Index theory on orbifolds via formal deformation quantization

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(joint work with M. Pflaum and X. Tang)

Symplectic orbifolds are naturally encountered in mathematical physics and Poisson geometry, e.g. as the result of symplectic reduction with respect to a locally free action of a compact Lie group. It is therefore very natural to try to extend known quantization schemes on symplectic manifolds to this category. The aim of this talk is to explain how this gives a useful approach to index theory on orbifolds and as such may serve as an example for dealing with more singular spaces. In all this we use the the theory of formal deformation quantization.

A symplectic orbifold X is, loosely speaking, a topological Hausdorff space which locally is homeomorphic to an open neighbourhood of 0 in $\mathbb{R}^{2n}/\Gamma$, where Γ is a finite group acting by linear symplectic transformations with respect to the standard symplectic form on $\mathbb{R}^{2n}$. Besides being an orbifold in the usual sense, the symplectic structure gives the sheaf of smooth functions on X, i.e., those functions that locally lift to smooth Γ-invariant functions in a chart as above, the structure of a sheaf of Poisson algebras. Instead of considering the deformation problem for this Poisson algebra, denoted A_X, we will do the folowing, in the spirit of noncommutative geometry:

Associated to a symplectic orbifold X is a proper étale groupoid G, with structure maps $s, t : G_1 \to G_0$ and $G_0/G_1 \cong X$, such that G_0 carries an invariant symplectic form ω, i.e., $s^*\omega = t^*\omega$. The convolution algebra of an étale groupoid is defined as $A_G := C_c^\infty(G_1)$ with the product

$$(f_1 * f_2)(g) = \sum_{g_1 g_2 = g} f_1(g_1) f_2(g_2),$$

for $g \in G_1$. Notice that the center of A_G equals $A_X := C_c^\infty(X)$, and when X happens to be a manifold, A_G is even Morita equivalent to its center. The symplectic nature of the orbifold amounts to a canonical Hochschild class $\pi \in H^2(A_G, A_G)$ which satisfies $[\pi, \pi] = 0$, cf. [8]. The upshot is that the "classical phase space" is already a noncommutative geometry!

A formal deformation quantization of the noncommutative Poisson algebra (A_G, π) consists of an associative product $\star_c$ on $A_G[[\hbar]]$, compatible with the $\hbar$-adic filtration, such that in zeroth order one recovers the convolution product above, and the Hochschild class of the first order approximation equals π. Such a deformation, denoted by $A_G^\hbar$, can be constructed as follows: Using Fedosov's method [1], one can construct a G-invariant deformation quantization of the sheaf of smooth functions on the symplectic manifold G_0, with characteristic class $[\Omega] \in H^2(X, \mathbb{C}[[\hbar]])$. Using a kind of crossed product construction, one obtains a deformation $A_G^\hbar$ of A_G, whereas the invariant section of this sheaf give a deformation quantization $A_X^\hbar$ of A_X, cf. [6]. In sharp contrast to the classical (i.e., undeformed) theory, it turns out that there is a canonical Morita equivalence

$$A_G^\hbar \overset{M}{\sim} A_X^\hbar, \tag{1}$$

when the orbifold X is reduced, so, in a sense, the quantization "resolves the singularities".

As a first step towards the index theorem, one computes the cyclic theory of the algebra $A_G^\hbar$. To state the following theorem, let $\tilde{X}$ be the "inertia orbifold" associated to X. This orbifold, canonically associated to X, was first considered in [3], and is referred to in the physics literature as the "twisted sectors" of X.

Theorem 1 ([5]). *The Hochschild and cyclic cohomology groups of the deformed groupoid algebra $A_G^\hbar$ are given by*

$$HH^\bullet(A_G^\hbar) \cong H^\bullet\left(\tilde{X}, \mathbb{C}((\hbar))\right)$$
$$HC^\bullet(A_G^\hbar) \cong \bigoplus_{k\geq 0} H^{\bullet-2k}\left(\tilde{X}, \mathbb{C}((\hbar))\right).$$

Similar results hold for Hochschild and cyclic homology [5] in terms of compactly supported cohomology, so that these are Poincaré dual over $\tilde{X}$ to the cohomology as stated above. Of special interest is the above result for HC^0 since cyclic cocycles of degree zero are nothing but traces on the algebra $A_G^\hbar$, i.e., linear maps $\mathrm{tr} : A_G^\hbar \to \mathbb{C}((\hbar))$ satisfying

$$\mathrm{tr}(a \star_c b) = \mathrm{tr}(b \star_c a).$$

Therefore we find that

$$\dim_{\mathbb{C}((\hbar))}\{\text{space of traces}\} = \#\ \mathrm{Components}(\tilde{X}),$$

in particular is not one-dimensional when X is a nontrivial orbifold. This is in sharp contrast with the case of smooth symplectic manifolds where it is well-known that there is a unique trace up to normalization, cf. [1, 4].

Let $K^0_{\mathrm{orb}}(X)$ be the Grothendieck group of formal differences of orbifold vector bundles, sometimes called "orbifold K-theory". A trace tr in the sense above induces an index map

$$\mathrm{tr}_* : K^0_{\mathrm{orb}}(X) \to \mathbb{C}((\hbar))$$

as follows. By taking the trace of idempotents in matrix algebras over $A_G^\hbar$, one gets a map $\mathrm{tr}_* : K_0(A_G^\hbar) \to \mathbb{C}((\hbar))$. To obtain the index map from this, one uses the isomorphisms

$$K_0(A_G^\hbar) \cong K_0(A_G) \cong K^0_{\mathrm{orb}}(X).$$

Here the first isomorphism states that K-theory is "rigid" under deformation quantization, whereas the second can be viewed as a kind of Serre–Swan theorem for orbifolds. The algebraic index theorem for orbifolds gives a topological formula for the value of an index map associated to a trace on a K-theory class given by a pair (E, F) of orbifold vector bundles, isomorphic outside a compact subset of X.

Theorem 2 ([7]). *Let tr_α be a trace corresponding to the connected component $\tilde{X}_\alpha$ of $\tilde{X}_\alpha$. Let $E \to X$ be an orbifold vector bundle. Then, up to a constant,*

$$(\mathrm{tr}_\alpha)_*\left([E]-[F]\right) \propto \int_{\tilde{X}_\alpha} \frac{\mathrm{Ch}_\theta(R_E - R_F)}{\det\left(1-\theta^{-1}\exp(-R^\perp)\right)}\hat{A}(R^T)e^{\iota_\alpha^*\Omega/\hbar}.$$

In the formula above, the right hand side consists of the usual characteristic classes which can be explicitly represented by differential forms on $\tilde{X}$ by choosing a Riemannian metric with curvature R and connections on E and F. The map ι_α is the canonical embedding of the connected component $\tilde{X}_\alpha$ into X, and θ is the canonical automorphism acting on vector bundles normal to this embedding, so that one can twist the Chern character of E and F restricted to $\tilde{X}_\alpha$ and write down the denominator familiar from the equivariant index theorem.

Notice that the formula in the theorem is only given up to a constant, since thus far the trace is only determined up to normalization by its support on $\tilde{X}_\alpha$. When X is a manifold, i.e., $\tilde{X} = X$ and there is a unique trace, the normalization can easily be fixed and the theorem above reduces to the algebraic index theorem in [1, 4]. For an orbifold, the normalization issue is much more nontrivial because of the non-uniqueness of the trace, and the actual statement proved in [7] is much stronger: using the concept of a "twisted trace density", a canonical trace with support on $\tilde{X}_\alpha$ is constructed for which the normalization can explicitly be fixed. The resulting formula was first conjectured in [2].

Finally, one can obtain the the classical Kawasaki index theorem [3], by considering the deformation quantization $A^\hbar_{T^*X}$ of of the cotangent bundle T^*X of an orbifold X induced by the asymptotic pseudo-differential calculus. Using the Morita equivalence (1), one can compare the operator trace to the canonical trace above. As can be suspected from the index theorem, this trace has support on all connected components of $\tilde{X}$. Since the index of an elliptic operator on a compact orbifold is determined by its symbol, considered as a class in the (compactly supported) orbifold K-theory of T^*X, one derives the cohomological formula of the Kawasaki index from the algebraic index theorem, cf. [7].

References

[1] Fedosov, B.: *Deformation Quantization and Index Theory*, Akademie-Verlag, Berlin, 1996.

[2] Fedosov, B., B.W. Schulze and N. Tarkhanov: *On index theorem for symplectic orbifolds*, Ann. Inst. Fourier (Grenoble) **54**, no. 5, 1601–1639 (2004).

[3] Kawasaki, T.: *The index of elliptic operators over V-manifolds*, Nagoya Math. J. **84**, 135-157 (1981).

[4] Nest, R., and B. Tsygan: *Algebraic index theorem*, Comm. Math. Phys **172**, 223–262 (1995).

[5] Neumaier, N., M. Pflaum, H. Posthuma and X. Tang: *Homology of of formal deformations of proper étale Lie groupoids*. `arXiv:math.KT/0412462`, J. Reine Angew. Math. to appear.

[6] Pflaum, M.J.: *On the deformation quantization of symplectic orbispaces*, Diff. Geom. Appl. **19** 343–368 (2003).

[7] M. Pflaum, H. Posthuma, and X. Tang. An algebraic index theorem for orbifolds. `arXiv:math.KT/0507546` (2005)

[8] Tang, X.: *Deformation quantization of pseudo Poisson groupoids*, `arXiv:math.QA/0405378`, (2004).