δ above, this is equivalent to giving objects $X_n \in \mathscr{C}$, $n = 0, 1, \ldots$, together with maps $\delta_i : X_{n-1} \to X_n$ and $\sigma_j : X_{n+1} \to X_n$ satisfying the identities (44) above.

EXERCISE 5.2. Show that the usual simplices

$$\Delta^n := \{(t_0, \ldots, t_n) \in \mathbb{R}^{n+1},\ 0 \le t_i \le 1,\ t_0 + \ldots + t_n = 1\},$$

form a cosimplicial topological space. What are the face maps and the degeneracies?

The *cyclic category* Λ has the same objects as Δ, but the morphisms contain a new generator, besides the face maps δ_i and the degeneracies σ_j, the *cyclic generator* $\tau_n : [n] \to [n]$ given by

$$\tau_n(i) := \begin{cases} i+1, & 0 \le i < n \\ 0 & i = n. \end{cases}$$

So the cyclic category Λ contains the simplicial category Δ as a subcategory; Λ has more morphisms. Therefore, besides the simplicial identities, we now have the following extra relations:

$$\begin{aligned} \tau_n \delta_i &= \delta_{i-1} \tau_{n-1}, && 1 \le i \le n, \\ \tau_n \delta_0 &= \delta_n, && \\ \tau_n \sigma_i &= \sigma_{i-1} \tau_{n+1}, && 1 \le i \le n \\ \tau_n \sigma_0 &= \sigma_n \tau_{n+1}^2, && \\ \tau_n^{n+1} &= id. && \end{aligned} \tag{45}$$

DEFINITION 5.3. Let $\mathscr{C}$ be a category. A *cyclic object* is a functor $\Lambda \to \mathscr{C}$.

Concretely, this means that we have define a simplicial object $(X_\bullet, \delta_\bullet, \sigma_\bullet)$ together with a morphism $\tau_n : X_n \to X_n$ satisfying the extra identities above.

REMARK 5.4. Let $X_\bullet$ be a simplicial set. Its *geometric realization* is defined as

$$|X_\bullet| := \left(\coprod_{n \ge 0} X_n \times \Delta^n \right) \Big/ \sim,$$

where $(x, s) \in X_m \times \Delta^n$ is equivalent (written $\sim$) to $(y, t) \in X_n \times \Delta^n$ is there exists a morphism $f : [m] \to [n]$ in Δ such that $f^* y = x$ and $f_* s = t$. So we glue simplices, labeled with elements in $X_\bullet$ according to the simplicial maps in $X_\bullet$. The geometric realization is, almost by definition, a CW-complex, so it has an induced topology coming from $\Delta^\bullet$, so we obtain a functor

$$|\ | : \underline{Simplicial\ Sets} \to \underline{Top}.$$

The following theorem explains the extra structure present on a cyclic set in comparison to simplicial sets:

THEOREM 5.5. *Let $X_\bullet$ be a cyclic set. Then its geometric realization carries a canonical (in the homotopy sense) action of S^1.*

We will not prove this theorem, see [**L**, Ch. 7]

EXERCISE 5.6. Let A be a unital algebra over $\mathbb{K}$. Show that the $\mathbb{K}$-vector spaces $A_n^\# < \ n = 0, 1, \ldots$ carry the structure of a cyclic $\mathbb{K}$-module with structure maps

$$\begin{aligned}
\delta_i(a_0 \otimes \ldots \otimes a_n) &= a_0 \otimes \ldots \otimes a_i a_{i+1} \otimes \ldots \otimes a_n, \quad 0 \leq i < n \\
\delta_n(a_0 \otimes \ldots \otimes a_n) &= a_n a_0 \otimes \ldots \otimes a_{n-1} \\
\sigma_i(a_0 \otimes \ldots \otimes a_n) &= a_0 \otimes \ldots \otimes a_i \otimes 1 \otimes a_{i+1} \otimes \ldots \otimes a_n \\
\tau_n(a_0 \otimes \ldots \otimes a_n) &= a_n \otimes a_0 \otimes \ldots \otimes a_{n-1}.
\end{aligned}$$

5.2. ***H*-unitality.** Let $(X_\bullet, \delta_\bullet, \sigma_\bullet, \tau_\bullet)$ be a cyclic $\mathbb{K}$-module. We form the following double complex, called *Tsygan's double complex*:

$$\begin{array}{ccccccc}
\cdots & & \cdots & & \cdots & & \\
\downarrow{\scriptstyle b} & & \downarrow{\scriptstyle -b'} & & \downarrow{\scriptstyle b} & & \\
X_2 & \xleftarrow{1-\lambda} & X_2 & \xleftarrow{N} & X_2 & \xleftarrow{1-\lambda} & \cdots \\
\downarrow{\scriptstyle b} & & \downarrow{\scriptstyle -b'} & & \downarrow{\scriptstyle b} & & \\
X_1 & \xleftarrow{1-\lambda} & X_1 & \xleftarrow{N} & X_1 & \xleftarrow{1-\lambda} & \cdots \\
\downarrow{\scriptstyle b} & & \downarrow{\scriptstyle -b'} & & \downarrow{\scriptstyle b} & & \\
X_0 & \xleftarrow{1-\lambda} & X_0 & \xleftarrow{N} & X_0 & \xleftarrow{1-\lambda} & \cdots
\end{array}$$

where:

$$\begin{aligned}
b := \sum_{i=0}^{n} (-1)^i \delta_i, &\quad b' := \sum_{i=0}^{n-1} (-1)^i \delta_i, \\
\lambda := (-1)^n \tau_n, &\quad N := 1 + \lambda + \ldots + \lambda^n.
\end{aligned}$$

DEFINITION 5.7. The cyclic homology of a cyclic $\mathbb{K}$-module is the homology of the total complex above.

PROPOSITION 5.8. *For a unital algebra A over $\mathbb{K}$, this definition agrees, via the cyclic $\mathbb{K}$-module of exercise 5.6, with Definition 3.3.*

PROOF. Let us write $CC_{p,q}(A) = C_q(A) = A^{\otimes(q+1)}$ for the entries of Tsygan's double complex. We have already seen in the proof of Theorem 2.2 that for a unital algebra A, the b'-complex is contractible with homotopy

$$s(a_0 \otimes \ldots \otimes a_n) = 1 \otimes a_0 \otimes \ldots \otimes a_n$$

satisfying $b's + sb' = 1$. We therefore "kill" this complex as follows: consider the map

$$(1, sN) : B_{p,q}(A) \to CC_{2p,q-p}(A) \oplus CC_{2p-1,q-p+1}.$$

On the right hand side, the differential of the total complex decomposes as

$$\begin{pmatrix} b & 1-\lambda \\ N & -b' \end{pmatrix}$$

Together with the identity $B = (1-\lambda)sN$ (check!), one checks that the map above is a morphism of chain complexes:

$$\begin{aligned}(1, sN)(b+B)a &= (ba + Ba, sNba + sNBa) \\ &= (ba + (1-\lambda)sNa, -sb'N + sN(1-\lambda)sNa) \\ &= (ba + (1-\lambda)sNa, (-b's+1)N + 0) \\ &= (ba + (1-\lambda)sNa, -b'sNa + Na).\end{aligned}$$

We skip the final part of the proof showing that this map is a quasi-isomorphism. □

Notice that this Proposition in some sense "explains" the strange formula for the B-differential. The key point is now the observation that in the definition of Tsygan's double complex $CC_{\bullet,\bullet}(A)$, the unit is not needed, so we can use it as a definition of cyclic homology of non-unital algebras.

REMARK 5.9. The good thing about this definition of Hochschild and cyclic homology for nonunital algebras is that it is the homology of an explicit complex. There is a "naive" definition as follows: let A be a nonuintal algebra and denote by $\tilde{A}$ its unitization. The canonical morphism $\mathbb{K} \to \tilde{A}$ induces maps

$$HH_k(\mathbb{K}) \to HH_k(\tilde{A}), \quad HC_k(\mathbb{K}) \to HC_k(\tilde{A}).$$

We can define $HH_k(A)$ and $HC_k(A)$ to be the cockernel of these maps. It can be shown, c.f.[**L**], that these definitions agree with the previous ones.

DEFINITION 5.10. Let A be an algebra over $\mathbb{C}$. We say that A is *H-unital* if the complex $(C_\bullet(A), b')$ is acyclic.

We have already seen that a unital algebra is H-unital. More generally, an algebra A is said to have *local units* if for every finite number of elements $a_i \in A$, $i = 1, \ldots, n$, there exists an element $u \in A$ such that $ua_i = a_i u = a_i$ for all i.

PROPOSITION 5.11. *An algebra with local units is H-unital.*

5.3. Excision. We have already seen the six term periodic exact sequence is K-theory associated to an algebra extension. A natural question is if a similar statement holds true for Hochschild and/or cyclic homology. Let us therefore consider an extension

$$0 \to I \to A \to A/I \to 0.$$

We assume that A is unital, but of course I not. We say that the ideal $I \subset A$ satisfies *excision* in Hochschild homology if there is an associated long exact sequence

$$\ldots \to HH_{n+1}(A/I) \to HH_n(I) \to HH_n(A) \to HH_n(A/I) \to HH_{n-1}(I) \to \ldots$$

THEOREM 5.12 (Wodzicky). *Let A be a unital algebra and $I \subset A$ an ideal. Then the following are equivalent:*

i) *I is H-unital,*
ii) *I satisfies excision.*

THEOREM 5.13 (excision in cyclic homology). *Let A be a unital algebra and $I \subset A$ an H-unital ideal. Then there exists a long exacts sequence*

$$\ldots \to HC_{n+1}(A/I) \to HC_n(I) \to HC_n(A) \to HC_n(A/I) \to HC_{n-1}(I) \to \ldots$$

REMARK 5.14. Periodic cyclic homology is much better behaved! By a result of Cuntz and Quillen, c.f. [**CQ**], it satisfies excision for any ideal, i.e., without the assumption of H-unitality.

CHAPTER 4

The Chern Character

The Chern character is a natural map from K-theory to cyclic homology of a an algebra. Before we discuss the general case (due to Connes), let us consider the classical case of the Chern character from topological K-theory to ordinary cohomology. When the underlying topological space is in fact a manifold, this character is conveniently constructed in de Rham cohomology: this is known as Chern–Weil theory.

1. The commutative case: Chern–Weil theory

1.1. Characteristic classes. Let us first give the formal definition of a characteristic class of a vector bundle. First we recall the definition of a morphism of vector bundles: Let $E \to X$ and $F \to Y$ be vector bundles over topological spaces X and Y. A morphism from F to E is an continuous map $\varphi : F \to E$ on the total spaces which commutes with the projections. It therefore induces a map $\bar{\varphi} : Y \to X$. We can therefore decompose φ as a pair of map $(\tilde{\varphi}, \bar{\varphi})$ with $\tilde{\varphi} : F \to \bar{\varphi}^* E$ a morphism of vector bundles over Y, i.e., linear on the fibers. We shall assume that this map $\tilde{\varphi}$ is invertible.

DEFINITION 1.1. A *characteristic class* of degree q is a natural assignment of a cohomology class $\xi(E) \in H^q(X)$ to a vector bundle E over a topological space X. By natural we mean that for any morphism $\varphi : F \to E$ we have the equality

$$\xi(F) = \bar{\varphi}^* \xi(E) \in H^q(Y).$$

Let us already make the remark that if we are able to find a characteristic class that is additive; $\xi(E \oplus F) = \xi(E) + \xi(F)$ for any pair of vector bundles E and F, we get an induced map

$$\xi : K^0(X) \to H^\bullet(X).$$

Indeed, by definition a characteristic class is well defined on the set of isomorphism classes of vector bundles. If it is additive as well, it gives a morphism from the semigroup $Vect(X)$ to the abelian group $H^\bullet(X)$. By the universal property of the Grothendieck group, we get the morphism.

1.2. Connections and curvature. Let X be a smooth manifold, and $E \to X$ a smooth vector bundle. We denote by $\Omega^k(X; E)$ the space of differential k-forms on X with values in E.

DEFINITION 1.2. A *connection* on E is a linear map

$$\nabla : \Gamma^\infty(X;E) \to \Omega^1(X;E),$$

satisfying the Leibniz rule

$$\nabla(fs) = f\nabla(s) + df \otimes s,$$

with $f \in C^\infty(X)$ and $s \in \Gamma^\infty(X;E)$.

LEMMA 1.3. *The space of connections on a vector bundle E is an affine space modeled on* $\Omega^1(X, \mathrm{End}(E))$.

PROOF. Let ∇ and ∇' be two connections on E. It follows form the Leibniz rule that

$$(\nabla - \nabla')fs = f(\nabla - \nabla')s, \quad \text{for all } f \in C^\infty(X),\ s \in \Gamma(X;E).$$

The operator $\nabla - \nabla' : \Gamma^\infty(X;E) \to \Omega^1(X;E)$ is therefore $C^\infty(X)$-linear, and it follows that $\nabla - \nabla' \in \Omega^1(X;\mathrm{End}(E))$ □

REMARK 1.4.

- For a trivial vector bundle $E = X \times \mathbb{C}^r$ we always have the trivial connection given by the de Rham operator d extended to vector valued functions. By the Lemma above, any other connedtion can be written as $\nabla = d + \alpha$ with $\alpha \in \Omega^1(X, \mathrm{End}(E))$ a matrix-valued one-form.
- For a general vector bundle, we can write $\nabla = d + \alpha_i$ in a local trivialization over U_i. On the overlap $U_i \cap U_j$ of two local trivializations the two one forms α_i and α_j are related by (check!)

$$\alpha_i = \psi_{ij}\alpha_j\psi_{ij}^{-1} + (d\psi_{ij})\psi_{ij}^{-1},$$

  with $\psi_{ij} : U_{ij} \to GL(r, \mathbb{C})$ the transition function.
- It can be shown by a standard partition of unity argument that a connection always exist on a vector bundle.

REMARK 1.5. Connections behave well with respect to the standard constructions with vector bundles: Let E and F be vector bundles over X, with connections ∇_E, ∇_F.

i) On the direct sum, we have the obvious connection

$$\nabla_{E\oplus F} = \begin{pmatrix} \nabla_E & 0 \\ 0 & \nabla_F \end{pmatrix},$$

ii) On the tensor product we have the connection $\nabla_{E\otimes F} = \nabla_E \otimes 1 + 1 \otimes \nabla_E$,

iii) On the dual E^*, we have the connection defined by the following equation

$$d\langle \alpha, s\rangle = \langle \nabla_{E^*}(\alpha), s\rangle + \langle \alpha, \nabla_E(s)\rangle, \quad \alpha \in \Gamma^\infty(X;E^*),\ s \in \Gamma^\infty(X;E),$$

using the dual pairing $\langle\ ,\ \rangle : \Gamma^\infty(X;E^*) \times \Gamma^\infty(X;E) \to C^\infty(X)$ between sections of E and E^*

iv) As a special case of *iii*), we obtain a connection on $\mathrm{End}(E) = E \otimes E^*$, defined by

$$\nabla_{\mathrm{End}(E)}(A)(s) := \nabla_E(A(s)) - A(\nabla_E(s)) \quad \text{for } A \in \Gamma^\infty(X, \mathrm{End}(E),\ s \in \Gamma(X; E).$$

v) On the pull-back bundle f^*E for a smooth map $f : Y \to X$, there is a natural pull-back connection $f^*\nabla_E$.

Using the Leibniz identity, we can extend a connection to an operator

$$\nabla : \Omega^k(X, E) \to \Omega^{k+1}(X; E), \quad \nabla(s \otimes \alpha) = \nabla s \wedge \alpha + s \otimes d\alpha$$

The operator ∇ thus defined doesn't turn $\Omega^\bullet(X; E)$ into a complex: $\nabla^2 \neq 0$. However we do have

$$\nabla^2(fs) = f\nabla^2(s),$$

so its curvature $R(\nabla) := \nabla^2 \in \Omega^2(X; \mathrm{End}(E))$ is exactly the obstruction to ∇ being a cochain operator.

LEMMA 1.6 (Bianchi identity).

$$\nabla(R(\nabla)) = 0.$$

PROOF. Write out:

$$\begin{aligned}\nabla(R(\nabla))(s) &= \nabla(R(\nabla)(s)) - R(\nabla)(\nabla(s)) \\ &= \nabla^3(s) - \nabla^3(s) \\ &= 0.\end{aligned}$$

□

1.3. The Chern–Weil homomorphism. We assume E has rank r. Denote by $\mathfrak{gl}(r, \mathbb{C})$ the Lie algebra of $GL(r, \mathbb{C})$.

DEFINITION 1.7. An *invariant homogeneous polynomial* of degree k on $\mathfrak{gl}(r, \mathbb{C})$ is given by a symmetric map

$$P : \underbrace{\mathfrak{gl}(r, \mathbb{C}) \times \ldots \times \mathfrak{gl}(r, \mathbb{C})}_{k \text{ times}} \to \mathbb{C}$$

which is invariant under the action of $GL(r, \mathbb{C})$:

$$P(gA_1g^{-1}, \ldots, gA_kg^{-1}) = P(A_1, \ldots, A_k), \quad \text{for } A_i \in \mathfrak{gl}(r, \mathbb{C}) \text{ and } g \in GL(r, \mathbb{C}).$$

We denote the graded algebra of invariant polynomials by $I(GL(r, \mathbb{C}))$.

For $P \in I^k(GL(r, \mathbb{C})$, we pick a connection on our vector bundle and consider the differential form

$$P(R(\nabla), \ldots, R(\nabla)) \in \Omega^{2k}(X).$$

To make sense of this expression, choose, for a point $x \in X$ an isomorphism $E_x \cong \mathbb{C}^r$ of the fiber of E. This induces an isomorphism $\mathrm{End}(E)_x \cong \mathfrak{gl}(r, \mathbb{C})$, so that we can apply P

to $R(\nabla)$ at that point. Since P is invariant, its value is in fact independent of the chosen isomorphism.

PROPOSITION 1.8.

i) *The form* $P(R(\nabla),\dots,R(\nabla))$ *is closed.*
ii) *The induced cohomology class in* $H^{2k}_{dR}(X)$ *is independent of the chosen connection.*

PROOF. First remark that by invariance of P we have

$$\sum_{i=1}^{k} P(A_1,\dots,[A,A_i],\dots,A_k)=0,\quad A,A_1,\dots,A_k\in\mathfrak{gl}(r,\mathbb{C}).$$

This identity can be obtained by using invariance with respect to conjugation with $g=\exp(tA)$ and differentiation. Therefore, in a local trivialization where we write $\nabla=d+A$, we find

$$\begin{aligned}
dP(R(\nabla),\dots,R(\nabla)) &= \sum_{i=1}^{k} P(R(\nabla),\dots,dR(\nabla),\dots,R(\nabla))\\
&= \sum_{i=1}^{k} P(R(\nabla),\dots,(\nabla-A)R(\nabla),\dots,R(\nabla))\\
&= \sum_{i=1}^{k} P(R(\nabla),\dots,\nabla R(\nabla),\dots,R(\nabla))\\
&= 0
\end{aligned}$$

by the Bianchi identity. This proves the first claim.

For the second, let ∇' be another connection. By Lemma 1.3 we write $\nabla'=\nabla+\alpha$ for $\alpha\in\Omega^1(X;\mathrm{End}(E))$. A small computation shows that

$$R(\nabla')=R(\nabla)+\nabla\alpha+\alpha\wedge\alpha.$$

Consider the one parameter family $\nabla_t=\nabla+t\alpha$ interpolating between ∇ and ∇'. Expanding $P(R(\nabla_t),\dots,R(\nabla_t))$ around $t=0$ we find

$$P(R(\nabla_t),\dots,R(\nabla_t))=P(R(\nabla),\dots,R(\nabla))+ktP(R(\nabla),\dots,R(\nabla),\nabla\alpha))+h.o.t.$$

By the Bianchi identity we can write this as

$$dP(R(\nabla),\dots,R(\nabla),\alpha).$$

Therefore derivative in t of the induced map to de Rham cohomology is zero, and the claim follows. ☐

COROLLARY 1.9 (Chern–Weil homomorphism). *There is a canonical homomorphism of graded algebras*

$$I^\bullet(GL(r,\mathbb{C}))\to H^{2\bullet}_{dR}(X).$$

1.4. Chern character. Consider the invariant polynomials $P_k \in I^k(GL(r,\mathbb{C}))$ defined by

$$\mathrm{tr}(e^A) = P_0(A) + P_1(A) + \dots, \quad A \in \mathfrak{gl}(r,\mathbb{C}).$$

The polynomial P_k defines the *k-th Chern character*[1]

$$ch_k(E) := P_k(R(\nabla)) \in H^{2k}_{dR}(X).$$

The *total Chern character* is defined as

$$ch(E) := \sum_{k\geq 0} ch_k(E).$$

Now for two vector bundles E and F, connections ∇_E, ∇_F on E and F define an obvious connection on $E \oplus F$ whose curvature can be written as

$$\begin{pmatrix} R(\nabla_E) & 0 \\ 0 & R(\nabla_F) \end{pmatrix}.$$

From this it follows now that

$$ch(E \oplus F) = ch(E) + ch(F),$$

so the Chern character induces a natural map

$$ch : K^0(X) \to H^{ev}_{dR}(X).$$

2. The noncommutative case: The Chern character in cyclic homology

2.1. The Dennis trace map. Let A be a unital algebra, and consider the algebra $M_n(A)$ of $n \times n$ matrices with values in A. Consider the following map, called the *Dennis trace map* $\mathrm{tr}_* : C_\bullet(M_n(A)) \to C_\bullet(A)$ defined by

$$\mathrm{tr}_*(a_0 \otimes \ldots \otimes a_k) := \sum_{i_0,\ldots,i_k=1}^{n} (a_0)_{i_0 i_1} \otimes (a_k)_{i_1 i_2} \otimes \ldots \otimes a_{i_k i_0}. \tag{46}$$

If we write $a_i = u_i \otimes a'_i$ with $a'_i \in A$ and $u_i \in M_n(\mathbb{K})$ using the isomorphism $M_n(A) \cong M_n(\mathbb{K}) \otimes A$, then the map is simply given by

$$\mathrm{tr}_*(a_0 \otimes \ldots \otimes a_k) = tr(u_0 \cdots u_k) \otimes a'_0 \otimes \ldots \otimes a'_k,$$

where tr is now the trace of an $n \times n$-matrix. With this, the following Lemma is easily proved:

LEMMA 2.1. *tr_* is a morphism of mixed complexes.*

We therefore have a map

$$tr_* : HC_\bullet(M_n(A)) \to HC_\bullet(A).$$

[1] The standard convention is to define $ch_k(E) := P_k(\frac{\sqrt{-1}}{2\pi} R(\nabla))$. We do not do this here to smoothen the compatibility with the noncommutative Chern-character.

2.2. The reduced complex. For the precise definition of the Chern character, it turns out to be convenient to use a slightly different version of the cyclic bicomplex, called the *reduced cyclic complex*. This is defined by

$$\overline{C}_\bullet(A) := A \otimes (A/\mathbb{K})^{\otimes \bullet}.$$

We equip it with Hochschild and cyclic differentials b and B defined by the same formulas as for the unnormalized chains. (Warning: this is slightly nontrivial: $A/\mathbb{K}$ is not an algebra, but since the formula for the b-operator involves an alternating sum, this is not a problem.

PROPOSITION 2.2. *The projection map $C_\bullet(A) \to \overline{C}_\bullet(A)$ is a morphism of mixed complexes that induces isomorphisms on Hochschild and cyclic homology.*

2.3. The character from K_0. Connes' version of the Chern character in the noncommutative setting is a map

$$\mathrm{Ch}_{2k} : K_0(A) \to HC_{2k}(A)$$

for an algebra A. Let us first give the definition. Since A now need not be a C^*-algebra, we have to be a bit careful to define $K_0(A)$. Of course a safe way to define it is as the Grothendieck group of the monoid of isomorphism classes of finitely generated projective modules over A. Such a module is represented by an idempotent $e \in M_n(A)$, with $e^2 = e$. In the following, we write $[e]_p$ for the element

$$[e]_p := \underbrace{\overline{e} \otimes \ldots \otimes \overline{e}}_{p \text{ factors}} \in (M_n(A)/\mathbb{K})^{\otimes p}$$

With this notation we define

$$\mathrm{Ch}_{2k}(e) := \sum_{i=1}^{k} (-1)^i \frac{(2i)!}{i!} \mathrm{tr}_* \left((e - \frac{1}{2}) \otimes [e]_{2i} \right) \tag{47}$$

LEMMA 2.3. *$\mathrm{Ch}_{2k}(e) \in \mathrm{Tot}_{2k}(\overline{BA})$ is a closed cycle: $(b+B)\mathrm{Ch}_{2k}(e) = 0$.*

PROOF. Using the fact that $e^2 = e$, we have

$$\begin{aligned} b((e-\frac{1}{2}) \otimes [e]_{2p}) &= \frac{1}{2} e \otimes [e]_{2p-1} - (e - \frac{1}{2}) \otimes [e]_{2p-1} + \frac{1}{2} e \otimes [e]_{2p-1} \\ &= \frac{1}{2} \otimes [e]_{2p-1}. \end{aligned}$$

On the other hand, since we are working in the reduced complex, we find

$$B((e-\frac{1}{2}) \otimes [e]_{2(p-1)}) = (2p-1) \otimes [e]_{2p-1}$$

The statement now follows from the precise normalization of the Chern character above and Lemma 2.1 □

PROPOSITION 2.4. *The class of $\mathrm{Ch}_{2k}(e) \in HC_{2k}(A)$ only depends on the class of e in $K_0(A)$.*

PROOF. Let us first proof that the class in cyclic homology only depends on the isomorphism class of the module, i.e., that $\mathrm{Ch}_{2k}(geg^{-1}) = \mathrm{Ch}_{2k}(e) \in HC_{2k}(A)$ for $g \in GL_n(A)$. This follows from the following Lemma:

LEMMA 2.5. *Let A be a unital algebra and $g \in A^*$ be invertible. The conjugation action*

$$\alpha_g(a_0 \otimes \ldots \otimes a_k) = ga_0g^{-1} \otimes \ldots \otimes ga_kg^{-1}$$

on $C_\bullet(A)$ commutes with b and B. The induced action on Hochschild and cyclic homology is trivial.

PROOF OF LEMMA. It is easily checked that the action commutes with the Hochschild and cyclic operators b and B. By Lemma 3.5, it suffices to check that the induced map on Hochschild homology is the identity. Consider the maps $h_i : C_k(A) \to C_{k+1}(A)$, $i = 0, \ldots, k$ defined by

$$h_i(a_0 \otimes \ldots \otimes a_k) := a_0g^{-1} \otimes ga_1g^{-1} \otimes \ldots \otimes ga_ig^{-1} \otimes g \otimes a_{i+1} \otimes \ldots \otimes a_n$$

The alternating sum $h := \sum_{i=0}^{k}(-1)^i h_i$ defines a homotopy between the identity map and α_g i.e.,

$$bh + hb = id - \alpha_g.$$

(Check this identity!) This proves that the action on Hochschild homology is trivial. □

To complete the proof of the Proposition, we need to show that Chern character is additive under the direct sum of modules. If we have two module represented by idempotents $e \in M_n(A)$ and $e' \in M_m(A)$, their direct sum is represented by

$$\begin{pmatrix} e & 0 \\ 0 & e' \end{pmatrix} \in M_{m+n}(A).$$

Since the shape of this matrix is diagonal, the Dennis trace map (46) will map such elements to the sum, so indeed we have that $\mathrm{Ch}_{2k}(e \oplus e') = \mathrm{Ch}_{2k}(e) + \mathrm{Ch}_{2k}(e')$. This completes the proof. □

Next, recall the periodicity operator $S : HC_2k(A) \to HC_{2k-2}(A)$ in cyclic homology, c.f. §**??**. If a cyclic cycle is represented by chains $(\varphi_0, \varphi_2, \ldots, \varphi_{2k})$ with $\varphi_{2i} \in C_{2i}(A)$ with $b\varphi_{2i} = B\varphi_{2i-2}$, then

$$S(\varphi_0, \varphi_2, \ldots, \varphi_{2k}) = (\varphi_0, \varphi_2, \ldots, \varphi_{2k-2}).$$

Therefore we immediately see from (47) that

$$S(\mathrm{Ch}_{2k}(e)) = \mathrm{Ch}_{2k-2}(e).$$

It follows that the *total Chern character*

$$\mathrm{Ch}(e) = \sum_{i=1}^{\infty}(-1)^i \frac{(2i)!}{i!} \mathrm{tr}_* \left((e - \frac{1}{2}) \otimes [e]_{2i} \right)$$

is well defined in periodic cyclic homology. This defines the Chern character

$$\mathrm{Ch} : K_0(A) \to HP_0(A).$$

2.4. Algebraic K-theory. Cyclic homology is a purely algebraic theory. By contrast, the definition of K-theory of C^*-algebras given in the previous chapters uses the topology of the algebra in an essential way. There is an algebraic version of K-theory, that does not need any topology. However, only for K_0 the definition agrees with the previous one.

Let A be a unital algebra. We denote by $GL_\infty(A)$ the group of finite size invertible matrices with entries in A.

DEFINITION 2.6. The algebraic K-theory groups in degree 0 and 1 are defined as

$$K_0^{alg}(A) = \mathrm{Idem}(M_\infty(A))/GL_\infty(A)$$
$$K_1^{alg}(A) = GL_\infty(A)/[GL_\infty(A), GL_\infty(A)].$$

We record the following facts: when A is a Banach algebra, $K_0^{alt}(A) \cong K_0(A)$, and when A is also *commutative*, there exists a surjective map

$$K_1^{alg}(A) \to K_1(A).$$

(Recall that for a Banach algebra $K_1(A) := \pi_0(GL_\infty(A))$.)

2.5. The odd Chern character. Analogous to the even case, there is an odd Chern character

$$\mathrm{Ch} : K_1^{alg}(A) \to HP_1(A).$$

We only give the relevant formulas, and omit the details. A class in $K_1^{alg}(A)$ is represented by a matrix $u \in GL_\infty(A)$. Then

$$\mathrm{Ch}_{2k+1}(u) := (-1)^k k! \mathrm{tr}_*(u^{-1} \otimes u \otimes u^{-1} \otimes u \otimes \ldots \otimes u^{-1} \otimes u).$$

One checks that

$$B(\mathrm{Ch}_{2k-1}(u)) = -b\mathrm{Ch}_{2k+1}(u),$$

so the element $\sum_{i=0}^k \mathrm{Ch}_{2i+1}(u) \in HC_{2k+1}(A)$ is well-defined. The total Chern character is defined as

$$\mathrm{Ch}(u) := \sum_{k \geq 0} \mathrm{Ch}_{2k+1}(u) \in HP_1(A).$$

3. Compatibility with the HKR-isomophism

The aim of this section is to prove the following theorem:

THEOREM 3.1. *Let X be a compact smooth manifold. Then the following diagram commutes:*

$$\begin{array}{ccc} K_0^{alg}(C^\infty(X)) & \xrightarrow{\cong} & K^0(X) \\ \downarrow \mathrm{Ch} & & \downarrow \mathrm{ch} \\ HP_{ev}^{top}(C^\infty(X)) & \xrightarrow[\Psi_X]{} & H_{dR}^{ev}(X) \end{array}$$

PROOF. We start with the following observation: an idempotent $e \in M_n(C^\infty(X))$ is the same as giving a vector bundle E over X *equippped with a connection*. Indeed we can think of e as a matrix valued function on X and a section of the vector bundle it defines is given by a $s \in C^\infty(X, \mathbb{C}^n)$ with the property $es = s$. The associated *Grassmannian connection* is then defined by

$$\nabla(s) = eds \in \Omega^1(X; E).$$

Its curvature is easily computed as

$$\begin{aligned} \nabla^2(s) &= ede \wedge ds \\ &= ede \wedge des, \end{aligned}$$

where we have used that $ds = des + eds$ and $edee = 0$. Therefore

$$R(\nabla) = ede \wedge de.$$

With this, the Chern character in de Rham cohomology is represented by the differential form

$$\begin{aligned} \mathrm{ch}(E) &= \mathrm{tr}(e^{R(\nabla)}) \\ &= \mathrm{tr}(\sum_{n\geq 0} \frac{R(\nabla)^n}{n!} \\ &= \sum_{n\geq 0} \frac{\mathrm{tr}(ede \wedge \ldots \wedge de)}{n!} \end{aligned}$$

On the other hand we have that the noncommutative Chern character is given by the periodic cyclic cycle given by

$$\mathrm{Ch}(e) = \sum_{n\geq 0} \frac{(2n)!}{n!} \mathrm{tr}\left((e - \frac{1}{2})[e]_{2n} \right).$$

Under the analytic version of the HKR-mprphism (39) this cyclic cycle is mapped to the differential form

$$\Psi_X(\mathrm{Ch}(e)) = \sum_{n\geq 0} \frac{1}{n!} \mathrm{tr}\left((e - \frac{1}{2}) de \wedge \ldots \wedge de \right).$$

In de Rham cohomology, the two expressions agree because

$$de \wedge \ldots \wedge de = d(ede \wedge \ldots de)$$

is exact. This proves the theorem. □

4. Stability under the holomorphic functional calculus

We have seen that K-theory is most naturally defined for C^*-algebras, whereas cylic (co)homology is only interesting for algebras which are *not* C^*-algebras. This dichotomy becomes a bit of a problem when one considers the Chern character. The standard procedure is then to consider a dense "smooth" subalgebra of a C^*-algebra and to consider the cyclic theory of the one and the K-theory of the other. The typical case is given by the inclusion of $C^\infty(X)$ in $C(X)$ for a smooth manifold X.

Let A be a Banach algebra. For $a \in A$, spectrum $\mathrm{sp}(a) \subset \mathbb{C}$ is well-defined as in Defintion 1.2. Let f be a holomorphic function on a neighbourhood $\mathrm{sp}(a)$, and consider

$$f(a) := \frac{1}{2\pi i} \oint_C \frac{f(z)}{(z-a)} dz,$$

where C is a contour going around $\mathrm{sp}(a)$ once. It can be shown that this defines a well-defined element in A. This is called the *holomorphic functional calculus*. This is a weakening of the continuous functional calculus for C^*-algebras, valid for arbitrary unital Banach algebras.

DEFINITION 4.1. A unital subalgebra $\mathcal{A} \subset A$ is said to be *stable under the holomorphic functional calculus* if for all $a \in \mathcal{A}$ and all holomorphic functions f on a neighborhood of $\mathrm{sp}(a)$, we have that $f(a) \in \mathcal{A}$.

PROPOSITION 4.2. *Let $\mathcal{A} \subset A$ be stable under the holomorphic functional calculus. The the map $K_0(\mathcal{A}) \to K_0(A)$ induced by the inclusion is an isomorphism.*

PROOF. Let $e \in A$ be an idempotent. (The general case follows by taking $M_N(A)$ instead of A.) Since $\mathcal{A}$ is dense in A, we can approximate e by elements $a_n \in \mathcal{A}$, $n = 1, 2, \ldots$. Since $\mathrm{sp}(e) = \{0, 1\}$, we have that for n large enough $\mathrm{sp}(a_n)$ will be concentrated in neighborhoods of 0 and 1, and therefore separated by the line $\mathrm{Re}(z) = 1/2$. Now choose a holomorphic function f that is locally constant in a neighborhood of $\mathrm{sp}(a_n)$ and equal to 0 at 0 and equal to 1 in 1, this is possible by the fact that the spectrum is separated as mentioned above. Since $f^2 = f$, we see that $f(a_n)$ is an idempotent in $\mathcal{A}$. By the proof of Proposition 3.5, we see that e and $f(a_n)$ define the same element in K_0. This proves surjectivity of the map $K_0(\mathcal{A}) \to K_0(A)$.

□

EXAMPLE 4.3.

i) The inclusion $C^\infty(X) \subset C(X)$, for any compact manifold X, is stable under the holomorphic functional calculus.

ii) The subalgebra $\mathcal{T}^\infty \subset \mathcal{T}$ of the Toeplitz algebra generated by $T_f, f \in C^\infty(S^1)$, is stable under the holomorphic calculus.

CHAPTER 5

Index theorems

1. The index pairing

1.1. Cyclic cohomology. In the previous chapters we have considered cyclic homology of algebras. The dual theory, called cyclic cohomology is defined as follows. The space of Hochschild cochains of degree k is given by

$$C^k(A) := \mathrm{Hom}_{\mathbb{K}}(A^{\otimes(k+1)}, \mathbb{K}).$$

We dualize the Hochschild and cyclic differentials (27) (for $M = A^*$) and (33) to operators $b : C^k(A) \to C^{k+1}(A)$ and $B : C^k(A) \to C^{k-1}(A)$ given by:

$$\begin{aligned}
b\phi(a_0 \otimes \ldots \otimes a_{k+1}) &= \sum_{i=0}^{k} (-1)^i \phi(a_0 \otimes \ldots \otimes a_i a_{i+1} \otimes \ldots \otimes a_{k+1}) \\
&\quad + (-1)^k \phi(a_{k+1} a_0 \otimes \ldots \otimes a_k), \\
B\phi(a_0, \ldots, a_{k-1}) &= \sum_{i=0}^{k-1} (-1)^{(k-1)i} \phi\big(1, a_i, \ldots, a_{k-1}, a_0, \ldots, a_{i-1}\big) \\
&\quad - \sum_{i=0}^{k-1} (-1)^{(k-1)i} \phi\big(a_i \otimes 1 \otimes a_{i+1} \otimes \ldots a_n \otimes a_0 \otimes \ldots \otimes a_{i-1}\big).
\end{aligned}$$

By construction we have $b^2 = B^2 = bB + Bb = 0$, so we can arrange the two into a double complex that now looks like:

$$\begin{array}{ccccccc}
\cdots & & \cdots & & \cdots & & \cdots \\
\uparrow{\scriptstyle b} & & \uparrow{\scriptstyle b} & & \uparrow{\scriptstyle b} & & \uparrow{\scriptstyle b} \\
C^3(A) & \xrightarrow{B} & C^2(A) & \xrightarrow{B} & C^1(A) & \xrightarrow{B} & C^0(A) \\
\uparrow{\scriptstyle b} & & \uparrow{\scriptstyle b} & & \uparrow{\scriptstyle b} & & \\
C^2(A) & \xrightarrow{B} & C^1(A) & \xrightarrow{B} & C^0(A) & & \\
\uparrow{\scriptstyle b} & & \uparrow{\scriptstyle b} & & & & \\
C^1(A) & \xrightarrow{B} & C^0(A) & & & & \\
\uparrow{\scriptstyle b} & & & & & & \\
C^0(A) & & & & & &
\end{array}$$

Then we make the obvious definitions: the *Hochschild cohomology* is defined as the cohomology of the complex $(C^\bullet(A), b)$, whereas the *cyclic cohomology* is defined as the cohomology of the total complex of the double complex above. As for homology, we can use the *reduced cochains* to compute these cohomologies: these are the elements in $\phi \in C^k(A)$ that satisfy

$$\phi(a_0, \ldots, a_k) = 0, \quad \text{if } a_i = 0 \text{ for some } i \geq 1.$$

Since it is constructed from the dual complex, there is an obvious pairing $C^k(A) \times C_k(A) \to \mathbb{K}$, which descends to (co)homology: it induces a pairing between Hochschild homology and cohomology, and on the level of cyclic (co)homology:

$$\langle\ ,\ \rangle : HC_k(A) \times HC^k(A) \to \mathbb{K}.$$

1.2. Pairing with K-theory.

1.3. Excision. Recall that *topological* K-theory of C^*-algebra satisfies excision in the sense that a short exact sequence of C^*-algebras gives rise to a six-term periodic exact sequence of K-groups. For algebraic K-theory, a similar result holds true: given a short exact sequence

$$0 \to I \to A \to A/I \to 0, \tag{48}$$

where I is a two-sided ideal in a unital algebra A, there exists a long exact sequence []

$$K_1^{alg}(I) \longrightarrow K_1^{alg}(A) \longrightarrow K_1^{alg}(A/I) \xrightarrow{\text{Ind}} K_0^{alg}(I) \longrightarrow K_0^{alg}(A) \longrightarrow K_0^{alg}(A/I). \tag{49}$$

Here, all the maps are induced by the maps in the short exact sequence, except for the connecting map $\text{Ind} : K_1^{alg}(A/I) \to K_0^{alg}(I)$, called the *index map*. Let us describe this map in more detail: Let $u \in GL_1(A/I)$ and choose a $v \in GL_2(A)$ that projects onto the matrix

$$\begin{pmatrix} u & 0 \\ 0 & u^{-1} \end{pmatrix} \in GL_2(A/I).$$

Define

$$e_0 := \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad e_1 := v e_0 v^{-1}.$$

e_1 is an idempotent that projects to $1 \oplus 0$ in $M_2(A/I)$, so it defines a unique idempotent $e_1 \in M_2(\tilde{I})$. The formal difference $[e_1] - [e_0]$ lies in the kernel of the map $K_0(\tilde{I}) \to \mathbb{Z}$, which allows us to define

$$\text{Ind}([u]) := [e_1] - [e_0] \in K_0(I).$$

More specifically, choose liftings $a, b \in A$ of u and u^{-1}. Define x and y to be such that $ba = 1 - x$ and $ab = 1 - y$. By exactness of the sequence (48), we see that $x, y \in I$. Also we have the property that $xb = by$, and by induction $x^i b = by^i$ for any $i \in \mathbb{N}$. Let now $n \geq 1$, and define

$$b_n := (1 + x + \ldots + x^{2n-1})b = b(1 + y + \ldots + y^{2n-1}).$$

One checks that $b_n a = 1 - x^{2n}$ and $ab_n = 1 - y^{2n}$. Therefore

$$(b_n, x^n) \cdot \begin{pmatrix} a \\ x^n \end{pmatrix} = 1,$$

and the matrix

$$e_n := \begin{pmatrix} a \\ x^n \end{pmatrix} (b_n, x^n) = \begin{pmatrix} 1 - y^{2n} & ax^n \\ x^n b_n & x^{2n} \end{pmatrix} \tag{50}$$

is an idempotent. The matrices

$$v_n := \begin{pmatrix} a & -y^n \\ x^n & b_n \end{pmatrix}, \quad v_n^{-1} := \begin{pmatrix} b_n & x^n \\ -y^n a & \end{pmatrix}$$

are inverses of each other in $GL_2(A)$ and we have $v_n e_0 v_n^{-1} = e_n$.

The analogue of the long exact sequence in periodic cyclic homology is given by the six-term exact sequence:

$$\begin{array}{ccccc} HP_0(I) & \longrightarrow & HP_0(A) & \longrightarrow & HP_0(A/I) \\ {\scriptstyle\partial}\uparrow & & & & \downarrow{\scriptstyle\partial} \\ HP_1(A/I) & \longleftarrow & HP_1(A) & \longleftarrow & HP_1(I) \end{array}$$

This was proved by Cuntz and Quillen in **[CQ]**. Subsequently, Nistor **[Ni]** proved that the Chern character is compatible with the connecting maps ∂ and Ind. Dually, for cyclic cohomology this means that for $[u] \in K_1^{alg}(A/I)$ and $\phi \in HP_0(I)$ we have

$$\langle \mathrm{Ch}(\mathrm{Ind}([u])), \phi \rangle = \langle \mathrm{Ch}([u]), \partial(\phi) \rangle . \tag{51}$$

Let us describe the map $\partial : HP_0(I) \to HP_1(A/I)$, when the periodic cyclic cocycle is given by a map $\tau : I^p \to \mathbb{C},\ p > 0$ satisfying $\tau([A, I^p]) = 0$. Such a map is called an *odd higher trace* on A/I. First remark:

LEMMA 1.1. *Let τ be an odd higher trace on A/I. Then, for $2n - 1 \geq p$,*

$$\tau_{2n}(a_0, \ldots, a_{2n}) := \tau(a_0 \cdots a_{2n})$$

defines a cyclic cocycle on I satisfying $S\tau_{2n+2} = \tau_{2n}$.

PROOF. Exercise. □

Next we choose a linear lifting $l : A/I \to A$, and consider its "curvature"

$$R(a,b) := l(ab) - l(a)l(b) \in I, \quad a,b \in A/I,$$

measuring the defect from l being an actual homomorphism. Next, we choose n such that $2n+1 \geq p$ and we define

$$\begin{aligned}\phi_\tau(a_0,\ldots,a_{2n+1}) :=& \tau(R(a_0,a_1)\cdot R(a_2,a_3)\cdots R(a_{2n},a_{2n+1})) \\ & - \tau(R(a_{2n+1},a_0)\cdot R(a_1,a_2)\cdots R(a_{2n-1},a_{2n})).\end{aligned}$$

LEMMA 1.2. $b\phi_\tau = 0 = B\phi_\tau$.

PROOF. □

REMARK 1.3 (Schatten classes). Let $\mathcal{H}$ be a separable Hilbert space with basis $\{e_i\}_{i\in\mathbb{N}}$. For a positive bounded operator S on $\mathcal{H}$, define its trace to be

$$\mathrm{Trace}(S) = \sum_{i=0}^{\infty} \langle Se_i, e_i\rangle_{\mathcal{H}} \in [0,\infty].$$

One can prove (check!) that this trace is independent of the chosen basis. For $1 \leq p < \infty$, define the *p-th Schatten class* $B_p(\mathcal{H})$ to be the set of bounded operators for which

$$||T||_p := \sqrt[p]{\mathrm{Trace}(|T|^p)} < \infty,$$

where $|T| = \sqrt{T^*T}$ defined by means of the functional calculus. For $p = 2$, these operators are called *Hilbert–Schmidt*. One can prove that each $B_p(\mathcal{H}$ is an ideal inside the compact operators $K(\mathcal{H})$. For $T \in B_1(\mathcal{H})$, the sum $\sum_i \langle Te_i, e_i\rangle$ converges absolutely to a linear functional

$$\mathrm{Trace} : B_1(\mathcal{H}) \to \mathbb{C},$$

satisfying

$$\mathrm{Trace}(ST) = \mathrm{Trace}(TS), \quad \text{for all } S \in B(\mathcal{H}),\ T \in B_1(\mathcal{H}). \tag{52}$$

PROPOSITION 1.4. *For any* $p \geq 1$ *we have*

$$HP^\bullet(B_p(\mathcal{H})) = \begin{cases} \mathbb{C} & \bullet = 0, \\ 0 & \bullet = 1. \end{cases}$$

We can write down the generator for $HP_0(B_p(\mathcal{H})$: let $2n-1 \geq p$, and define

$$\mathrm{Trace}_{2n}(a_0,\ldots,a_{2n}) := \mathrm{Trace}(a_0\cdots a_{2n}), \quad a_i \in B_p(\mathcal{H}).$$

Since we have

$$\frac{1}{2n} B\mathrm{Trace}_{2n+1} = \mathrm{Trace}_{2n} = \frac{1}{(2n+2)} b\mathrm{Trace}_{2n-1}, \tag{53}$$

its class does not depend on the choice of n.

PROPOSITION 1.5 (Fedosov/Calderon). *Let T be a bounded operator on a Hilbert space such that T is invertible modulo $B_p(\mathcal{H})$, i.e., there exists a bounded operator S such that $1 - ST$ and $1 - TS$ are p-summable. Then we have*

$$\mathrm{index}(T) = \mathrm{Trace}\left((1 - ST)^p\right) - \mathrm{Trace}\left(((1 - TS)^p\right).$$

PROOF. We know from the short exact sequence (23) that a Fredholm operator determines a class $[u] \in K_1(C(\mathcal{H}))$. By the long exact sequence in K-theory we have a map

$$\mathrm{Ind} : K_1(C(\mathcal{H})) \to K_0(K(\mathcal{H})) \cong \mathbb{Z},$$

which is exactly the index of the operator. The last isomorphism is proved using the stability-property of K-theory, c.f. Example 4.5, using the dense inclusion $M_\infty(\mathbb{C}) \subset K(\mathcal{H})$. It is easy to see that the matrix trace induces this isomorphism

$$\mathrm{tr}_* : K_0(M_\infty(\mathbb{C})) \xrightarrow{\cong} \mathbb{Z}.$$

The Hilbert space trace is an extension of this trace to the intermediate algebra of trace class operators $M_\infty(\mathbb{C}) \subset B_1(\mathcal{H}) \subset K(\mathcal{H})$. Since the map induced by the trace is an example of the pairing (**??**) of K-theory and cyclic cohomology, its value only depends on the class in periodic cyclic cohomology, and by (53) we may just as well use Trace_{2n} for any $n \geq 1$. But this cocycle actually extends to $B_p(\mathcal{H})$, for $1 \leq p \leq 2n + 1$.

If we evaluate Trace_{2n} on the idempotent (50), we find

$$\mathrm{Index}(T)\mathrm{Trace}_{2n}(e_n) = \mathrm{Trace}(x^{2n} - y^{2n})$$

with $x = 1 - ST$ and $y = 1 - TS$. Finally, for $i \geq p$ we have

$$\begin{aligned}
\mathrm{Trace}(x^{i+1} - y^{i+1}) &= \mathrm{Trace}(x^i - STx^i - y^i + TSy^i) \\
&= \mathrm{Trace}(x^i - y^i - ATx^i + Tx^iS) \\
&= \mathrm{Trace}(x^i - y^i),
\end{aligned}$$

because of property (52). □

2. Fredholm modules

The notion of a *Fredholm module* emerged in the works of Atiyah and Kasparov as an abstract version of an elliptic operator on a compact manifold.

2.1. Definition.

DEFINITION 2.1. Let $\mathcal{A}$ be an algebra over $\mathbb{C}$. An *odd Fredholm module* over $\mathcal{A}$ is given by:

i) A Hilbert space $\mathcal{H}$,
ii) a representation of $\mathcal{A}$ on $\mathcal{H}$ given by a homomorphism $\rho : \mathcal{A} \to B(\mathcal{H})$,

ii) A bounded selfadjoint operator $F : \mathcal{H} \to \mathcal{H}$ such that $F^2 = 1$ and with $[F, \rho(a)]$ a compact operator for all $a \in \mathcal{A}$.

An *even Fredholm module* is given by the data above, together with a $\mathbb{Z}/2$-grading given by a selfadjoint operator ϵ on $\mathcal{H}$, satisfying $\epsilon^2 = 1$, which commutes with $\mathcal{A}$ and anti-commutes with F.

Let $(\mathcal{A}, \mathcal{H}, F)$ be an odd Fredholm module. It is easy to see that the operator $P = (F+1)/2$ is a projection, namely it projects onto the $+1$ eigenspace of F on $\mathcal{H}$. Let $u \in GL_n(\mathcal{A})$, and consider the matrix

$$PuP = (Pu_{ij}P) : P\mathcal{H} \oplus \mathbb{C}^n \to P\mathcal{H} \oplus \mathbb{C}^n.$$

PROPOSITION 2.2. *PuP is Fredholm and its index defines a homomorphism*

$$K_1^{alg}(\mathcal{A}) \to \mathbb{Z}.$$

PROOF. Remark that $F = 2P - 1$, so $[P, u]$ is compact. Consider the operator $Pu^{-1}P$. This is a two-sided inverse to PuP modulo the compact operators:

$$\begin{aligned}(PuP)(Pu^{-1}P) &= PuPu^{-1}P \\ &= P[u,P]u^{-1}P - P.\end{aligned}$$

By Atkinson's theorem 6.4, this proves that PuP is Fredholm.

Let $u, v \in GL_n(\mathcal{A})$. Since

$$\begin{aligned}P[u,v]P &= PuvP - PvuP \\ &= P[P,u]vP - P[P,v]uP - [PuP, PvP].\end{aligned}$$

So, modulo the compact operators $P[u,v]P$ is equal to $[PuP, PvP]$. Since the index defines a homomorphism

$$\text{index} : B(\mathcal{H})/C(\mathcal{H}) \to \mathbb{Z},$$

we find that $\text{index}(P[u,v]P) = 0$ and the claim follows. □

In the even case, using ϵ we can decompose $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$ into eigenspaces, and with this

$$F = \begin{pmatrix} 0 & T^* \\ T & 0 \end{pmatrix}, \quad \rho(a) = \begin{pmatrix} \rho_+(a) & 0 \\ 0 & \rho_-(a) \end{pmatrix}.$$

Given an even Fredholm module, and an idempotent $e \in M_N(\mathcal{A})$, consider the operator

$$\rho_+(e)F\rho_-(e) : \rho_+(e)(\mathcal{H}_+ \oplus \mathbb{C}^N) \to \rho_-(e)(\mathcal{H}_- \oplus \mathbb{C}^N).$$

PROPOSITION 2.3. *$\rho_+(e)F\rho_-(e)$ is Fredholm and its index defines a homomorphism*

$$\text{index}^F : K_0^{alg}(\mathcal{A}) \to \mathbb{Z}. \tag{54}$$

PROOF. The proof is similar to the previous Proposition. This time we use the operator $\rho_-(e)F\rho_+(e)$ as an inverse modulo the compact operators:

$$\begin{aligned}\rho_+(e)F\rho_-(e)\rho_-(e)F\rho_+(e) &= \rho_+(e)F\rho_-(e)F\rho_+(e)\\ &= \rho_+(e)([F,\rho_(e)] + \rho_+(e)F)F\rho_+(e)\\ &= \rho_+(e)[F,\rho_(e)]Fe + \rho_+(e).\end{aligned}$$

We omit the details. □

2.2. The Chern character of a Fredholm module.

THEOREM 2.4 (Connes). *Let $(\mathcal{A},\mathcal{H},F)$ be an odd Fredholm module and suppose there exists an odd integer such that the product $[F,a_0]\cdots[F,a_n]$ is trace class for all $a_0,\ldots,a_n\in\mathcal{A}$. The the formula*

$$\mathrm{ch}_n^F(a_0,\ldots,a_n) := \frac{\Gamma(n/2+1)}{2\cdot n!}\mathrm{Trace}\left(F[F,a_0]\cdots[F,a_n]\right)$$

defines a cyclic cocycle whose class is independent of n. For $[u]\in K_1^{alg}(\mathcal{A})$, the following equality holds true:

$$\left\langle [u],\mathrm{ch}_n^F\right\rangle = \mathrm{index}^F(u).$$

REMARK 2.5. A Fredholm module satisfying the condition of the theorem is called *n-summable*.

PROOF. With the property

$$[F,a_1a_2] = a_1[F,a_2] + [F,a_1]a_2,\quad a_1,a_2\in\mathcal{A},$$

one easily verifies that $b\mathrm{ch}_n^F = 0$. Similarly, $B\mathrm{ch}_n^F = 0$, since $[F,1]=0$. We therefore see that ch_n^F is a cocycle in the (b,B)-complex. Define

$$\psi_{n+1}(a_0,\ldots,a_{n+1}) := \frac{\Gamma(n/2+2)}{(n+2)!}\mathrm{Trace}\left(a_0[F,a_1]\cdots[F,a_{n+1}]\right).$$

Then $b\psi_{n+1} = -\mathrm{ch}_{n+2}^F$ and $B\psi_{n+1} = \mathrm{ch}_n^F$, so ch_{n+2}^F and ch_n^F differ a coboundary in the (b,B)-complex.

To prove the last claim, notice that an odd Fredholm module with the property stated in the Theorem determines an extension

$$0\to B_{n/2}(\mathcal{H})\to (B_{n/2}(\mathcal{H})+\sigma(\mathcal{A}))\to \mathcal{A}/\ker(\sigma)\to 0,$$

where $\sigma(a) := P\rho(a)P$. This extension has a canonical linear splitting $l(a)=\sigma(a)$, with associated curvature

$$R(a,b) = \sigma(ab)-\sigma(a)\sigma(b) = \frac{1}{2}P[F,a][F,b]P\in B_{n/2}(\mathcal{H}).$$

Next, we consider the class $\partial(\mathrm{Trace}_{2m}) \in HP^1(\mathcal{A}/\ker\sigma)$. By Lemma 1.1, it is represented by the cycle

$$\begin{aligned}\phi_{2m+1}(a_0,\ldots,a_{2n+1}) =& \mathrm{Trace}\left(R(a_0,a_1)\cdots R(a_{2n}a_{2n+1}\right)\\ &\quad - \mathrm{Trace}\left(R(a_{2n+1},a_0)\cdots R(a_{2n-1},a_{2n})\right)\\ &= \mathrm{Trace}\left(F[F,a_0][F,a_1]\cdots[F,a_{2n+1}]\right).\end{aligned}$$

For $u \in K_1^{alt}(\mathcal{A})$, we now have

$$\begin{aligned}\left\langle [u], \mathrm{ch}_n^F \right\rangle &= \phi_{2m+1}(u,u^{-1},u,u^{-1},\ldots,u,u^{-1})\\ &= \langle [u], \partial(\mathrm{Trace}_{2m})\rangle\\ &= \langle \mathrm{Ind}(u), \mathrm{Trace}_{2m}\rangle\\ &= \mathrm{Index}(PuP)\\ &= \mathrm{index}^F(u).\end{aligned}$$

This finishes the proof of the Theorem. □

The even case is as follows:

THEOREM 2.6 (Connes). *Let $(\mathcal{A},\mathcal{H},F,\epsilon)$ be an even Fredholm module and suppose there exists an even integer such that the product $[F,a_0]\cdots[F,a_n]$ is trace class for all $a_0,\ldots,a_n \in \mathcal{A}$. The the formula*

$$\mathrm{ch}_n^F(a_0,\ldots,a_n) := \frac{\Gamma(n/2+1)}{2\cdot n!}\mathrm{Trace}\left(\epsilon F[F,a_0]\cdots[F,a_n]\right)$$

defines a cyclic cocycle whose class is independent of n. For $[e] \in K_0^{alg}(\mathcal{A})$, the following equality holds true:

$$\left\langle [e], \mathrm{ch}_n^F \right\rangle = \mathrm{index}^F(e).$$

We omit the proof. Finally, we state the following important technical result:

THEOREM 2.7 (Connes). *Let $(\mathcal{H},F)$ be a Fredholm module over a Banach algebra $\mathcal{A}$. Then, for $p \geq 1$, the subalgebra*

$$\{a \in \mathcal{A},\ [F,a] \in B_p(\mathcal{H})\}$$

is stable under the holomorphic functional calculus.

2.3. The Toeplitz index theorem revisited. Let us now consider an example of this formalism, connected with the following simple example of a Fredholm module: we take as our algebra $\mathcal{A} = C^\infty(S^1)$, acting on the Hilbert space $\mathcal{H} = L^2(S^1)$ by multiplication. As our operator F we take the so-called *Hilbert transform* given on the orthonormal basis $e_n = z^n$ by

$$F(e_n) = \begin{cases} e_n & n \geq 0\\ -e_n & n < 0.\end{cases}$$

The associated projection operator P is therefore the projection onto the Hardy subspace.

LEMMA 2.8. *Let $a, b \in C^\infty(S^1)$. Then*

$$\text{Trace}\,(F[F,a][F,b]) = \frac{1}{2\pi}\int_{S^1} adb.$$

PROOF. Do this yourself by approximating a and b by polynomials. □

THEOREM 2.9. *Let $f \in C^\infty(S^1)$ be nonvanishing. Then we have*

$$\text{Index}(T_f) = -\frac{1}{2\pi i}\int_S^1 \frac{du}{u}.$$

3. The Kadison conjecture

As an application we describe Connes' proof of the Kadison conjecture, previously proved by Pimsner–Voiculescu. The argument uses the integrality of the map (**??**) associated to a fredholm module. We begin by describing the conjecture.

3.1. Group C^*-algebra. Let Γ be a discrete group. Its *algebraic group algebra* $\mathbb{C}\Gamma$ is given by functions $f : \Gamma \to \mathbb{C}$ with compact (hence finite) support. Its multiplication is defined by the *convolution product*:

$$(f_1 * f_2)(\gamma) := \sum_{\gamma_1\gamma_2=\gamma} f_1(\gamma_1)f_2(\gamma_2).$$

This is a unital algebra with unit given by the delta function at the unit element of Γ. It also carries a $*$-operator defined as $f^*(\gamma) := \overline{f(\gamma^{-1})}$.

Next we consider the Hilbert space $\ell^2(\Gamma)$ of square summable functions on Γ. The group algebra has a left regular representation on $\ell^2(\Gamma)$ given by

$$(f \cdot \xi)(\gamma) := \sum_{\gamma_1\gamma_2=\gamma} f(\gamma_1)\xi(\gamma_2), \quad f \in \mathbb{C}\Gamma,\ \xi \in \ell^2(\Gamma).$$

It is easy to check that this defines an injective $*$-homomorphism $\pi : \mathbb{C}\Gamma \to B(\ell^2(\Gamma))$. The *reduced group C^*-algebra $C^*_r(\Gamma)$* is defined to be the norm closure of $\mathbb{C}\Gamma$ inside $B(\ell^2(\Gamma))$. We can now describe the Kadison conjecture. Recall that a group Γ is said to be torsion-free if the only element of finite order is the unit.

CONJECTURE 3.1 (Kadison). *Let Γ be a torsion-free discrete group. Then $C^*_r(\Gamma)$ does not have a nontrivial projection.*

EXERCISE 3.2. Why is a C^*-algebra without nontrivial projections called *connected*? (Hint: think of the commutative case.) Give an example of a discrete group with torsion whose group C^*-algebra does contain a projection.

Below, we shall outline a proof of the Kadison conjecture for *free* groups. The general case of the conjecture is still open, and it has been positively verified for a number of classes of groups.

3.2. The trace. Consider the functional $\tau : \mathbb{C}\Gamma \to \mathbb{C}$ defined by the evaluation at the unit:

$$\tau(f) := f(e), \quad f \in \mathbb{C}\Gamma.$$

If we write δ_e for the delta function at the unit e of Γ, we can write this as

$$\tau(f) := \langle \pi(f)\delta_e, \delta_e \rangle ,$$

which shows that τ extends to $C_r^*(\Gamma)$. In fact this trace is positive and faithful meaning that

$$\tau(f^* f) \geq 0,$$

and

$$\tau(f^* f) = 0 \Longleftrightarrow f = 0.$$

Of course, a trace is a cyclic cocycle of degree 0, so it induces a map

$$\tau_* : K_0(C_r^*(\Gamma)) \to \mathbb{C}.$$

The idea of Connes is to use a Fredholm module to prove that this map is integral, meaning that it takes values in $\mathbb{Z}$.

3.3. A Fredholm module associated to a tree. We now restrict attention to the case of a finitely generated *free* group Γ. Recall that this means that there exists a finite subset $S \subset \Gamma$ with the property that any element $\gamma \in \Gamma$ can be written in a unique way as a product of finitely many elements in S and their inverses. This implies that there cannot exist any relations between the elements of S, apart from the existence of inverses.

We can construct a tree associated to Γ (its *Caley graph*) as follows: this is a graph G with vertices $V(G) = \Gamma$ given by elements in Γ. Its edges $E(G)$ consists of pairs of vertices given by $(\gamma, \gamma s)$ with $s \in S$. It is easy to check that because the group is free, the graph is a tree, i.e., has no loops.

The group Γ has a canonical action on this tree, meaning that it acts on both edges and vertices, in a compatible way, from the left. One easily checks that this action is free and transitive.

We now fix $\gamma_0 \in \Gamma = V(G)$, and we consider the map

$$\varphi : V(G)/\{\gamma\} \to E(G),$$

defined as follows: for $\gamma \in \Gamma$, $\varphi(\gamma)$ is the unique edge connecting γ which lies in the unique path from γ_0 to γ. Clearly, φ is a bijection and it is *almost equivariant*:

$$\gamma\varphi(\gamma') = \varphi(\gamma\gamma'), \quad \text{except for finitely many } \gamma' \in \Gamma.$$

We now define $\mathcal{H}_+ := \ell^2(V(G))$ and $\mathcal{H}_- := \ell^2(E(G)) \oplus \mathbb{C}$. Using the action of Γ on the tree, we obtain a representation of $C_r^*(\Gamma)$ on $\mathcal{H}_+$ and $\mathcal{H}_-$. For $\mathcal{H}_+$ this is just the usual representation on $\ell^2(\Gamma)$ For $\mathcal{H}_-$ we have

$$a \cdot (\xi, z) = (a\xi, 0), \quad a \in C_r^*(\Gamma),\ \xi \in \ell^2(E(G)),\ z \in \mathbb{C}.$$

Denote by $\delta_\gamma \in \ell^2(V(G)$ the basis element given by the delta-function at $\gamma \in V(G)$. Then we define the following operator $P : \mathcal{H}_+ \to \mathcal{H}_-$

$$P(\delta_\gamma) = \begin{cases} (0,1) & \gamma = \gamma_0 \\ \delta_{\varphi(\gamma)} & \gamma \neq \gamma_0. \end{cases}$$

Because φ is a bijection, this map is unitary. We now put $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$,

$$F := \begin{pmatrix} 0 & P^{-1} \\ P & 0 \end{pmatrix}, \quad \gamma := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

PROPOSITION 3.3.

i) *The triple $(\mathcal{H}, F, \gamma)$ forms an even Fredholm module over $\mathbb{C}\Gamma$.*
ii) *The induced Chern character equals, in degree zero, the trace τ on $C_r^*(\Gamma)$.*

PROOF. i) Let $\gamma \in \Gamma$. consider the commutator

$$[F, \gamma](\delta_{\gamma'} = \delta_{\varphi(\gamma\gamma')} - \gamma\delta_{\varphi(\gamma')},$$

which is zero except for finitely many γ'. Therefore $[F, \gamma]$ is a finite rank operator. Therefore the same holds true for $\mathbb{C}\Gamma$

ii) By the argument above we see that

$$\mathbb{C}\Gamma \subset \{a \in C_r^*(\Gamma),\ [F, a] \in B_1(\mathcal{H})\},$$

so we can take $n = 0$ in Theorem 2.6. For $a \in \mathbb{C}\Gamma$ we have that

$$\epsilon F[F, a] = \begin{pmatrix} a - P^{-1}aP & 0 \\ 0 & -a + PaP^{-1} \end{pmatrix}.$$

Therefore

$$\frac{1}{2}\mathrm{Trace}_{\mathcal{H}}(\epsilon F[F, a]) = \mathrm{Trace}_{\mathcal{H}_+}(a - P^{-1}aP).$$

Consider now the element $a - \tau(a)e \in \mathbb{C}\Gamma$. Since the action of Γ on $V(G)$ and $E(G)$ is free, we have

$$\langle \delta_q, (a - \tau(a)e)\delta_q \rangle = 0, \quad \text{for all } q \in V(G), E(G).$$

It follows then that

$$\mathrm{Trace}_{\mathcal{H}_+}(a - P^{-1}aP) = \tau(a)\mathrm{Trace}_{\mathcal{H}_+}(e - P^{-1}eP) = \tau(a),$$

proving the second claim. □

We can now prove the Kadison conjecture for finitely generated free groups: Let $e \in C_r^*\Gamma$ be a projection. Since the trace is positive, we have that $\tau(a) \geq 0$ if $a \geq 0$. For a projection $0 \leq e \leq 1$ and therefore $0 \leq \tau(e) \leq 1$. By the proposition above, we have $\tau(e) \in \mathbb{Z}$, so either $\tau(e) = 0$ or $\tau(e) = 1$. First assume $\tau(e) = 0$. Then we have

$$0 = \tau(e) = \tau(e^*e) \Longrightarrow e = 0,$$

by faithfulness of τ. Next, if $\tau(e) = 1$, we have

$$0 = \tau(e - 1) = \tau((e - 1)^*(e - 1)) \Longrightarrow e = 1,$$

since $e - 1$ is also a projection. The conjecture is proved.

4. Spectral triples

4.1. Definition and examples.

DEFINITION 4.1. A *spectral triple* $(\mathcal{A}, \mathcal{H}, D)$ consists of an algebra $\mathcal{A}$ acting on a Hilbert space $\mathcal{H}$ by bounded linear operators, together with an unbounded operator D, such that:

i) D is self adjoint and has compact resolvant $(\lambda I - D)^{-1}$,
ii) the commutator $[D, a]$ is a bounded operator on $\mathcal{H}$. (But need not be an element of $\mathcal{A}$.)

There are some standard summability assumptions:

DEFINITION 4.2. A spectral triple $(\mathcal{A}, \mathcal{H}, D)$ is said to be

- p-summable if $(1 + D^2)^{-1/2}$ belongs to the p-th Schatten ideal,
- θ-summable if e^{-tD^2} is trace class for all $t > 0$.

LEMMA 4.3. *If a spectral triple is p-summable, it is θ-summable.*

PROOF. We can write

$$e^{-tD^2} = (1 + D^2)^{p/2} e^{-tD^2} (1 + D^2)^{-p/2}.$$

By the spectral theorem applied to the bounded Borel function $f(\lambda) = (1 + \lambda^2)^{p/2} e^{-t\lambda^2}$, the first two factors combine to give a bounded operator on $\mathcal{H}$. The last factor is trace class by assumption, and the result follows. □

The relation between spectral triples and Fredholm modules is explained by the following:

PROPOSITION 4.4. *Let $(\mathcal{A}, \mathcal{H}, D)$ be a spectral triple and suppose that D is invertible. Then $F = D/|D|$ defines a Fredholm module.*

The operator $D/|D|$ is defined by means of the functional calculus for unbounded operators. The condition that D should be invertible is not too restrictive: given a spectral triple, we can always perturb D a little bit to make sure that zero is not in the spectrum.

EXAMPLE 4.5.

i) The easiest example is given by the commutative geometry on S^1: we consider the algebra $\mathcal{A} = C^\infty(S^1)$ acting by multiplication on the Hilbert space $L^2(S^1)$. The unbounded operator is given by $D = \sqrt{-1}d/d\theta$. Clearly D is not invertible, but $D + 1/2$ is. The resulting Fredholm module is that of §2.3.

ii) More generally, let X be a *spin*^{c} manifold, with Dirac operator D acting on the sections of the spinor bundle $\mathbb{S}$. In this case $(L^2(M;\mathbb{S}), D)$ forms a spectral triple over $C^\infty(X)$. The geodesic distance on X w.r.t. the underlying riemannian metric can be recovered from this spectral triple by Connes' remarkable formula

$$d(x,y) = \sup\{|f(x) - f(y)|,\ \|[D,f]\| \leq 1\}.$$

Bibliography

[At] M. Atiyah. *K*-theory.

[BT] R. Bott and L. Tu. Differential Forms in Algebraic Topology. GTM

[Co82] A. Connes. Noncommutative differential geometry. *Publ. Math. IHES* 1982

[CQ] J. Cuntz and D. Quillen. Excision in bivariant periodic cyclic cohomology. Invent. Math., 127(1):67–98, 1997.

[HR] N. Higson and J. Roe. *Analytic K-homology*. OUP

[K] M. Karoubi. *K*-theory.

[L] J.L. Loday. *Cyclic homology*

[Ni] V. Nistor, Higher index theorems and the boundary map in cyclic cohomology, Documenta Mathematica 2 (1997), 263296.