Localized index theory on Lie groupoids and the van Est map

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(joint work with M. Pflaum, X. Tang)

Lie groupoids are natural generalisations of manifolds, Lie groups, actions of Lie groups on manifolds and foliations. As such, they are models for singular spaces and there are several connections to the theory of stratified spaces. Here we are concerned with the index theory of longitudinally elliptic operator on such Lie groupoids, generalizing the Atiyah–Singer families index theorem.

Let G be a Lie groupoid over the unit space M, and we denote the source and target map by $s, t : \mathsf{G} \to M$. The composition $g_1 g_2$ of two elements $g_1, g_2 \in \mathsf{G}$ is defined only if $t(g_1) = s(g_2)$. A longitudinal pseudodifferential operator [3, 4] on G is a family of pseudodifferential operators on the t-fibers $t^{-1}(x),\ x \in M$ that is smooth in x and invariant under the action of G. With the right conditions on the support of such pseudodifferential operators, they form an algebra denoted by $\Psi^\infty(\mathsf{G})$. For us, two facts of this pseudodifferential calculus are important:

i) The universal enveloping algebra $\mathcal{U}(A)$, where A is the Lie algebroid associated to G embeds into $\Psi^\infty(\mathsf{G})$ as families of invariant differential operators on the t-fibers.

ii) The ideal of smoothing operators $\Psi^{-\infty}(\mathsf{G}) \subset \Psi^\infty(\mathsf{G})$ is isomorphic to the convolution algebra $\mathcal{A}$ of G. Recall that the convolution algebra is given by $\mathcal{A} = \Gamma_c^\infty(\mathsf{G}; s^* \bigwedge^{\mathrm{top}} A^*)$ equipped with the product

$$(a_1 * a_2)(g) := \int_{h \in \mathsf{G}_{t(g)}} a_1(gh^{-1}) a_2(h),$$

where G_x is the submanifold of all arrows $g \in \mathsf{G}$ having target $x \in M$.

All this is easily extended to the case of operators acting on sections on a vector bundle pulled back from M. Therefore, for $E \to M$ a vector bundle, we write $\mathcal{U}(A; E)$ for the tensor product $\mathcal{U}(A) \otimes \mathrm{End}(E)$. We say an element $D \in \mathcal{U}(A; E)$ is *elliptic* if it defines an elliptic differential operator D_x on $t^{-1}(x)$ for each $x \in M$.

Standard arguments using this pseudodifferential calculus construct an *index class*

$$[\mathrm{Ind}(D)] \in K_0(\mathcal{A})$$

of such an elliptic operator. Unfortunately, the K-theory of $\mathcal{A}$ is still very poorly understood in general, with the exception of so-called *foliation groupoids* where the Connes–Skandalis index theorem gives a topological construction of the index class above out of the symbol of D. To go beyond the case of foliations, we therefore apply the Chern-Connes character to cyclic homology and study the class

$$\mathrm{Ch}([\mathrm{Ind}(D)]) \in HC_\bullet(\mathcal{A}).$$

In general, the cyclic homology of $\mathcal{A}$ is (again) not understood beyond the foliation case where there is a complete computation due to Brylinski–Nistor and Crainic.

To circumvent this lack of understanding, we shall construct certain cyclic *co*homology classes, and compute the pairing with the homology class above. The result is an index theorem valid for all Lie groupoids, not just the foliation ones, cf. [5].

First of all, we construct the line bundle $L = \bigwedge^{\rm top} T^*M \otimes \bigwedge^{\rm top} A$ of "transversal densities". It was first noticed by Evens–Lu–Weinstein that the groupoid G naturally acts on this line bundle and therefore we can consider its *differentiable groupoid cohomology* $H^\bullet_{\rm diff}(\mathsf{G}; L)$. This is a straightforward generalization of differentiable group cohomology with values in a representation of a Lie group, viz. the case when M is a point. This cohomology is the domain of a canonical map

$$\chi : \bigoplus_{i\geq 0} H^{\bullet+2i}_{\rm diff}(\mathsf{G}; L) \to HC^\bullet(\mathcal{A}).$$

This map can be thought of as the characteristic map associated to an action of a Hopf algebroid on $\mathcal{A}$. It enables us to pair elements in $K_0(\mathcal{A})$, such as the index class, with differentiable groupoid cohomology classes.

Second, we construct the index class in $K_0(\mathcal{A})$ is such a way that it is represented by idempotents in $\mathcal{A}$ with support arbitrarily close to the unit. In fact, one can construct a "localized K-theory" $K_0^{\rm loc}(\mathcal{A})$ build from idempotents with exactly this property, equipped with a canonical forgetful map $K_0^{\rm loc}(\mathcal{A}) \to K_0(\mathcal{A})$. The remark above then boils down to the statement that there is a natural refinement

$$[{\rm Ind}(D)]_{\rm loc} \in K_0^{\rm loc}(\mathcal{A})$$

of the index class. The crucial feature of the localized K-theory is that it naturally pairs with *Lie algebroid cohomology*:

$$\langle\ ,\ \rangle : K_0^{\rm loc}(\mathcal{A}) \times H^{\rm ev}_{\rm Lie}(A; L) \to \mathbb{C}.$$

Similar to differentiable groupoid cohomology, Lie algebroid cohomology generalizes the cohomology theory of Lie algebras and the representation of A on L is just the infinitesimal part of the representation of G. As for Lie groups, there is a natural "van Est" map for Lie groupoids

$$E : H^\bullet_{\rm diff}(\mathsf{G}; L) \to H^\bullet_{\rm Lie}(A; L).$$

The first result relates the global pairing with the localized one via this van Est map:

Theorem 1. *Let G be a Lie groupoid, $E \to M$ a vector bundle over the unit space, and $D \in \mathcal{U}(A, E)$ an elliptic element. Then, for $\alpha \in H^{2k}_{\rm diff}(\mathsf{G}; L)$,*

$$\langle \chi(\alpha), {\rm Ch}([{\rm Ind}(D)])\rangle = \langle E(\alpha), [{\rm Ind}(D)]_{\rm loc}\rangle\,.$$

This reduces the computation of the index to a local computation near the unit space. We perform this computation using the fact that the pseudodifferential calculus on G is a quantization of the Lie–Poisson structure on A^*, and reduce it to the *algebraic index theorem* for this Poisson manifold. The final result is given as follows:

Theorem 2. *Let $A \to M$ be an integrable Lie algebroid, E a vector bundle over M and $D \in \mathcal{U}(A, E)$ an elliptic element. For $c \in H^{2k}_{\rm Lie}(A; L)$ we have*

$$\langle c, [\mathrm{Ind}(D)]_{\rm loc}\rangle = \frac{1}{(2\pi\sqrt{-1})^k}\int_{A^*} \pi^* c \wedge \hat{A}(\pi^! A) \wedge \rho^*_{\pi^! A}\mathrm{ch}(\sigma(D)).$$

Here, the right hand side is a topological expression using the usual characteristic classes, only now given in Lie algebroid cohomology rather than de Rham cohomology. The notation $\pi^! A$ denotes the pull-back (in the category of Lie algebroids) of A along the projection $\pi : A^* \to M$. This is a Lie algebroid over A^* with anchor map $\rho^*_{\pi^! A} : \pi^! A \to TA^*$, which has the same Lie algebroid cohomology as A.

Together, these two theorems give a complete understanding of the pairing between the index class and Lie groupoid cohomology classes for any Lie groupoid. Possibly, the localized index Theorem 0.2 is much more powerful and has more applications. We can consider some special cases to get some more insight:

i) The pair groupoid $M \times M$ of any manifold is proper, and there is therefore only one nonzero differentiable groupoid cohomology class which lives in degree zero. In this case, we find the Atiyah–Singer index theorem for elliptic operators on M. On the other hand, the associated Lie algebroid is simply TM and its Lie algebroid cohomology is given by $H^\bullet_{dR}(M)$. With this, Theorem 0.2 recovers Connes–Moscovici's localized index theorem [2]. The *covering index theorem* of Connes–Moscovici is a very natural statement in the present framework about two Lie groupoid that induce the same Lie algebroid.

ii) For a foliation $\mathcal{F} \subset TM$, we can apply this theory to the holonomy groupoid $\mathsf{G}_{\mathcal{F}}$ of $\mathcal{F}$. In this case we find Connes' index theorem [1, §III.7.γ] for the pairing between the index class and elements in $H^\bullet(B\mathsf{G}_{\mathcal{F}})$, but only for those classes that come from differentiable groupoid cohomology of the holonomy groupoid. This restriction is the price we have to pay for being able to extend the index theorem from foliation groupoids to arbitrary Lie groupoids.

References

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