LECTURE NOTES ON KOORNWINDER POLYNOMIALS

JASPER V. STOKMAN

Preface

The present contribution to the Proceedings of the SIAG OP-SF summer school 2000 on Orthogonal Polynomials and Special Functions gives a detailed account of my lectures at the summer school. It treats a very general family of classical, basic hypergeometric orthogonal polynomials in several variables, known as the Koornwinder polynomials. The basic properties of the Koornwinder polynomials are derived using recent powerful techniques of Cherednik, Macdonald, Koornwinder, Noumi, Sahi and others, in which the affine Hecke algebra plays an important role.

These lecture notes are accompagnied by a second, independent article in this volume which is written jointly with Masatoshi Noumi. In this second contribution we discuss the Hecke algebra techniques for the Askey-Wilson polynomials, which are the Koornwinder polynomials in one variable. The one variable set-up allows us to go beyond the derivation of the basic properties of the associated orthogonal polynomials, without putting to much effort in the necessary preparations; we are therefore able to derive more properties of the orthogonal polynomials than is achieved in the lecture notes for the multivariable set-up. Especially the readers who are not familiar to the affine Hecke algebra techniques, or who are well acquinted with the classical theory on basic hypergeometric orthogonal polynomials in one variable, are advised to read the second contribution first. The technicalities are less involved, and the connection with the familiar notations and results from basic hypergeometric series theory is made explicit.

1. Introduction

1.1. **Classical hypergeometric orthogonal polynomials.** If μ is a positive Borel measure on $\mathbb{R}$ with finite moments, then the corresponding orthogonal polynomials $\{p_n(\cdot)\}_{n\in\mathbb{Z}_+}$ satisfy a three term recurrence relation of the form

$$\begin{aligned} xp_k(x) &= a_kp_{k+1}(x) + b_kp_k(x) + c_kp_{k-1}(x), \qquad k \geq 1, \\ xp_0(x) &= a_0p_1(x) + b_0p_0(x) \end{aligned} \tag{1.1}$$

with $a_i, b_i, c_i \in \mathbb{R}$ and $a_ic_{i+1} > 0$.

In many applications of orthogonal polynomials in mathematics and physics, the orthogonal polynomials are in addition joint eigenfunctions of a second order differential operator. For instance, in one-dimensional quantum physical systems, the Hamiltonian is an (essentially self-adjoint) linear operator with respect to a positive Borel measure μ which is usually given explicitly by a second order differential operator. Its eigenvalues describe the discrete energy spectrum of the physical system, while the eigenfunctions represent the corresponding physical states. Sometimes the discrete states (which are automatically pair-wise orthogonal with respect to μ) can be represented by the corresponding orthogonal polynomials.

In this way, the Laguerre polynomials enter in the description of the energy spectrum of the hydrogen atom, the Hermite polynomials enter in the description of the energy spectrum of the harmonic oscillator, and the Jacobi polynomials enter in the description of the energy spectrum of the physical system of two particles on a closed wire with respect to certain square inverse potentials.

Mathematically, orthogonal polynomials which are in addition eigenfunctions of a second order differential operator occur for instance in representation theory and harmonic analysis. The basic example is Fourier analysis on the unit sphere (the theory of spherical harmonics), which is described in terms of a degenerate family of Jacobi polynomials (the so called Legendre polynomials). The corresponding second order differential operator then relates to the Laplace-Beltrami operator on the unit sphere.

Altogether, orthogonal polynomials which are joint eigenfunctions of a second order differential operator form an important sub-class of orthogonal polynomials, which nowadays are considered to be classical, see [1]. A classification of these families of orthogonal polynomials follows from Bochner's [3] paper: it consists (up to an affine linear transformation of the geometric variable) of the Hermite, Laguerre and Jacobi polynomials. These families are all expressible in terms of hypergeometric series.

1.2. **Classical orthogonal polynomials.** A richer class of orthogonal polynomials is obtained by requiring the orthogonal polynomials $\{p_n\}_{n\in\mathbb{Z}_+}$ to satisfy a three term recurrence relation of the form

$$\begin{aligned} (Lp_n)(x) &= \gamma_n p_n, \qquad n\in\mathbb{Z}_+,\\ (Lf)(x) &:= a(x)(f(qx)-f(x)) + b(x)(f(x/q)-f(x)), \end{aligned} \tag{1.2}$$

where $\gamma_n\in\mathbb{C}$, and $q\in\mathbb{C}\setminus\{0\}$ is an additional deformation parameter which we will assume to be generic. Orthogonal polynomials which in addition satisfy a second order q-difference equation of the form (1.2), can be expressed in terms of basic hypergeometric series.

In the terminology of Andrews and Askey [1], families of orthogonal polynomials satisfying a second order q-difference operator of the form (1.2) are also considered to be classical. The most general family of such classical basic hypergeometric orthogonal polynomials is the celebrated family of Askey-Wilson polynomials, which were introduced by Askey and Wilson in the famous memoir [2] in 1985. All other families of classical basic hypergeometric orthogonal polynomials are obtained from the Askey-Wilson polynomials by limit transitions or specializations. The corresponding hierarchy of classical basic hypergeometric orthogonal polynomials (known as the q-Askey scheme) is of a considerable size, see [18]. The classical hypergeometric orthogonal polynomials are obtained from the basic hypergeometric ones by sending the base q to one.

Classical basic hypergeometric orthogonal polynomials have been used extensively in mathematical physics and in representation theory. The physical background of the setting usually provides natural interpretations of the deformation parameter q, as well as of the "canonical" limit $q\to 1$. For instance, in problems related to quantization (i.e. the transition from classical to quantum mechanics), q is related to $e^{\hbar}$, where $\hbar$ is the Planck constant, so that $q\to 1$ corresponds with $\hbar\to 0$. In the transition from non-relativistic to relativistic models, q is related to $e^{1/c}$, where c is the speed of light, so that $q\to 1$ corresponds with $c\to\infty$.

1.3. **Self-dual orthogonal polynomials.** The Askey-Wilson polynomials p_n ($n \in \mathbb{Z}_+$), which are the most general classical basic hypergeometric orthogonal polynomials, see [12], satisfy a remarkable symmetry property, which is usually referred to as "duality". It roughly states that the spectral and geometric parameter of the Askey-Wilson polynomial are interchangeable. To be a little bit more precise at this stage: we take the Askey-Wilson polynomial $p_n(x)$ to be a Laurent polynomial in x, invariant under $x \leftrightarrow x^{-1}$, so that $p_n(x)$ ($n \in \mathbb{Z}_+$) is actually a set of orthogonal polynomials with respect to the variable $x + x^{-1}$ (With this convention one has to replace multiplication by x in (1.1) by multiplication by $x + x^{-1}$). Then for a suitable normalization of the Askey-Wilson polynomials, there exist complex numbers $x, \gamma \in \mathbb{C} \setminus \{0\}$ such that $p_n(xq^m) = \widetilde{p}_m(\gamma q^n)$ for all $m, n \in \mathbb{Z}_+$, where $\widetilde{p}_m$ is the Askey-Wilson polynomial of degree m with respect to a different ("dual") choice of parameters.

Observe that a family of orthogonal polynomials satisfying such a duality property is classical and of basic hypergeometric type, since application of duality to its three term recurrence relation (1.1) yields a second order q-difference equation of the form (1.2) for the "dual" orthogonal polynomials $\{\widetilde{p}_m\}_{m \in \mathbb{Z}_+}$! Hence families of "self-dual" orthogonal polynomials form a distinguished sub-class of the classical basic hypergeometric orthogonal polynomials.

1.4. **Macdonald polynomials.** Due to work of Cherednik and Macdonald, the class of self-dual orthogonal polynomials can be extended in a natural way to families of multivariable self-dual orthogonal polynomials (known nowadays as the Macdonald polynomials). The rich symmetry structure of the polynomials, which can be incorporated in an algebraic object called the double affine Hecke algebra, is a crucial tool for understanding the basic properties of these families of multivariable orthogonal polynomials. In fact, the existence of a large class of mutually commuting difference operators for which the Macdonald polynomials are joint eigenfunctions is the reason that the symmetry techniques are very powerful tools in unraveling the structure of the Macdonald polynomials.

The theory of Macdonald polynomials plays a fundamental role in several different branches of mathematics and mathematical physics, such as algebraic combinatorics, representation theory of quantum groups and affine Hecke algebras, quantum Khnizhnik-Zamolodohikov equations, conformal field theory, relativistic quantum integrable systems (Calogero-Moser systems), etc.

As an example, I will be a little bit more concrete here on the connection with Calogero-Moser systems. The Macdonald polynomials are q-analogs of the Heckman-Opdam polynomials, which are families of multivariable "classical" orthogonal polynomials generalizing the Jacobi polynomials. The Heckman-Opdam polynomials solve the (completely integrable) quantum physical system of a bosonic gas on a closed wire with pair-wise inverse quadratic potentials (the so-called quantum Calogero-Moser system); the second order differential operator for which the Heckman-Opdam polynomials are joint eigenfunctions is (up to a conjugation with the ground state) the Hamiltonian of the system. The Macdonald polynomials then have a similar interpretation, now in terms of a relativistic version of the quantum Calogero-Moser system, see e.g. Ruijsenaars & Schneider [28]. As in §1.2, the deformation parameter q is related to the speed of light c, in such a way that the classical limit $q \to 1$ corresponds with the non-relativistic limit $c \to \infty$.

1.5. **Koornwinder polynomials.** The main goal of these lectures is to explain Cherednik's and Macdonald's theory for Koornwinder's [17] multivariable generalization of the Askey-Wilson polynomials, known nowadays as the Koornwinder polynomials. Just as in the one-variable case, the family of Koornwinder polynomials may be seen as the "grand-father" who provided a considerable off-spring of degenerate families of multivariable analogs of classical (basic) hypergeometric orthogonal polynomials, see e.g. [33] and [11] (these form part of a multivariable generalization of the (q-)Askey scheme). In particular, most families of the Macdonald polynomials are special cases or limit cases of the family of Koornwinder polynomials.

So restricting our attention to the special class of Koornwinder polynomials does not provide a major loss of generality. On the other hand, it simplifies certain technical aspects of the theory, which hopefully helps the reader to get quicker acquainted with the theory.

1.6. **Literature.** The material treated in these lecture notes follow closely the papers of Noumi [26], Sahi [29], [30] and Stokman [32] (see also Noumi & Stokman [27] for the Askey-Wilson polynomials). The precise references of the material in the main body of the text will be postponed to the end of the lecture notes, see §10.

The only prerequisites for these lecture notes are some basic facts on Coxeter groups and Hecke algebras, which I will recall as soon as they are needed. I give precise reference to the literature for the further details on these results (I will mainly use Humphreys' book [14] as reference).

After a detailed introduction in §2 of the root data related to the theory of Koornwinder polynomials, I have added a long section §3 in which I give the main definitions, and state the main theorems. In order to justify definitions, some elementary proofs are already given at this stage. I hope that this section helps the reader to get acquainted with the main aims of the lecture notes, without being distracted by detailed proofs. The subsequent sections §4–§9 then provide the necessary details for a full understanding of the proofs.

Finally I would like to remark already at this point that various techniques in these lecture notes are similar to Cherednik's [5]–[9] and Macdonald's [22] treatment of Macdonald polynomials and affine Hecke algebras. Several excellent surveys on this theory are available now, see for instance Macdonald [23] and Kirillov Jr. [16]. Again, for further precise references to the literature, see §10.

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2. The root data

In this section we introduce the root systems and the corresponding Weyl groups which naturally arise in the theory of Koornwinder polynomials. Some relevant basic results about these root systems are discussed briefly and in a rather ad hoc fashion. As a basic reference for more details on root systems we use the book [14] of Humphreys.

2.1. **The root system Σ.** Let $\big(V=\mathbb{R}^n,(\cdot,\cdot)\big)$ be Euclidean n-space with fixed orthonormal basis $\{\epsilon_i\}_{i=1}^n$ with respect to the scalar product $(\cdot,\cdot)$ on V. We assume that $n\geq 2$ throughout the lecture notes. We write $|v|=\sqrt{(v,v)}$ for the norm of $v\in V$.

Consider the following finite sub-set $\Sigma\subset V$,

$$\Sigma=\{\pm 2\epsilon_i\}_{i=1}^n\cup\{\pm\epsilon_i\pm\epsilon_j\,|\,1\leq i<j\leq n\},\tag{2.1}$$

where all sign combinations occur. For any $v\in\Sigma$, let $H_v=\{w\in V\,|\,(v,w)=0\}$ be the hyperplane in V orthogonal to v, and denote $s_v\in\mathrm{GL}_{\mathbb{R}}(V)$ for the orthogonal reflection in H_v. Then

$$s_v(w)=w-\big(w,v^\vee\big)v,\qquad w\in V\tag{2.2}$$

where $v^\vee=2v/|v|^2$ is the co-root of v. Observe that s_v is an involution, i.e. $s_v^2=1$ ($v\in\Sigma$), and that s_v is orthogonal with respect to $(\cdot,\cdot)$, i.e. $\big(s_v(w),s_v(w')\big)=\big(w,w'\big)$ for all $w,w'\in V$.

Let $W=W(\Sigma)\subset\mathrm{GL}_{\mathbb{R}}(V)$ be the sub-group generated by the orthogonal reflections s_v ($v\in\Sigma$). The following lemma can be checked by direct computations.

Lemma 2.1. *$\Sigma\subset V$ is a so called root system, i.e.*

- *Σ is a finite set which spans V,*
- *$\big(\alpha,\beta^\vee\big)\in\mathbb{Z}$ for all $\alpha,\beta\in\Sigma$,*
- *$w(\Sigma)=\Sigma$ for all $w\in W$.*

Elements $\alpha\in\Sigma$ are called *roots*, and the group $W=W(\Sigma)$ is called the *Weyl group* of Σ. In the standard terminology on root systems (see for instance [14]), Σ is the root system of type C_n.

Let $\Pi_0=\{a_1,\ldots,a_n\}\subset\Sigma$ be the sub-set of roots

$$a_i:=\epsilon_i-\epsilon_{i+1}\ \ (i=1,\ldots,n-1),\qquad a_n:=2\epsilon_n.\tag{2.3}$$

Lemma 2.2. **(i)** *Any $\alpha\in\Sigma$ can be uniquely written as a $\mathbb{Z}_+$-linear or a $\mathbb{Z}_-$-linear combination of the roots $a_i\in\Pi_0$ ($i=1,\ldots,n$).*

(ii) *The Weyl group W is generated (as a group) by the reflections $s_i:=s_{a_i}$ ($i=1,\ldots,n$).*

Proof. **(i)** Direct verification.

(ii) This is a standard property of root systems, see [14, Thm. 1.5]. We sketch here an ad hoc proof for the root system Σ.

Observe that the simple reflection s_i ($i=1,\ldots,n-1$) acts on V by interchanging ϵ_i and ϵ_{i+1} and keeping ϵ_j ($j\neq i,i+1$) fixed. In particular, the sub-group of W generated by the simple reflections $s_1,\ldots,s_{n-1}$ act as the permutation group S_n in n letters on the standard basis $\{\epsilon_i\}_{i=1}^n$. On the other hand, s_n maps ϵ_n to $-\epsilon_n$ and keeps ϵ_j fixed ($j=1,\ldots,n-1$). Hence the sub-group $W'\subset W$ generated by the reflections s_i ($i=1,\ldots,n$) is naturally isomorphic to $S_n\ltimes(\pm1)^n$, where S_n acts on $(\pm1)^n$ by permuting its entries.

From this description of W' it is now easy to check that $s_\alpha\in W'$ for all $\alpha\in\Sigma$, hence $W\subset W'$. The other inclusion being obvious, we obtain the desired result. $\square$

The set $\Pi_0\subset\Sigma$ is called a *basis* of the root system Σ, and the elements $a_i\in\Pi_0$ are called the *simple roots* of Σ. The choice of basis Π_0 induces a natural decomposition of Σ in positive roots Σ^+ and negative roots $\Sigma^-:=-\Sigma^+$, where Σ^+

is the set of roots $\alpha \in \Sigma$ which can be written as a $\mathbb{Z}_+$-linear combination of the simple roots. The reflections s_i $(i = 1, \ldots, n)$ are called the *simple reflections* in W with respect to Π_0.

Lemma 2.3. *Σ decomposes into two W-orbits, namely $\Sigma = \Sigma_l \cup \Sigma_m$, where*

$$\begin{aligned}\Sigma_l &= Wa_n = \{\pm 2\epsilon_i \,|\, i = 1, \ldots, n\},\\ \Sigma_m &= Wa_k = \{\pm\epsilon_i \pm \epsilon_j \,|\, 1 \leq i < j \leq n\}, \qquad (k \in \{1, \ldots, n-1\} \text{ arbitrary}),\end{aligned}$$

where all sign combinations occur.

Proof. An ad hoc proof can be easily given using the description of W as the semi-direct product $S_n \ltimes (\pm 1)^n$, see the proof of lemma 2.2.

The lemma is also a direct consequence of a general fact from the theory of irreducible root systems, which says that the decomposition of Σ in W-orbits coincides with the decomposition of Σ according to the norm of the roots, see [15, Lem. 10.4C]. The lemma follows from this, since Σ_l (respectively Σ_m) is the set of roots in Σ of squared length 4 (respectively 2). □

2.2. **The reduced affine root system R.** Let $\widehat{V}$ be the set of affine linear mappings from V to $\mathbb{R}$. As a vector-space over $\mathbb{R}$, $\widehat{V}$ is isomorphic to $V \bigoplus \mathbb{R}\delta$, where $\delta : V \to \mathbb{R}$ is the affine linear function defined by $\delta(v) := 1$ for all $v \in V$. This follows from the fact that any affine linear transformation $f \in \widehat{V}$ can be written as

$$f(w) = \big(v, w\big) + \lambda\delta(w), \qquad w \in V$$

for some $v \in V$ and some $\lambda \in \mathbb{R}$.

We extend the scalar product $\big(\cdot, \cdot\big)$ on V to a positive semi-definite bilinear form on $\widehat{V}$ by setting

$$\big(v + \lambda\delta, w + \mu\delta\big) = \big(v, w\big), \qquad v, w \in V, \quad \lambda, \mu \in \mathbb{R}. \tag{2.4}$$

In particular, the norm $|f| = \sqrt{\big(f, f\big)}$ of $f \in \widehat{V}$ is zero if and only if f is a constant function.

Fix a non-constant function $f \in \widehat{V} \setminus \mathbb{R}\delta$. We define an involution $s_f \in \mathrm{GL}_{\mathbb{R}}(\widehat{V})$ by

$$s_f(g) = g - \big(g, f^\vee\big)f, \qquad g \in \widehat{V}, \tag{2.5}$$

where $f^\vee = 2f/|f|^2$ is the co-root of f. It can be easily checked that $s_f(g) = g \circ \widetilde{s}_f$, where $\widetilde{s}_f : V \to V$ is the orthogonal reflection in the affine hyperplane $f^{-1}(\{0\}) \subset V$. Let $R \subset \widehat{V}$ be the sub-set

$$R = \Sigma + \mathbb{Z}\delta, \tag{2.6}$$

and let $\mathcal{W} = \mathcal{W}(R) \subset \mathrm{GL}_{\mathbb{R}}(\widehat{V})$ be the sub-group generated by the reflections s_f $(f \in R)$. Alternatively, we may think of $\mathcal{W}$ as the sub-group of affine linear transformations of V generated by the orthogonal affine reflections $\widetilde{s}_f$ $(f \in R)$. Observe that $W \subset \mathcal{W}$ is a sub-group in a natural way: it acts on V as in the previous subsection, and it fixes the constant functions.

For $v \in V$, we let $\tau(v) \in \mathrm{GL}_{\mathbb{R}}(\widehat{V})$ be given by

$$\tau(v)f = f + \big(f, v\big)\delta, \qquad f \in \widehat{V}.$$

Observe that $\tau(v)f = f \circ \widetilde{\tau}(v)$, where $\widetilde{\tau}(v) : V \to V$ is the affine linear mapping given by $\widetilde{\tau}(v)(w) = w - v$ for all $w \in V$. We write

$$\Lambda_0 = \bigoplus_{i=1}^{n} \mathbb{Z}\epsilon_i \subset V,$$

which is a W-stable $\mathbb{Z}$-lattice in V.

Lemma 2.4. $\mathcal{W} \simeq W \ltimes \tau(\Lambda_0)$.

Proof. Observe that $w\tau(\lambda) = \tau(w\lambda)w$ for $w \in W$ and $\lambda \in \Lambda_0$, and that W has trivial intersection with $\tau(\Lambda_0)$ since W fixes the constant functions. Hence the sub-group of $\mathrm{GL}_{\mathbb{R}}(\widehat{V})$ generated by W and $\tau(\Lambda_0)$ is naturally isomorphic to the semi-direct product $W \ltimes \tau(\Lambda_0)$.

Let $f = \alpha + c\delta \in R \setminus \mathbb{R}\delta$ with $\alpha \in \Sigma$ and $c \in \mathbb{Z}$. Then

$$s_f(g) = g - \big(g, f\big)f^\vee = g - \big(g, \alpha\big)\alpha^\vee - c\big(g, \alpha^\vee\big)\delta = s_\alpha\big(\tau(-c\alpha^\vee)(g)\big). \qquad (2.7)$$

Since $\alpha^\vee \in \Lambda_0$ for all $\alpha \in \Sigma$, we see that $\mathcal{W} \subset W\tau(\Lambda_0)$.

On the other hand, for any $\alpha \in \Sigma$ it follows from (2.7) that $\tau(-\alpha^\vee) = s_\alpha s_{\alpha+\delta} \in \mathcal{W}$. Since the co-roots $\alpha^\vee$ ($\alpha \in \Sigma$) span Λ_0 (indeed, $\epsilon_j = (2\epsilon_j)^\vee$ for $j = 1, \ldots, n$ already span Λ_0), we conclude that $\tau(\Lambda_0) \subset \mathcal{W}$. Since also $W \subset \mathcal{W}$, we conclude that $W\tau(\Lambda_0) \subset \mathcal{W}$. Hence $\mathcal{W} \simeq W \ltimes \tau(\Lambda_0)$, as desired. □

Remark 2.5. We have used in the proof of lemma 2.4 that Λ_0 is the so-called *co-root lattice* of Σ, i.e.

$$\Lambda_0 = \mathbb{Z} - \mathrm{span}\{\alpha^\vee \mid \alpha \in \Sigma\}.$$

It is also easily verified that Λ_0 is the so-called *weight lattice* of Σ, i.e.

$$\Lambda_0 = \{\lambda \in V \mid \langle \lambda, \alpha^\vee \rangle \in \mathbb{Z}, \quad \forall \alpha \in \Sigma\}.$$

Lemma 2.6. *$R \subset \widehat{V}$ is an affine root system, i.e.*

- *R does not contain constant functions and spans $\widehat{V}$,*
- *$\big(f, g^\vee\big) \in \mathbb{Z}$ for all $f, g \in R$,*
- *$w(R) = R$ for all $w \in \mathcal{W}$,*
- *$\mathcal{W}$ acts properly on V. In other words, for all $K_1, K_2 \subset V$ compact, there are only finitely many $w \in \mathcal{W}$ for which $w(K_1)$ has non-empty intersection with K_2.*

Proof. Only the third property requires proof (the second property is a direct consequence of lemma 2.1). By lemma 2.1, $w(\Sigma) = \Sigma$ for all $w \in W$. Since W fixes constant functions, we obtain $w(R) = R$ for all $w \in W$. Let now $\lambda \in \Lambda_0$. Then $\tau(\lambda)(R) = R$ if $\big(\lambda, \beta\big) \in \mathbb{Z}$ for all $\beta \in \Sigma$. This again follows from lemma 2.1, since any $\lambda \in \Lambda_0$ can be written as a $\mathbb{Z}$-linear combination of the co-roots $\alpha^\vee$ ($\alpha \in \Sigma$), cf. remark 2.5. The structure of $\mathcal{W}$ as described in lemma 2.4 now shows that $w(R) = R$ for all $w \in \mathcal{W}$, as desired. □

The sub-group $\mathcal{W} = \mathcal{W}(R) \subset \mathrm{GL}_{\mathbb{R}}(\widehat{V})$ is called the *affine Weyl group* associated with the affine root system R.

Let $\Pi = \{a_0, a_1, \ldots, a_n\} = \{a_0\} \cup \Pi_0 \subset R$, where

$$a_0 = \delta - 2\epsilon_1 \in R. \qquad (2.8)$$

We have now the following analogue of lemma 2.2.

Lemma 2.7. **(i)** *Any $f \in R$ can be uniquely written as a $\mathbb{Z}_+$-linear or a $\mathbb{Z}_-$-linear combination of the roots $a_i \in \Pi$ $(i = 0, \ldots, n)$.*

(ii) *The affine Weyl group $\mathcal{W}$ is generated (as a group) by the reflections $s_i := s_{a_i}$ $(i = 0, \ldots, n)$.*

Proof. **(i)** Let $f \in R$. If $f(0) = 0$, then $f \in \Sigma$, and we can use lemma 2.2. If $f(0) = m \in \mathbb{Z}_{>0}$, then $f = \alpha + m\delta$ for some $\alpha \in \Sigma$. But then $f = (\alpha + 2m\epsilon_1) + ma_0$, and $\alpha + 2m\epsilon_1$ is a $\mathbb{Z}_+$-linear combination of the simple roots Π_0 of Σ^+. Hence if $f(0) > 0$, then f can be written as a $\mathbb{Z}_+$-linear combination of the roots $a_i \in \Pi$ $(i = 0, \ldots, n)$. Finally if $f(0) < 0$, then $-f \in R$ satisfies $(-f)(0) > 0$, so we can use the previous result to show that f can be written as a $\mathbb{Z}_-$-linear combination of the roots $a_i \in \Pi$ $(i = 0, \ldots, n)$.

(ii) This is again a general fact on affine root systems, see [14, Prop. 4.3]. For convenience, we give here another, ad hoc proof in case of the affine root system R.

Let $\mathcal{W}' \subset \mathcal{W}$ be the sub-group generated by the reflections s_i $(i = 0, \ldots, n)$. Then $\mathcal{W}'$ contains W by lemma 2.2. By (2.7), $\tau(\epsilon_1) = s_{\epsilon_1} s_0 \in \mathcal{W}'$, and consequently

$$\tau(\epsilon_j) = s_{j-1} \cdots s_2 s_1 \tau(\epsilon_1) s_1 s_2 \cdots s_{j-1} \in \mathcal{W}' \tag{2.9}$$

for all $j = 1, \ldots, n$. By lemma 2.4, we conclude that $\mathcal{W} \subset \mathcal{W}'$. Hence $\mathcal{W}' = \mathcal{W}$, as desired. $\square$

In analogy with finite root systems, we call the set $\Pi \subset R$ a *basis* of the affine root system R, and the elements $a_i \in R$ the *simple roots* of R. The choice of basis Π induces a decomposition of R in positive roots R^+ and negative roots $R^- = -R^+$, where R^+ is the set of roots $f \in R$ which can be written as a $\mathbb{Z}_+$-linear combination of the simple roots $a_i \in \Pi$ $(i = 0, \ldots, n)$. Observe that by the proof of lemma 2.7**(i)**,

$$R^+ = \Sigma^+ \cup \{f \in R \,|\, f(0) > 0\}. \tag{2.10}$$

Remark 2.8. The connected components of the complement $V \backslash \cup \{f^{-1}(\{0\}) \,|\, f \in R\}$ of the hyperplane configuration $\cup\{f^{-1}(\{0\}) \,|\, f \in R\}$ are called chambers. For a fixed chamber C, we let $\Pi(C) \subset R$ be the set of roots $f \in R$ for which the affine hyperplane $f^{-1}(\{0\})$ is a wall of C, and which takes positive values in the chamber C. Then $\Pi(C)$ is called a basis of the affine root system R. The fixed basis Π which we have chosen in this section corresponds to the chamber

$$C = \{v = \sum_{i=1}^{n} v_i \epsilon_i \mid 0 < v_n < v_{n-1} < \cdots < v_1 < 1/2\}.$$

It is known that any two choices of basis of R are conjugate to each other under $\mathcal{W}$, see [14, Chapter 4] for more details.

The $\mathcal{W}$-orbit structure of the affine root system R can now be given explicitly as follows.

Lemma 2.9. *There are three $\mathcal{W}$-orbits in R, namely*

$$\begin{aligned}
R_m &= \mathcal{W}a_k = \Sigma_m + \mathbb{Z}\delta, \qquad (k \in \{1, \ldots, n-1\} \text{ arbitrary}) \\
R_l^1 &= \mathcal{W}a_0 = \Sigma_l + (1 + 2\mathbb{Z})\delta, \\
R_l^2 &= \mathcal{W}a_n = \Sigma_l + 2\mathbb{Z}\delta.
\end{aligned}$$

Proof. We fix an arbitrary $k \in \{1, \ldots, n-1\}$. Then $\tau(m\epsilon_k)a_k = a_k + m\delta$ for all $m \in \mathbb{Z}$, hence $\tau(\Lambda_0)a_k = a_k + \mathbb{Z}\delta$. By lemma 2.3 and lemma 2.4, it then follows that

$$\mathcal{W}a_k = Wa_k + \mathbb{Z}\delta = \Sigma_m + \mathbb{Z}\delta.$$

For $a_0 = \delta - 2\epsilon_1$, we have $\tau(\Lambda_0)a_0 = a_0 + (1+2\mathbb{Z})\delta$, hence by lemma 2.3 and by lemma 2.4,

$$\mathcal{W}a_0 = W2\epsilon_1 + (1+2\mathbb{Z})\delta = \Sigma_l + (1+2\mathbb{Z})\delta.$$

In a similar fashion one can show that $\mathcal{W}a_n = \Sigma_l + 2\mathbb{Z}\delta$. The lemma now follows since the three orbits $\mathcal{W}a_0, \mathcal{W}a_k$ and $\mathcal{W}a_n$ are pair-wise different and they exhaust the affine root system R. □

Proposition 2.10. *The simple reflections s_i $(i = 0, \ldots, n)$ satisfy the braid relations*

$$\begin{aligned} s_is_{i+1}s_is_{i+1} &= s_{i+1}s_is_{i+1}s_i, \qquad i = 0, n-1,\\ s_is_{i+1}s_i &= s_{i+1}s_is_{i+1}, \qquad i = 1, \ldots, n-2,\\ s_is_j &= s_js_i, \qquad |i-j| > 1. \end{aligned}$$

In fact, the braid relations together with the quadratic relations $s_i^2 = 1$ $(i = 0, \ldots, n)$ give a presentation of the affine Weyl group $\mathcal{W}$.

Proof. The braid relations for the s_i $(i = 0, \ldots, n)$ are easily checked by hand.

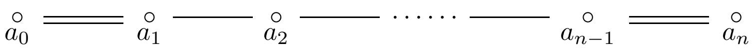
If $\mathcal{W}'$ is the abstract group with generators s_i $(i = 0, \ldots, n)$ satisfying the braid relations and the quadratic relations $s_i^2 = 1$ $(i = 0, \ldots, n)$, then $\mathcal{W}'$ is an example of a *Coxeter group*. The canonical, surjective group homomorphism $\mathcal{W}' \to \mathcal{W} \subset \mathrm{GL}_{\mathbb{R}}(\widehat{V})$ is called the geometric representation of the Coxeter group $\mathcal{W}'$ on $\widehat{V}$. It is a fundamental result in the theory of Coxeter groups that its geometric representation is faithful (see for instance [14, Cor 5.4]), which implies the second statement of the proposition.

□

Remark 2.11. The information of the affine Weyl group $\mathcal{W}$ can be depicted by a so-called *extended Dynkin diagram.* The vertices are labeled by the simple roots $\Pi = \{a_0, \ldots, a_n\}$. We draw 0, 1 and 2 edges between the vertices a_i and a_j if the braid relation of the corresponding simple roots s_i and s_j are of the form

$$s_is_js_i\cdots \quad (m(i,j) \text{ terms}) = s_js_is_j\cdots \quad (m(i,j) \text{ terms})$$

with $m(i,j) = 2, 3$ and 4, respectively. Thus the extended Dynkin diagram of $\mathcal{W}$ looks like



2.3. **The non-reduced affine root system R_{nr}.** The present choice of affine root system R is still not sufficient for our purposes. In order to arrive at the Koornwinder level, we need to add two more $\mathcal{W}$-orbits to R in the following way.

We set $R_s^i = \frac{1}{2}R_l^i$ for $i = 1, 2$, so

$$\begin{aligned} R_s^1 &= \mathcal{W}a_0/2 = \{\pm\epsilon_j + (\frac{1}{2} + \mathbb{Z})\delta \,|\, j = 1, \ldots, n\},\\ R_s^2 &= \mathcal{W}a_n/2 = \{\pm\epsilon_j + \mathbb{Z}\delta \,|\, j = 1, \ldots, n\}. \end{aligned} \tag{2.11}$$

The union of R with these two $\mathcal{W}$-orbits is denoted by R_{nr}, so

$$R_{nr} = R_s^1 \cup R_s^2 \cup R = \{\pm\epsilon_i + \frac{m}{2}\delta, \pm 2\epsilon_i + m\delta \,|\, m \in \mathbb{Z},\, i = 1, \ldots, n\} \\ \cup \{\pm\epsilon_i \pm \epsilon_j + m\delta \,|\, m \in \mathbb{Z},\; 1 \leq i < j \leq n\}. \tag{2.12}$$

We now have the following extension of lemma 2.6.

Lemma 2.12. *The set $R_{nr} \subset \widehat{V}$ is an affine root system.*

Proof. By the previous observations, we only need to check that $\big(f, g^\vee\big) \in \mathbb{Z}$ for $f, g \in R_{nr}$. But this is an immediate consequence of the fact that

$$\Sigma_{nr} = \frac{1}{2}\Sigma_l \cup \Sigma_m \cup \Sigma_l \subset V$$

is a (finite) root system in V. In fact, since the lattice Λ_0 is the co-root lattice of the root system Σ (see remark 2.5) and $\langle \Lambda_0, a_n\rangle = 2\mathbb{Z}$, we immediately have that $\langle \alpha, \beta^\vee\rangle \in \mathbb{Z}$ for all $\alpha, \beta \in \Sigma_{nr}$. □

Remark 2.13. The (finite) root system Σ and the affine root system R are *reduced*, in the sense that for any $\alpha \in \Sigma$ (respectively $f \in R$),

$$\mathbb{R}\alpha \cap \Sigma = \pm\alpha, \qquad (\mathbb{R}f \cap R = \pm f, \;\text{ respectively}).$$

The (finite) root system Σ_{nr} and the affine root system R_{nr} do not satisfy this property, and are therefore called *non-reduced*. Observe that Σ is the set of unmultiplyable roots in Σ_{nr} (i.e. $\Sigma = \{\alpha \in \Sigma_{nr} \,|\, 2\alpha \notin \Sigma_{nr}\}$). Similarly, R is the set of unmultiplyable roots in R_{nr}.

3. Statement of the main theorems

3.1. **The fundamental action of $\mathcal{W}$.** Let $x_1, \ldots, x_n$ be n independent indeterminates and write

$$\mathcal{A} = \mathbb{C}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$$

for the corresponding algebra of Laurent polynomials. We occasionally use the notations $p = p(x) = p(x_1, \ldots, x_n)$ for a Laurent polynomial $p \in \mathcal{A}$.

Associated with the lattice element $\lambda = \sum_i \lambda_i \epsilon_i \in \Lambda_0$ ($\lambda_i \in \mathbb{Z}$) we have the monomial

$$x^\lambda = x_1^{\lambda_1} x_2^{\lambda_2} \cdots x_n^{\lambda_n}.$$

The set of monomials $\{x^\lambda \,|\, \lambda \in \Lambda_0\}$ form a linear basis of $\mathcal{A}$.

Let $\Lambda \subset \widehat{V}$ be the lattice

$$\Lambda = \Lambda_0 + \frac{1}{2}\mathbb{Z}\delta \subset \widehat{V}. \tag{3.1}$$

Observe that the lattice $\Lambda \subset \widehat{V}$ is stable under the action of $\mathcal{W} \subset \mathrm{GL}_{\mathbb{R}}(\widehat{V})$, and that $R_{nr} \subset \Lambda$.

We fix now a generic parameter $q \in \mathbb{C} \setminus \{0\}$, and we write $q^{\frac{1}{2}}$ for its square root with respect to the branch of $\sqrt{\cdot}$ which is positive on $\mathbb{R}_{>0}$. We now define

$$x^{\lambda + \frac{m}{2}\delta} = q^{\frac{m}{2}} x^\lambda \in \mathcal{A}, \qquad \lambda \in \Lambda_0,\;\; m \in \mathbb{Z}.$$

Lemma 3.1. *The map $w(x^\lambda)=x^{w\lambda}$ for $w\in\mathcal{W}$ and $\lambda\in\Lambda_0$ extends by linearity to a well defined left action of $\mathcal{W}$ on $\mathcal{A}$ (so $(ww')p=w(w'p)$ for all $w,w'\in\mathcal{W}$ and all $p\in\mathcal{A}$). For any Laurent polynomial $p\in\mathcal{A}$, we have*

$$\begin{aligned}\big(s_0p\big)(x)&=p(qx_1^{-1},x_2,\ldots,x_n),\\ \big(s_ip\big)(x)&=p(x_1,\ldots,x_{i-1},x_{i+1},x_i,x_{i+2},\ldots,x_n),\qquad (i=1,\ldots,n-1),\\ \big(s_np\big)(x)&=p(x_1,\ldots,x_{n-1},x_n^{-1}).\end{aligned}$$

Furthermore, the elements $\tau(\lambda)\in\mathcal{W}$ ($\lambda\in\Lambda_0$) act as q-difference operators: $\tau(\lambda)(x^\mu)=q^{(\lambda,\mu)}x^\mu$ for all $\mu\in\Lambda_0$.

Proof. We observe that $w(\lambda+m\delta)=w\lambda+m\delta$ for all $w\in\mathcal{W}$, $\lambda\in\Lambda_0$ and $m\in\frac{1}{2}\mathbb{Z}$. In particular, the endomorphism $\phi_w\in\mathrm{End}_{\mathbb{C}}(\mathcal{A})$ ($w\in\mathcal{W}$) defined by $\phi_w(x^\lambda)=x^{w\lambda}$ for all $\lambda\in\Lambda_0$ actually satisfies $\phi_w(x^\mu)=x^{w\mu}$ for all $\mu\in\Lambda$. Hence for all $\lambda\in\Lambda_0$ and all $w,w'\in\mathcal{W}$ we have

$$\phi_w\big(\phi_{w'}(x^\lambda)\big)=\phi_w(x^{w'\lambda})=x^{w(w'\lambda)}=x^{(ww')\lambda}=\phi_{ww'}(x^\lambda),$$

i.e. $\phi_w\phi_{w'}=\phi_{ww'}$ for all $w,w'\in\mathcal{W}$. This proves the first statement of the lemma.

The remaining formulas are now direct consequences of the explicit formulas for the action of $\mathcal{W}$ on $\Lambda\subset\widehat{V}$. In particular, for $\tau(\epsilon_1)$, we have by (2.7) that $\tau(\epsilon_1)=s_{\epsilon_1}s_0$, so that $\tau(\epsilon_1)x^\lambda=q^{(\lambda,\epsilon_1)}x^\lambda$. Now by (2.9), this leads to $\tau(\epsilon_j)x^\lambda=q^{(\lambda,\epsilon_j)}x^\lambda$ for all j. □

Remark 3.2. Observe that the action of $\mathcal{W}$ on $\mathcal{A}$ which we defined in the previous lemma, depends on the deformation parameter q. Since the action of the finite Weyl group W on $\mathcal{A}$ is independent of the deformation parameter q, we may think of the parameter q as an extra degree of freedom which is associated to the translation part $\tau(\Lambda_0)$ of the affine Weyl group $\mathcal{W}$.

Remark 3.3. Observe that $\mathcal{W}$ acts on $\mathcal{A}$ by algebra automorphisms. This implies that the action of $\mathcal{W}$ uniquely extends to an action by automorphisms on the quotient field $\mathcal{Q}$ of $\mathcal{A}$. Here the quotient field $\mathcal{Q}$ consists of the rational functions in the n indeterminates $x_1,\ldots,x_n$.

3.2. **Noumi's difference-reflection operators.** We call a complex valued function $\mathbf{t}=\{t_f\,|\,f\in R_{nr}\}$ on R_{nr} a *multiplicity function* if $t_{wf}=t_f$ for all $w\in\mathcal{W}$ and all $f\in R_{nr}$. By the $\mathcal{W}$-orbit structure of R_{nr} (see lemma 2.9 and §2.3), we see that $\mathbf{t}$ is completely determined by the five values

$$\begin{aligned}t_0&:=t_{a_0},\qquad t_n:=t_{a_n}\\ t&:=t_k:=t_{a_k}\qquad (k\in\{1,\ldots,n-1\}\ \text{arbitrary}),\\ t_0^\vee&:=t_{a_0/2},\qquad t_n^\vee:=t_{a_n/2}.\end{aligned}\tag{3.2}$$

On the other hand, it is obvious that any choice of parameters $(t_0,t_n,t,t_0^\vee,t_n^\vee)\in\mathbb{C}^5$ uniquely extends to a multiplicity function $\mathbf{t}=\{t_f\,|\,f\in R_{nr}\}$. We assume throughout the lecture notes that the values t_f ($f\in R_{nr}$) of the multiplicity function $\mathbf{t}$ are generically complex.

We define now rational functions $c_f = c_f(\cdot;\mathbf{t}|q) \in \mathcal{Q}$ for $f \in R$ by

$$c_f(x) = \begin{cases} \dfrac{(1 - t_f t_{f/2} x^{f/2})(1 + t_f t_{f/2}^{-1} x^{f/2})}{(1 - x^f)}, & \text{if } f/2 \in R_{nr}, \\ \dfrac{(1 - t_f^2 x^f)}{(1 - x^f)}, & \text{if } f/2 \notin R_{nr}. \end{cases} \tag{3.3}$$

It is convenient to have a uniform formula for c_f. For this purpose we extend the multiplicity function $\mathbf{t}$ to a complex valued function on $\widehat{V}$ by declaring $t_f := 1$ if $f \in \widehat{V} \setminus R_{nr}$. Then we can formally write

$$c_f(x) = \frac{\big(1 - t_f t_{f/2} x^{f/2}\big)\big(1 + t_f t_{f/2}^{-1} x^{f/2}\big)}{\big(1 - x^f\big)}, \qquad \forall f \in R. \tag{3.4}$$

In order to give a completely rigorous meaning to the right hand side of (3.4) for $f \in R$ with $f/2 \notin R_{nr}$, one should consider it as element in the algebra $\mathcal{Q}^{\frac{1}{2}}$ of rational functions in $x_i^{\frac{1}{2}}$ $(i = 1, \ldots, n)$, where $x_i^{\frac{1}{2}}$ is the formal square root of the indeterminate x_i (which contains $\mathcal{Q}$ as a sub-field in a natural way).

As we shall see in §4 and §5, the rational functions c_f $(f \in R)$ naturally appear in the description of the algebra of symmetries associated with the Koornwinder polynomials. This interpretation of the rational functions c_f $(f \in R)$ will imply various special identities in $\mathcal{Q}$, some of which can also be easily verified directly. We give here two examples of such identities.

Lemma 3.4. **(i)** *For all $f \in R$ and $w \in \mathcal{W}$, we have $wc_f = c_{wf}$.*
(ii) *For all $f \in R$, we have $c_f + c_{-f} = t_f^2 + 1$ in $\mathcal{Q}$.*

Proof. **(i)** This follows from the definition of the action of $\mathcal{W}$ on $\mathcal{A}$ (see lemma 3.1), and from the fact that $t_f = t_{wf}$ for all $f \in R$ and $w \in \mathcal{W}$.

(ii) This follows by a direct computation, using that $t_f = t_{s_f f} = t_{-f}$ for all $f \in R$. □

For $f \in R$, Noumi's difference-reflection operator $T_f = T_f(\mathbf{t}|q) \in \mathrm{End}_{\mathbb{C}}(\mathcal{Q})$ is defined by

$$\big(T_f p\big)(x) = t_f p(x) + t_f^{-1} c_f(x)\big((s_f p)(x) - p(x)\big), \qquad p \in \mathcal{Q}. \tag{3.5}$$

We observe at this stage the following three elementary properties for the difference-reflection operators T_f.

Lemma 3.5. *Let $f \in R$.*
(i) *The restriction of T_f to $\mathcal{A}$ maps into $\mathcal{A}$ (i.e. $T_f|_{\mathcal{A}} \in End_{\mathbb{C}}(\mathcal{A})$).*
(ii) *The difference-reflection operator T_f satisfies the quadratic relation*

$$\big(T_f - t_f\big)\big(T_f + t_f^{-1}\big) = 0.$$

In particular, T_f is invertible, with inverse $T_f^{-1} = T_f - t_f + t_f^{-1}$.
(iii) *For all $w \in \mathcal{W}$ and $f \in R$, we have $wT_f w^{-1} = T_{wf}$ in $End_{\mathbb{C}}(\mathcal{Q})$.*

Proof. **(i)** Let $f \in R$. We define a linear operator $D_f \in \mathrm{End}_{\mathbb{C}}(\mathcal{Q})$ by

$$\big(D_f p\big)(x) = \frac{p(x) - \big(s_f p\big)(x)}{1 - x^f}, \qquad p \in \mathcal{Q}. \tag{3.6}$$

In order to show that $T_f|_{\mathcal{A}} \in \mathrm{End}_{\mathbb{C}}(\mathcal{A})$, it suffices to prove that $D_f|_{\mathcal{A}} \in \mathrm{End}_{\mathbb{C}}(\mathcal{A})$.

Fix an arbitrary $\lambda \in \Lambda_0$. Then $s_f(\lambda) = \lambda - m\, f$ with $m = \big(\lambda, f^\vee\big) \in \mathbb{Z}$, hence

$$x^\lambda - s_f(x^\lambda) = x^\lambda\big(1 - x^{-mf}\big).$$

Dividing the right hand side by $1 - x^f$ then shows that

$$D_f(x^\lambda) = \begin{cases} -x^{\lambda-f} - x^{\lambda-2f} - \cdots - x^{\lambda-(\lambda,f^\vee)f} & \text{if } \; m = \big(\lambda, f^\vee\big) > 0, \\ 0 & \text{if } \; m = \big(\lambda, f^\vee\big) = 0, \\ x^\lambda + x^{\lambda+f} + \cdots + x^{\lambda-(1+(\lambda,f^\vee))f} & \text{if } \; m = \big(\lambda, f^\vee\big) < 0. \end{cases} \tag{3.7}$$

In particular, $D_f|_{\mathcal{A}} \in \mathrm{End}_{\mathbb{C}}(\mathcal{A})$, as desired.

(ii) This follows from a direct computation using lemma 3.4.

(iii) Use that $t_{wf} = t_f$ and $ws_fw^{-1} = s_{wf}$ for all $w \in \mathcal{W}$ and $f \in R$. $\square$

Observe that the quadratic relations $(T_f - t_f)(T_f + t_f^{-1}) = 0$ for $f \in R$ can be seen as deformations of $s_f^2 = 1$ for the reflection $s_f \in \mathcal{W}$ since $T_f(\mathbf{1}|q) = s_f$, where $\mathbf{1} = \{1_f\}_{f \in R_{nr}}$ is the multiplicity function identically equal to one. The other crucial property of the difference-reflection operators to which we return to at a later stage (see §4.4), is that $T_i := T_{a_i}$ $(i = 0, \ldots, n)$ satisfy the same braid relations as the simple reflections s_i $(i = 0, \ldots, n)$, see proposition 2.10. The powerful implications of these relations for the T_i's will become apparent in §4 and §5.

3.3. **Cherednik-Dunkl type Y-operators.**

The quadratic relations and braid relations for Noumi's difference reflection operators T_i $(i = 0, \ldots, n)$ imply that they behave in a similar way as the simple reflections s_i $(i = 0, \ldots, n)$.

For instance the simple reflections s_i $(i = 0, \ldots, n)$ generate the affine Weyl group $\mathcal{W}$, which contains the large abelian sub-group $\tau(\Lambda_0)$ of translations over the $\mathbb{Z}$-lattice Λ_0 generated by the standard basis $\{\epsilon_i\}_{i=1}^n$. In a similar fashion, the operators T_i $(i = 0, \ldots, n)$ give rise to n algebraically independent, invertible, pair-wise commuting operators $Y_i \in \mathrm{End}_{\mathbb{C}}(\mathcal{A})$ $(i = 1, \ldots, n)$ in the following way.

For $i = 1, \ldots, n$, we can express the translation operator $\tau(\epsilon_i)$ in terms of simple reflections s_j $(j = 0, \ldots, n)$ by

$$\tau(\epsilon_i) = s_i \cdots s_{n-1}s_ns_{n-1} \cdots s_1s_0s_1 \cdots s_{i-1}. \tag{3.8}$$

We will see in §4 that this is in fact a *reduced expression*, which means that $\tau(\epsilon_i)$ cannot be written as a product of simple reflections s_j $(j = 0, \ldots, n)$ in less than $2n$ terms.

The *Cherednik-Dunkl type* Y-operator Y_i $(i = 1, \ldots, n)$ is now defined in terms of difference-reflection operators T_j $(j = 0, \ldots, n)$ by the expression

$$Y_i = T_i \cdots T_{n-1}T_nT_{n-1} \cdots T_1T_0T_1^{-1}T_2^{-1} \cdots T_{i-1}^{-1} \tag{3.9}$$

(observe the close resemblance with the reduced expression for $\tau(\epsilon_i)$ as given in (3.8)!). The result to which we were already referring to, is the following theorem.

Theorem 3.6. *The operators $Y_i \in End_{\mathbb{C}}(\mathcal{A})$ $(i = 1, \ldots, n)$ pair-wise commute.*

3.4. **Non-symmetric Koornwinder polynomials.** The non-symmetric Koornwinder polynomials can be defined as the common eigenfunctions of the commuting Cherednik-Dunkl Y-operators Y_i $(i=1,\ldots,n)$ in the following way. Let Σ_m^+ (respectively Σ_l^+) be the positive roots in Σ of squared length 2 (respectively 4). For any $\lambda\in\Lambda_0$, we define $\rho_m(\lambda),\rho_l(\lambda)\in\Lambda_0$ by

$$\rho_m(\lambda)=\sum_{\alpha\in\Sigma_m^+}\mathrm{sgn}\big((\lambda,\alpha)\big)\alpha^\vee,\qquad \rho_l(\lambda)=\sum_{\alpha\in\Sigma_l^+}\mathrm{sgn}\big((\lambda,\alpha)\big)\alpha^\vee, \tag{3.10}$$

where $\mathrm{sgn}:\mathbb{Z}\to\{\pm1\}$ maps a positive integer to 1 and a strictly negative integer to -1. Let $\gamma_\lambda=\gamma_\lambda(\mathbf{t}|q)\in\mathbb{C}^n$ $(\lambda\in\Lambda_0)$ be the vector with ith coordinate given by

$$\gamma_{\lambda,i}=\big(t_0t_n\big)^{(\rho_l(\lambda),\epsilon_i)}t^{(\rho_m(\lambda),\epsilon_i)}q^{(\lambda,\epsilon_i)},\qquad i=1,\ldots,n. \tag{3.11}$$

Theorem 3.7. *There exists a unique basis $\{P_\lambda=P_\lambda(\cdot;\mathbf{t}|q)\,|\,\lambda\in\Lambda_0\}$ of $\mathcal{A}$ for which the basis element P_λ $(\lambda\in\Lambda_0)$ satisfies the following two properties:*

- *$Y_iP_\lambda=\gamma_{\lambda,i}\,P_\lambda$ for all $i=1,\ldots,n$.*
- *The coefficient of the monomial x^λ in the expansion of P_λ as linear combination of monomials x^μ $(\mu\in\Lambda_0)$, is equal to one.*

Definition 3.8. *The Laurent polynomial $P_\lambda=P_\lambda(\cdot;\mathbf{t}|q)$ is called the monic, non-symmetric Koornwinder polynomial of degree $\lambda\in\Lambda_0$.*

Remark 3.9. The Koornwinder polynomial P_λ is a deformation of the monomial x^λ $(\lambda\in\Lambda_0)$. Indeed, recall that for the multiplicity function $\mathbf{t}=\mathbf{1}$ identically equal to one, we have $T_i(\mathbf{1}|q)=s_i$ for $i=0,\ldots,n$, and hence $Y_i=\tau(\epsilon_i)$ for $i=1,\ldots,n$. Furthermore, $\gamma_{\lambda,i}(\mathbf{1}|q)=q^{(\lambda,\epsilon_i)}$. By lemma 3.1, we conclude that $P_\lambda(x;\mathbf{1}|q)=x^\lambda$ for all $\lambda\in\Lambda_0$.

3.5. **Duality.** We associate a dual multiplicity function $\tilde{\mathbf{t}}$ to the multiplicity function $\mathbf{t}$ by interchanging the value of $\mathbf{t}$ at the $\mathcal{W}$-orbit $\mathcal{W}a_0$ with its value at the $\mathcal{W}$-orbit $\mathcal{W}a_n^\vee$. In other words, $\tilde{\mathbf{t}}$ is the unique multiplicity function such that

$$\tilde{t}_0=t_n^\vee,\qquad \tilde{t}_0^\vee=t_0^\vee,\qquad \tilde{t}=t,\qquad \tilde{t}_n^\vee=t_0,\qquad \tilde{t}_n=t_n. \tag{3.12}$$

We use the short-hand notation $x_\lambda=\gamma_\lambda(\tilde{\mathbf{t}}|q)\in\mathbb{C}^n$ $(\lambda\in\Lambda_0)$ for the spectrum of the Y-operators with respect to dual parameters. Since $P_\lambda(\cdot;\mathbf{t}|q)$ depends meromorphically on the parameters $\mathbf{t}$, and $P_\lambda(\gamma_0(\mathbf{1}|q)^{-1};\mathbf{1}|\,q)\neq0$ in view of remark 3.9, we see that $P_\lambda(x_0^{-1})=P_\lambda(x_0^{-1};\mathbf{t}|q)\neq0$ for generic parameters $\mathbf{t}$. This justifies the following definition.

Definition 3.10. *Let $\lambda\in\Lambda_0$. The renormalized non-symmetric Koornwinder polynomial $E(\gamma_\lambda;\cdot)=E(\gamma_\lambda;\cdot;\mathbf{t}|\,q)\in\mathcal{A}$ of degree λ is defined by*

$$E\big(\gamma_\lambda;x\big)=\frac{P_\lambda(x)}{P_\lambda(x_0^{-1})}. \tag{3.13}$$

In other words, $E(\gamma_\lambda;\cdot)$ is the constant multiple of the monic, non-symmetric Koornwinder polynomial $P_\lambda(\cdot)$ which takes the value one at $x=x_0^{-1}$.

We use the short-hand notation $\widetilde{E}(x_\lambda;\cdot)$ for the renormalized non-symmetric Koornwinder polynomial $E\big(\gamma_\lambda(\tilde{\mathbf{t}}|q);\cdot;\tilde{\mathbf{t}}|\,q\big)=E\big(x_\lambda;\cdot;\tilde{\mathbf{t}}|\,q\big)$ with respect to dual parameters.

The renormalized non-symmetric Koornwinder polynomials satisfy the following crucial symmetry property.

Theorem 3.11 (Duality)**.** *The renormalized non-symmetric Koornwinder polynomials satisfy*

$$E\big(\gamma_\lambda; x_\mu^{-1}\big) = \widetilde{E}\big(x_\mu; \gamma_\lambda^{-1}\big), \qquad \forall \lambda, \mu \in \Lambda_0.$$

As we shall see in §6, the duality for non-symmetric Koornwinder polynomials stems from an important algebraic property of the so-called *double affine Hecke algebra* $\mathcal{H}$, which is the sub-algebra of $\mathrm{End}_{\mathbb{C}}(\mathcal{A})$ generated by Noumi's difference-reflection operators T_j $(j = 0, \ldots, n)$ and $\mathcal{A}$ itself, considered as multiplication operators in $\mathrm{End}_{\mathbb{C}}(\mathcal{A})$. The relevant algebraic property is encoded by an isomorphism of $\mathcal{H}$ to the double affine Hecke algebra $\widetilde{\mathcal{H}}$ with respect to dual parameters which interchanges the Y-operators with the multiplication operators $\mathcal{A} \subset \mathrm{End}_{\mathbb{C}}(\mathcal{A})$.

3.6. **Bi-orthogonality relations.** The bi-orthogonality relations for the non-symmetric Koornwinder polynomials are defined with respect to an explicit complex valued weight function $\Delta(\cdot) = \Delta(\cdot; \mathbf{t}|\, q)$ on a compact n-torus. This weight function can be naturally expressed in terms of an infinite product of the coefficients c_f occurring in Noumi's difference-reflection operators T_f $(f \in R)$. In order to ensure convergence of this infinite product, we need to restrict our attention to the case that the deformation parameter q has modulus strictly less than one. The weight function $\Delta(\cdot) = \Delta(\cdot; \mathbf{t}|\, q)$ is then defined as

$$\Delta(x) = \Delta(x; \mathbf{t}|\, q) = \prod_{f \in R^+} \frac{1}{c_f(x; \mathbf{t}|\, q)}. \tag{3.14}$$

Using the fact that

$$R^+ = \Sigma^+ \cup \{f \in R \,|\, f(0) > 0\} = \{f \in R \,|\, f(0) \geq 0\} \setminus \Sigma^-,$$

we can decompose the weight function $\Delta(\cdot)$ accordingly as

$$\Delta(x) = \mathcal{C}(x)\Delta_+(x), \tag{3.15}$$

where the function $\mathcal{C}(x) = \mathcal{C}(x; \mathbf{t}|q)$ and $\Delta_+(x) = \Delta_+(x; \mathbf{t}|q)$ are given by

$$\mathcal{C}(x) = \prod_{\alpha \in \Sigma^-} c_\alpha(x), \qquad \Delta_+(x) = \prod_{f \in R:\, f(0) \geq 0} \frac{1}{c_f(x)}. \tag{3.16}$$

The upshot of this decomposition of $\Delta(x)$ is that the factor $\Delta_+(x)$ is W-invariant in a natural way. To be more precise, if we extend the action of W on $\mathcal{Q}$ (see lemma 3.1) to an action on sufficiently nice functions h in the n variables $x_1, \ldots, x_n$ by the formulas

$$\begin{aligned} (s_0 h)(x) &= h(q x_1^{-1}, x_2, \ldots, x_n), \\ (s_i h)(x) &= h(x_1, \ldots, x_{i-1}, x_{i+1}, x_i, x_{i+2}, \ldots, x_n), \qquad i = 1, \ldots, n-1, \\ (s_n h)(x) &= h(x_1, \ldots, x_{n-1}, x_n^{-1}), \end{aligned} \tag{3.17}$$

then it follows from the W-invariance of the set $\{f \in R \,|\, f(0) \geq 0\}$ and from lemma 3.4**(i)** that the weight Δ_+ is invariant under the action of W.

To get a better understanding of the explicit form of $\Delta(x)$ and $\Delta_+(x)$, and to convince ourselves that they are well defined, we rewrite $\Delta_+(x)$ now in terms of q-shifted factorials, which are defined by

$$\big(a; q\big)_k = (1-a)(1-aq)\cdots(1-aq^{k-1}), \qquad k \in \mathbb{Z}_+ \cup \{\infty\},$$

where we use the convention that empty products are equal to one. Observe here that the infinite product $\big(a; q\big)_\infty = \prod_{i=0}^\infty (1 - aq^i)$ is well defined by the assumption that $|q|<1$. We also use the following short-hand notations for product of q-shifted factorials:

$$\left(a_1,\ldots,a_r;q\right)_k=\left(a_1;q\right)_k\left(a_2;q\right)_k\cdots\left(a_r;q\right)_k,\qquad k\in\mathbb{Z}_+\cup\{\infty\}.$$

Furthermore, it will be convenient to use the following reparametrization of the multiplicity function $\mathbf{t}$,

$$\{a,b,c,d\}=\{t_0t_0^\vee q^{1/2},-t_0t_0^{\vee -1}q^{1/2},t_nt_n^\vee,-t_nt_n^{\vee -1}\},\tag{3.18}$$

which allows us to rewrite the coefficient c_f of Noumi's difference-reflection operator T_f $(f\in R)$ in the following explicit way:

$$c_f(x)=\begin{cases}\dfrac{(1-ax^{\alpha/2}q^m)(1-bx^{\alpha/2}q^m)}{(1-x^\alpha q^{2m+1})}, & \text{if } f=\alpha+(2m+1)\delta\in\mathcal{W}a_0,\\ \dfrac{(1-t^2x^\alpha q^m)}{(1-x^\alpha q^m)}, & \text{if } f=\alpha+m\delta\in\mathcal{W}a_k,\\ \dfrac{(1-cx^{\alpha/2}q^m)(1-dx^{\alpha/2}q^m)}{(1-x^\alpha q^{2m})}, & \text{if } f=\alpha+2m\delta\in\mathcal{W}a_n,\end{cases}\tag{3.19}$$

where $k\in\{1,\ldots,n-1\}$ is arbitrary, $m\in\mathbb{Z}$ and $\alpha\in\Sigma$ (to be more precise: $\alpha\in\Sigma_l$ in case that $f\in\mathcal{W}a_0$ and $f\in\mathcal{W}a_n$, and $\alpha\in\Sigma_m$ in case that $f\in\mathcal{W}a_k$).

Lemma 3.12. *The W-invariant part $\Delta_+(x)$ of the function $\Delta(x)$ can be rewritten as*

$$\begin{aligned}\Delta_+(x)=&\prod_{i=1}^n\frac{\left(x_i^2,x_i^{-2};q\right)_\infty}{\left(ax_i,ax_i^{-1},bx_i,bx_i^{-1},cx_i,cx_i^{-1},dx_i,dx_i^{-1};q\right)_\infty}\\ &\times\prod_{1\le i<j\le n}\frac{\left(x_ix_j,x_ix_j^{-1},x_i^{-1}x_i,x_i^{-1}x_j^{-1};q\right)_\infty}{\left(t^2x_ix_j,t^2x_ix_j^{-1},t^2x_i^{-1}x_j,t^2x_i^{-1}x_j^{-1};q\right)_\infty}.\end{aligned}$$

Proof. By (3.19) and by the explicit $\mathcal{W}$-orbit structure of R, see lemma 2.9, we have for arbitrary $k\in\{1,\ldots,n-1\}$:

$$\begin{aligned}\prod_{f\in\mathcal{W}a_0\colon f(0)\ge0}\frac{1}{c_f(x)}&=\prod_{i=1}^n\frac{\left(qx_i^2,qx_i^{-2};q^2\right)_\infty}{\left(ax_i,ax_i^{-1},bx_i^{-1},bx_i^{-1};q\right)_\infty},\\ \prod_{f\in\mathcal{W}a_k\colon f(0)\ge0}\frac{1}{c_f(x)}&=\prod_{1\le i<j\le n}\frac{\left(x_ix_j,x_ix_j^{-1},x_i^{-1}x_j,x_i^{-1}x_j^{-1};q\right)_\infty}{\left(t^2x_ix_j,t^2x_ix_j^{-1},t^2x_i^{-1}x_j,t^2x_i^{-1}x_j^{-1};q\right)_\infty},\\ \prod_{f\in\mathcal{W}a_n\colon f(0)\ge0}\frac{1}{c_f(x)}&=\prod_{i=1}^n\frac{\left(x_i^2,x_i^{-2};q^2\right)_\infty}{\left(cx_i,cx_i^{-1},dx_i^{-1},dx_i^{-1};q\right)_\infty}.\end{aligned}$$

The lemma now follows from the obvious identity $\left(y;q^2\right)_\infty\left(qy;q^2\right)_\infty=\left(y;q\right)_\infty$ for q-shifted factorials. $\square$

Remark 3.13. The function

$$w(y)=\frac{\left(y^2,y^{-2};q\right)_\infty}{\left(ay,ay^{-1},by,by^{-1},cy,cy^{-1},dy,dy^{-1};q\right)_\infty}$$

which occurs as a factor of the W-invariant weight function $\Delta_+(x)$ for every coordinate $y=x_i$ $(i=1,\ldots,n)$ is exactly the weight function of the one variable Askey-Wilson polynomials. In particular, it is the t-dependent factor of $\Delta_+(x)$

which provides the non-trivial multivariable extension of the one variable Askey-Wilson weight function (for $t = 1$, we are simply left with the coordinate-wise product of the Askey-Wilson weight function).

In order to keep the exposition as simple as possible, we now assume that all parameters a, b, c, d and t have moduli less than one. More flexible conditions can be allowed here, which causes though additional technical complications. Under these conditions on the parameters, we can check without difficulty that $\Delta_+(x)$ and $\Delta(x)$ depend analytically on $x \in \mathbb{T}^n$, where $\mathbb{T} = \{y \in \mathbb{C} \,|\, |y| = 1\}$ is the unit circle in the complex plane.

We can thus define a bilinear form $\langle \cdot, \cdot \rangle = \langle \cdot, \cdot \rangle_{\mathbf{t},q}$ on $\mathcal{A}$ by

$$\langle p_1, p_2 \rangle = \frac{1}{(2\pi i)^n} \iint\limits_{x \in \mathbb{T}^n} p_1(x) p_2(x^{-1}) \Delta(x) \frac{dx}{x}, \qquad p_1, p_2 \in \mathcal{A}, \tag{3.20}$$

where $x^{-1} = (x_1^{-1}, \ldots, x_n^{-1})$, $\frac{dx}{x} = \frac{dx_1}{x_1} \cdots \frac{dx_n}{x_n}$ and $\mathbb{T}$ is positively oriented.

In the following theorem we use the short-hand notation

$$E'\big(\gamma_\lambda^{-1}; \cdot\big) := E\big(\gamma_\lambda^{-1}; \cdot; \mathbf{t}^{-1} |\, q^{-1}\big), \qquad \lambda \in \Lambda_0 \tag{3.21}$$

for the renormalized non-symmetric Koornwinder polynomials with respect to inverse parameters, where $\mathbf{t}^{-1} = \{t_f^{-1} \,|\, f \in R_{nr}\}$ (observe that (3.21) makes sense since $\gamma_\lambda(\mathbf{t}^{-1} |\, q^{-1}) = \gamma_\lambda^{-1}$ for all $\lambda \in \Lambda_0$).

Theorem 3.14 (Bi-orthogonality)**.** *For $\lambda, \mu \in \Lambda_0$ with $\lambda \neq \mu$, we have*

$$\langle E(\gamma_\lambda; \cdot), E'(\gamma_\mu^{-1}; \cdot) \rangle = 0.$$

The main ingredient of the proof is to show that the Y-operators are (in a suitable sense) self-adjoint with respect to the bilinear form $\langle \cdot, \cdot \rangle$. The bi-orthogonality relations follow then from the fact that the non-symmetric Koornwinder polynomials diagonalize the Y-operators, and that the spectrum of the Y-operators is simple.

3.7. **Diagonal terms.** In view of the bi-orthogonality relations for the non-symmetric Koornwinder polynomials (see theorem 3.14), it is natural to study the diagonal terms $\langle E(\gamma_\lambda; \cdot), E'(\gamma_\lambda^{-1}; \cdot) \rangle$ for all $\lambda \in \Lambda_0$. These diagonal terms can be expressed in terms of multiple residues of the weight function $\Delta(\cdot)$ in the following way.

Recall that the lattice Λ_0 can be interpreted as the weight lattice of the root system Σ, see remark 2.5. From this interpretation of the lattice Λ_0, we can define the cone Λ_0^+ of dominant weight by

$$\Lambda_0^+ = \{\lambda \in \Lambda_0 \,|\, \big(\lambda, \alpha^\vee\big) \in \mathbb{Z}_+ \qquad \forall \alpha \in \Sigma^+\}. \tag{3.22}$$

The cone Λ_0^+ exactly corresponds with the partitions of length $\leq n$, i.e.

$$\Lambda_0^+ = \{\sum_{i=1}^n \lambda_i \epsilon_i \in \Lambda_0 \,|\, \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq 0\}.$$

Since $W \simeq S_n \ltimes (\pm 1)^n$ via the restriction of the canonical action of W on V (see the proof of lemma 2.2), we have that the W-orbit of any $\lambda \in \Lambda_0$ intersects the cone of dominant weights Λ_0^+ in exactly one element $\lambda^+ \in \Lambda_0^+$:

$$\Lambda_0^+ \cap W\lambda = \{\lambda^+\}, \qquad \lambda \in \Lambda_0. \tag{3.23}$$

This fact holds true for general finite root systems, see [14, Thm. 1.12]. The element $\gamma_\lambda \in \mathbb{C}^n$ (see (3.11)) of the spectrum of the Y-operators simplify for $\lambda \in \Lambda_0^+$ dominant since then $\mathrm{sgn}\big((\lambda,\alpha)\big) = 1$ for all $\alpha \in \Sigma^+$. In particular, $\rho_m(\lambda)$ and $\rho_l(\lambda)$ respectively are equal to ρ_m and ρ_l for all dominant weights $\lambda \in \Lambda_0^+$, where ρ_m and ρ_l are given by

$$\rho_m = 2\sum_{i=1}^n (n-i)\epsilon_i, \qquad \rho_l = \sum_{i=1}^n \epsilon_i. \tag{3.24}$$

We obtain

$$\gamma_\lambda = \big(t_0t_nt^{2(n-1)}q^{\lambda_1}, t_0t_nt^{2(n-2)}q^{\lambda_2}, \ldots, t_0t_nq^{\lambda_n}\big), \qquad \lambda \in \Lambda_0^+, \tag{3.25}$$

where we write $\lambda_i = \big(\lambda, \epsilon_i\big) \in \mathbb{Z}_+$ for all $i = 1, \ldots, n$. Going over to dual parameters, we get

$$x_\lambda = \big(t_n^\vee t_nt^{2(n-1)}q^{\lambda_1}, t_n^\vee t_nt^{2(n-2)}q^{\lambda_2}, \ldots, t_n^\vee t_nq^{\lambda_n}\big), \qquad \lambda \in \Lambda_0^+. \tag{3.26}$$

We can now define the multiple residue $w_+(x_\lambda^{-1}) = w_+(x_\lambda^{-1}; \mathbf{t}|\, q)$ for $\lambda \in \Lambda_0^+$ by

$$w_+(x_\lambda^{-1}) = \operatorname*{Res}_{x_1=x_{\lambda,1}^{-1}}\left(\operatorname*{Res}_{x_2=x_{\lambda,2}^{-1}}\left(\cdots \operatorname*{Res}_{x_n=x_{\lambda,n}^{-1}}\left(\frac{\Delta_+(x)}{x_1\cdots x_n}\right)\cdots\right)\right), \tag{3.27}$$

where $x_{\lambda,i}$ is the ith coordinate of $x_\lambda \in \big(\mathbb{C}\setminus\{0\}\big)^n$.

Lemma 3.15. *The discrete weights $w_+(x_\lambda^{-1})$ ($\lambda \in \Lambda_0^+$) are non-zero for generic values of the multiplicity function $\mathbf{t}$.*

Proof. We fix $\lambda \in \Lambda_0^+$. Recall that this implies that the coefficients $\lambda_i \in \mathbb{Z}$ of λ satisfy $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq 0$. We define now n distinct positive roots $f_1, \ldots, f_n \in R^+$ by

$$f_i = a_i + (\lambda_i - \lambda_{i+1})\delta \quad (i = 1, \ldots, n-1), \quad f_n = a_n + 2\lambda_n\delta. \tag{3.28}$$

The new variables

$$y_i = x^{f_i} = q^{\lambda_i - \lambda_{i+1}}x_ix_{i+1}^{-1} \quad (i = 1, \ldots, n-1), \quad y_n = x^{f_n/2} = q^{\lambda_n}x_n$$

generate the Laurent algebra $\mathcal{A}$, i.e. $\mathcal{A} = \mathbb{C}[y_1^{\pm 1}, \ldots, y_n^{\pm 1}]$. We translate now the definition of $w_+(x_\lambda^{-1})$ in terms of these new variables $y = (y_1, \ldots, y_n)$. Observe that the residue point $x = x_\lambda^{-1}$ corresponds with

$$(t^{-2}, t^{-2}, \ldots, t^{-2}, t_n^{\vee -1}t_n^{-1}) \in \big(\mathbb{C}\setminus\{0\}\big)^n$$

in the y-coordinates under the above change of variables. Combined with the definition of c_f (see (3.3)), we see that $c_f(x_\lambda^{-1}) = 0$ for $f \in R^+$ iff $f = f_i$ for some $i = 1, \ldots, n$. Furthermore,

$$\begin{aligned} c_{f_i}(x) &= \frac{(1-t^2y_i)}{(1-y_i)} =: c_i(y_i) \qquad (i = 1, \ldots, n-1),\\ c_{f_n}(x) &= \frac{(1-t_nt_n^\vee y_n)(1+t_nt_n^{\vee -1}y_n)}{(1-y_n^2)} =: c_n(y_n), \end{aligned}$$

so that $c_i(y_i)$ has a simple zero at $y_i = t^{-2}$ for $i = 1, \ldots, n-1$ and $c_n(y_n)$ has a simple zero at $y_n = t_n^{\vee -1} t_n^{-1}$. Hence

$$\begin{aligned}\operatorname*{Res}_{x_1 = x_{\lambda,1}^{-1}} \left(\operatorname*{Res}_{x_2 = x_{\lambda,2}^{-1}} \left(\cdots \operatorname*{Res}_{x_n = x_{\lambda,n}^{-1}} \left(\prod_{i=1}^{n} \frac{1}{c_{f_i}(x) x_i} \right) \cdots \right) \right) = \\ = \operatorname*{Res}_{y_n = (t_n^\vee)^{-1} t_n^{-1}} \left(\frac{1}{c_n(y_n) y_n} \right) \prod_{i=1}^{n-1} \operatorname*{Res}_{y_i = t^{-2}} \left(\frac{1}{c_i(y_i) y_i} \right) = \\ = \frac{\left(t_n^{\vee -2} t_n^{-2} - 1\right)}{\left(1 + t_n^{\vee -2}\right)} (t^{-2} - 1)^{n-1},\end{aligned}$$

so that $w_+(x_\lambda^{-1})$ can be written as

$$w_+(x_\lambda^{-1}) = \frac{\left(t_n^{\vee -2} t_n^{-2} - 1\right)}{\left(1 + t_n^{\vee -2}\right)} (t^{-2} - 1)^{n-1} \prod_{f \in R_{\geq 0} \setminus \{f_1, \ldots, f_n\}} c_f(x_\lambda^{-1})^{-1}, \tag{3.29}$$

where $R_{\geq 0} = \{ f \in R \,|\, f(0) \geq 0 \}$. In particular, $w_+(x_\lambda^{-1})$ is non-zero for generic values of the multiplicity function $\mathbf{t}$. $\square$

We have dealt now with the multiple residue of the W-invariant part $\Delta_+(x)$ of $\Delta(x)$ at the residue point $x = x_\lambda^{-1}$ for $\lambda \in \Lambda_0^+$. We use this to define the multiple residue $w(x_\lambda^{-1}) = w(x_\lambda^{-1}; \mathbf{t} | \, q)$ of $\Delta(x)$ at $x = x_\lambda^{-1}$ ($\lambda \in \Lambda_0$) by

$$w(x_\lambda^{-1}) = \mathcal{C}(x_\lambda^{-1}) w_+(x_{\lambda^+}^{-1}), \qquad \lambda \in \Lambda_0. \tag{3.30}$$

This formula should be compared with the analogous decomposition (3.15) of the complex weight function $\Delta(x)$ into its W-invariant part $\Delta_+(x)$ and the correction term $\mathcal{C}(x)$. Observe that $w(x_\lambda^{-1})$ is non-zero by the previous lemma, since $\mathcal{C}(x)$ is regular and non-zero at $x = x_\lambda^{-1}$ for generic values of the multiplicity function $\mathbf{t}$.

Let $\widetilde{w}(\gamma_\lambda^{-1}) = w(\gamma_\lambda^{-1}; \tilde{\mathbf{t}} | \, q)$ be the discrete weight (3.30) with respect to the dual multiplicity function $\tilde{\mathbf{t}}$. The diagonal terms $\langle E(\gamma_\lambda; \cdot), E'(\gamma_\lambda^{-1}; \cdot) \rangle$ can now be expressed in the following way.

Theorem 3.16 (Diagonal terms)**.** *For all* $\lambda \in \Lambda_0$*, we have*

$$\frac{\langle E(\gamma_\lambda; \cdot), E'(\gamma_\lambda^{-1}; \cdot) \rangle}{\langle 1, 1 \rangle} = \frac{\widetilde{w}(\gamma_0^{-1})}{\widetilde{w}(\gamma_\lambda^{-1})}.$$

There are two essentially different ways to proceed with the proof of this theorem. The first way is by relating the diagonal terms to the quadratic norms of the symmetric Koornwinder polynomials (see the next subsection for the definition of the symmetric Koornwinder polynomials), and then using so-called *shift operators* to shift the parameters to the trivial case $\mathbf{t} = \mathbf{1}$. This approach has the advantage that it also leads to an evaluation of $\langle 1, 1 \rangle$ (which is essentially Gustafson's [13] multivariable q-analogue of the beta-integral). This method does not lead though to the expression of the diagonal terms in terms of residues of the weight function in a natural way.

The second method, which we follow in these lecture notes, makes use of the double affine Hecke algebra $\mathcal{H}$ and the duality of the non-symmetric Koornwinder polynomials. The proof is based on the following observation. Let $\lambda \in \Lambda_0$ and

$p \in \mathcal{A}$. Then the Laurent polynomial $p(x)E'(\gamma_\lambda^{-1}; x^{-1}) \in \mathcal{A}$ can again be written as linear combination of the non-symmetric Koornwinder polynomials:

$$p(x)E'(\gamma_\lambda^{-1}; x^{-1}) = \sum_\mu d_p^\lambda(\mu) E'(\gamma_\mu^{-1}; x^{-1}), \qquad d_p^\lambda(\mu) \in \mathbb{C}.$$

The upshot of duality is that multiplication of $E'(\gamma_\lambda^{-1}; x^{-1})$ by $p(x)$ can be rewritten in terms of a linear operator acting on the spectral parameter γ_λ^{-1} of the Koornwinder polynomial. In particular, the coefficients $d_p^\lambda(\mu)$ can be realized as coefficients of explicit difference-reflection operators acting on functions with support on the discrete spectrum $\{\gamma_\mu^{-1} \,|\, \mu \in \Lambda_0\}$, which make them computable in certain specific cases. We will show that this line of argument naturally leads to the explicit expressions

$$d^\lambda_{E(\gamma_\lambda;\cdot)}(0) = \frac{\widetilde{w}(\gamma_0^{-1})}{\widetilde{w}(\gamma_\lambda^{-1})}, \qquad \lambda \in \Lambda_0,$$

which is equivalent to the expression of the diagonal term $\langle E(\gamma_\lambda; \cdot), E'(\gamma_\lambda^{-1}; \cdot)\rangle$ as given in theorem 3.16.

3.8. **Symmetric Koornwinder polynomials.** We now discuss the symmetric Koornwinder polynomials, which satisfy orthogonality relations with respect to the symmetric part $\Delta_+(x)$ of the weight function $\Delta(x)$. We emphasize in this preliminary introduction the close connections with the non-symmetric theory, as well as with the classical orthogonal polynomial theory (as discussed in the introduction of these lecture notes).

Let $\mathcal{A}^W$ be the subalgebra of $\mathcal{A}$ consisting of Laurent polynomials $p \in \mathcal{A}$ satisfying $wp = p$ for all $w \in W$. In view of (3.23), we have a linear basis $m_\lambda(x)$ ($\lambda \in \Lambda_0^+$) of $\mathcal{A}^W$, where m_λ is the orbit sum defined by

$$m_\lambda(x) = \sum_{\mu \in W\lambda} x^\mu, \qquad \lambda \in \Lambda_0^+.$$

For any $\lambda \in \Lambda_0$, say $\lambda = \sum_i \lambda_i \epsilon_i$ with $\lambda_i \in \mathbb{Z}$, we set

$$Y^\lambda = Y_1^{\lambda_1} Y_2^{\lambda_2} \cdots Y_n^{\lambda_n}$$

which is well defined since the Y-operators Y_i ($i = 1, \ldots, n$) pair-wise commute and are invertible. We can extend this construction linearly, by setting

$$p(Y) = \sum_\lambda d_\lambda Y^\lambda, \qquad \big(p(x) = \sum_\lambda d_\lambda x^\lambda \in \mathcal{A}\big).$$

Theorem 3.17. *For any $p \in \mathcal{A}^W$, we have that $p(Y)|_{\mathcal{A}^W} \in End_{\mathbb{C}}(\mathcal{A}^W)$. Furthermore, for generic values of the multiplicity function $\mathbf{t}$, there exists a unique basis $\{P_\lambda^+ = P_\lambda^+(\cdot; \mathbf{t}|\, q) \,|\, \lambda \in \Lambda_0^+\}$ of $\mathcal{A}^W$ satisfying the two properties*

- *$p(Y)P_\lambda^+ = p(\gamma_\lambda)P_\lambda^+$ for all $p \in \mathcal{A}^W$,*
- *The coefficient of m_λ in the expansion of P_λ^+ as linear combination of orbit sums m_μ ($\mu \in \Lambda_0^+$), is equal to one.*

Definition 3.18. *The W-invariant Laurent polynomial P_λ^+ ($\lambda \in \Lambda_0^+$) is called the monic (symmetric) Koornwinder polynomial of degree λ.*

Any $p(Y)$ with $p \in \mathcal{A}$ can be realized as a difference-reflection operator with rational coefficients, i.e. $p(Y)$ is a finite $\mathcal{Q}$-linear combination of operators $\tau(\lambda)w$ with $\lambda \in \Lambda_0$ and $w \in W$. Restricted to $\mathcal{A}^W$, we thus see that $p(Y)|_{\mathcal{A}^W}$ can be realized as a q-difference operator with coefficients in the field $\mathcal{Q}$. In particular, $\{p(Y)|_{\mathcal{A}^W} \mid p \in \mathcal{A}^W\}$ is a commutative sub-algebra of $\mathrm{End}_{\mathbb{C}}(\mathcal{A}^W)$ consisting of q-difference operators with coefficients in $\mathcal{Q}$, which is diagonalized by the symmetric Koornwinder polynomials.

As an example, we give in the following theorem the explicit form of the q-difference operator

$$L := \big(m_{\epsilon_1}(Y) - m_{\epsilon_1}(\gamma_0)\big)|_{\mathcal{A}^W}.$$

The term $m_{\epsilon_1}(\gamma_0)$ is added so that $L(1) = 0$, where $1 \in \mathcal{A}^W$ is the Laurent polynomial identically equal to one (indeed, observe that $Y_i(1) = \gamma_{0,i}$ for $i = 1, \ldots, n$ by (3.9) and (3.25) since $T_j(1) = t_j$ for all $j = 0, \ldots, n$).

Theorem 3.19. *The q-difference operator L is explicitly given by*

$$L = \sum_{j=1}^{n} \big(\phi_j(x)(\tau(\epsilon_j) - 1) + \phi_j(x^{-1})(\tau(-\epsilon_j) - 1)\big),$$

with the rational coefficient $\phi_j(x)$ given by

$$\begin{aligned}\phi_j(x) =& t_0^{-1} t_n^{-1} t^{2(1-n)} \frac{(1 - ax_j)(1 - bx_j)(1 - cx_j)(1 - dx_j)}{(1 - x_j^2)(1 - qx_j^2)} \\ &\times \prod_{i \neq j} \frac{(1 - t^2 x_i x_j)(1 - t^2 x_i^{-1} x_j)}{(1 - x_i x_j)(1 - x_i^{-1} x_j)}.\end{aligned}$$

Remark 3.20. In remark 3.13 we saw that the explicit form of the weight function $\Delta_+(x)$ indicated a connection with the theory of one-variable Askey-Wilson polynomials. The same is the case for the q-difference operator L: for $n = 1$ (which we have excluded, but can in fact be treated in a similar fashion) it reduces to a multiple of the Askey-Wilson second order q-difference operator

$$\sum_{\epsilon = \pm 1} \frac{(1 - ay^{\epsilon})(1 - by^{\epsilon})(1 - cy^{\epsilon})(1 - dy^{\epsilon})}{(1 - y^{2\epsilon})(1 - qy^{2\epsilon})} (T_q^{\epsilon} - 1),$$

for which the Askey-Wilson polynomials are joint eigenfunctions (here $(T_q^{\epsilon} p)(y) = p(q^{\epsilon} y)$ are the multiplicative q^{ϵ}-shift in the one variable y). In particular, the symmetric Koornwinder polynomials form a multivariable generalization of the one-variable Askey-Wilson polynomials, with one extra degree of freedom t.

We can renormalize the symmetric Koornwinder polynomials now in a similar manner as their non-symmetric counterparts. Following the same reasoning as in remark 3.9, we see that $P_\lambda^+(x; \mathbf{1}| q) = m_\lambda(x)$ for all $\lambda \in \Lambda_0^+$. In particular, $P_\lambda^+(\gamma_0(\mathbf{1}| q); \mathbf{1}| q) \neq 0$. Since P_λ^+ depends meromorphically on the multiplicity function $\mathbf{t}$, we thus see that $P_\lambda^+(x_0) = P_\lambda^+(x_0; \mathbf{t}| q) \neq 0$ for generic values of the multiplicity function $\mathbf{t}$.

Definition 3.21. *The renormalized symmetric Koornwinder polynomial $E^+(\gamma_\lambda; \cdot) = E^+(\gamma_\lambda; \cdot; \mathbf{t}| q)$ of degree $\lambda \in \Lambda_0^+$ is defined by*

$$E^+(\gamma_\lambda; x) = \frac{P_\lambda^+(x)}{P_\lambda^+(x_0)}, \qquad \lambda \in \Lambda_0^+.$$

In other words, $E^+(\gamma_\lambda; x)$ is the constant multiple of the symmetric Koornwinder polynomial $P_\lambda^+(x)$ which takes the value one at $x = x_0^{\pm 1}$.

Let $\widetilde{E}^+(x_\lambda;\cdot) = E^+(x_\lambda;\cdot;\tilde{\mathbf{t}}|\,q)$ for $\lambda \in \Lambda_0^+$ be the renormalized Koornwinder polynomial of degree λ with respect to dual parameters. The duality of the non-symmetric Koornwinder polynomials has the following counterpart for symmetric Koornwinder polynomials.

Theorem 3.22. *For all $\lambda,\mu \in \Lambda_0^+$, we have*

$$E^+(\gamma_\lambda;x_\mu) = \widetilde{E}^+(x_\mu;\gamma_\lambda).$$

The second order q-difference equation which is satisfied by the symmetric Koornwinder polynomials (see theorem 3.17 and theorem 3.19) can be converted into a recurrence relation for the symmetric Koornwinder polynomials using duality. Formulated in terms of the symmetric Koornwinder polynomials with dual parameters, this can be stated as follows.

Corollary 3.23. *The symmetric Koornwinder polynomials $\widetilde{E}^+(x_\lambda;\cdot)$ ($\lambda \in \Lambda_0^+$) satisfy the recurrence relation*

$$\sum_{j=1}^n \Big(\phi_j(x_\lambda)\big(\widetilde{E}^+(x_{\lambda+\epsilon_j};\cdot) - \widetilde{E}^+(x_\lambda;\cdot)\big) + \phi_j(x_\lambda^{-1})\big(\widetilde{E}^+(x_{\lambda-\epsilon_j};\cdot) - \widetilde{E}^+(x_\lambda;\cdot)\big)\Big) =$$
$$= \big(m_{\epsilon_1}(\cdot) - m_{\epsilon_1}(\gamma_0)\big)\widetilde{E}^+(x_\lambda;\cdot)$$

for all $\lambda \in \Lambda_0^+$, where the contribution of the term $\phi_j(x_\lambda^{\pm 1})\big(\widetilde{E}^+(x_{\lambda\pm\epsilon_j};\cdot) - \widetilde{E}^+(x_\lambda;\cdot)\big)$ in the left hand side is taken to be zero if $\lambda \pm \epsilon_j \notin \Lambda_0^+$.

Proof. Let $\lambda \in \Lambda_0^+$. We will rewrite the eigenvalue equation

$$\big(L\,E^+(\gamma_\mu;\cdot)\big)(x_\lambda) = \big(m_{\epsilon_1}(\gamma_\mu) - m_{\epsilon_1}(\gamma_0)\big)E^+(\gamma_\mu;x_\lambda), \qquad \mu \in \Lambda_0^+ \tag{3.31}$$

(see theorem 3.17) using theorem 3.19 and the duality of the symmetric Koornwinder polynomials. It follows from the explicit form (3.26) of x_λ that $\lambda \pm \epsilon_j \notin \Lambda_0^+$ implies $\phi_j(x_\lambda^{\pm 1}) = 0$. On the other hand, if $\lambda \pm \epsilon_j \in \Lambda_0^+$, then

$$\big(\tau(\pm\epsilon_j)E^+(\gamma_\mu;\cdot)\big)(x_\lambda) = E^+(\gamma_\mu;x_{\lambda\pm\epsilon_j}) = \widetilde{E}^+(x_{\lambda\pm\epsilon_j};\gamma_\mu)$$

by the duality of the symmetric Koornwinder polynomials. So substitution of the explicit expression of L in (3.31) as given in theorem 3.19 combined with the above remarks and duality, imply the desired recurrence relation when both sides are evaluated at γ_μ for arbitrary $\mu \in \Lambda_0^+$. Since both sides of the desired identity are in $\mathcal{A}^W$, we conclude that the identity must also be true in $\mathcal{A}^W$. □

Remark 3.24. In the one variable case, corollary 3.23 gives the three term recurrence relation for the Askey-Wilson polynomials.

For the orthogonality relations of the symmetric Koornwinder polynomials, we define the bilinear form $\langle\cdot,\cdot\rangle_+ = \langle\cdot,\cdot\rangle_{+,\mathbf{t},q}$ by

$$\langle p_1,p_2\rangle_+ = \frac{1}{(2\pi i)^n}\iint_{x\in\mathbb{T}^n} p_1(x)p_2(x)\Delta_+(x)\frac{dx}{x}, \qquad p_1,p_2 \in \mathcal{A},$$

see (3.16) for the definition of $\Delta_+(x)$.

The orthogonality relations and diagonal terms for the symmetric Koornwinder polynomials are now given as follows.

Theorem 3.25. *For all $\lambda, \mu \in \Lambda_0^+$ we have*

$$\frac{\langle E^+(\gamma_\lambda;\cdot), E^+(\gamma_\mu;\cdot)\rangle_+}{\langle 1,1\rangle_+} = \delta_{\lambda,\mu}\frac{\widetilde{w}_+(\gamma_0^{-1})}{\widetilde{w}_+(\gamma_\lambda^{-1})},$$

where $\delta_{\lambda,\mu}$ is the Kronecker delta.

Remark 3.26. Observe that the eigenvalue $m_{\epsilon_1}(\gamma_\lambda) - m_{\epsilon_1}(\gamma_0)$ of the symmetric Koornwinder polynomial $E^+(\gamma_\lambda;\cdot)$ with respect to the second order q-difference operator L ($\lambda \in \Lambda_0^+$) are pair-wise different. Hence $E^+(\gamma_\lambda;\cdot)$ is, up to a constant, the unique W-invariant Laurent polynomial which is an eigenfunction of L with eigenvalue $m_{\epsilon_1}(\gamma_\lambda) - m_{\epsilon_1}(\gamma_0)$. From the explicit form of L, see theorem 3.19, we then easily derive that

$$E^+(\gamma_\lambda; x; \mathbf{t}|\, q) = E^+(\gamma_\lambda^{-1}; x; \mathbf{t}^{-1}|\, q^{-1}), \qquad \lambda \in \Lambda_0^+.$$

This fact already indicates that the bi-orthogonality relations for the non-symmetric Koornwinder polynomials (see theorem 3.14) become orthogonality relations for the symmetric Koornwinder polynomials.

The advantage of working with $\langle\cdot,\cdot\rangle_+$ is that the weight function $\Delta_+(x)$ is positive on $\mathbb{T}^n$ when the parameters $q, \mathbf{t}$ are furthermore assumed to be real (in contrast with $\Delta(x)$). In particular, the orthogonality relations for the symmetric Koornwinder polynomials are then formulated with respect to a positive orthogonality measure, and hence we can really speak of multivariable orthogonal polynomials in the sense of "classical" orthogonal polynomial theory.

We will see in §8.4, that the restriction of the bilinear form $\langle\cdot,\cdot\rangle$ to the subspace $\mathcal{A}^W$ coincides with $\langle\cdot,\cdot\rangle_+$ up to a constant multiple. This is caused by the fact that the correction term $\mathcal{C}(x)$ of $\Delta(x)$ with respect to its W-invariant part $\Delta_+(x)$ symmetrizes to a constant, i.e. $\sum_{w\in W}\big(w\mathcal{C}\big)(x)$ lies in the base field $\mathbb{C}$ of $\mathcal{Q}$. The fact that $\mathcal{C}$ symmetrizes to a constant is a consequence of an identity of Macdonald, who introduced it as a generalization of the Poincaré series of the Weyl group W. This fact is crucial for establishing the precise connections between the symmetric and the non-symmetric theory. In particular, it leads to the following explicit expansion of the symmetric Koornwinder polynomial as linear combination of the non-symmetric Koornwinder polynomials.

Theorem 3.27. *We have*

$$E^+(\gamma_\lambda; x) = \widetilde{\mathcal{C}}(\gamma_0^{-1})^{-1}\sum_{\mu\in W\lambda}\widetilde{\mathcal{C}}(\gamma_\mu^{-1})E(\gamma_\mu; x), \qquad \lambda \in \Lambda_0^+$$

where $\widetilde{\mathcal{C}}(x) = \mathcal{C}(x; \tilde{\mathbf{t}}|q)$.

4. The affine Hecke algebra and Noumi's representation

We discuss the affine Hecke algebra of type $\widetilde{C}_n$, which turns out to describe the algebra of symmetries for the Koornwinder polynomials. For this we need to recall some basic results from the theory of Coxeter groups and Hecke algebras, for which we give precise references to the literature (we refer to Humphreys' book [14] as much as possible).

4.1. **The length function.** In this subsection we recall some of the basic properties of the length function $l:\mathcal{W}\to\mathbb{C}$ on the affine Weyl group $\mathcal{W}$, which is defined by

$$l(w)=\#(R^+\cap w^{-1}R^-),\qquad w\in\mathcal{W}. \tag{4.1}$$

We start by recalling the following two well known facts, which are valid for an arbitrary Coxeter group.

Proposition 4.1. **(i)** *The length $l(w)$ of $w\in\mathcal{W}$ is equal to the minimal possible length of an expression $w=s_{i_1}s_{i_2}\cdots s_{i_r}$ of w as a product of simple reflections (such an expression is called a reduced expression of w).*

(ii) *For $w\in W$ in the finite Weyl group W, there exists a reduced expression $w=s_{i_1}\cdots s_{i_r}$ with all indices i_j in $\{1,\ldots,n\}$. In particular, $l(w)=\#(\Sigma^+\cap w^{-1}\Sigma^-)$ for $w\in W$ (so the length function of $\mathcal{W}$, restricted to the finite Weyl group W, coincides with the length function of W).*

Proof. For **(i)**, see for instance [14, §5.6]. For **(ii)** one observes that $W\subset\mathcal{W}$ is a parabolic sub-group, which means that W is a sub-group of $\mathcal{W}$ generated by a subset I of the simple reflections s_i $(i=0,\ldots,n)$. In our present setting we have $I=\{1,\ldots,n\}$. For the compatibility of the length functions with respect to parabolic sub-groups one can for instance consult [14, thm. 5.5]. □

The length of an affine Weyl group element $w\in\mathcal{W}$ can be explicitly computed in the following manner.

Proposition 4.2. *For $\lambda\in\Lambda_0$ and $w\in W$, we have*

$$l(\tau(\lambda)w)=\sum_{\alpha\in\Sigma^+}|-\big(\lambda,w\alpha\big)+\chi(w\alpha)|,$$

where $\chi(\alpha)=1$ if $\alpha\in\Sigma^-$ and $=0$ if $\alpha\in\Sigma^+$.

Proof. We use that $R^\pm=\Sigma^\pm\cup\{f\in R\,|\,f(0)\gtrless 0\}$, that $l(u)=\#(R^-\cap uR^+)$ for all $u\in\mathcal{W}$, and that

$$(\tau(\lambda)w)(\alpha+k\delta)=w\alpha+(k+\big(\lambda,w\alpha\big))\delta,\qquad \alpha\in\Sigma,\ \ k\in\mathbb{Z} \tag{4.2}$$

for all $w\in W$ and $\lambda\in\Lambda_0$. We distinguish now between four cases, while making use of (4.2): If $\alpha\in\Sigma^+$ and $\big(\lambda,w\alpha\big)\le 0$, then

$$\#\{k\in\mathbb{Z}_+\,|\,(\tau(\lambda)w)(\alpha+k\delta)\in R^-\}=\chi(w\alpha)-\big(\lambda,w\alpha\big).$$

If $\alpha\in\Sigma^+$ and $\big(\lambda,w\alpha\big)>0$, then

$$\#\{k\in\mathbb{Z}_+\,|\,(\tau(\lambda)w)(\alpha+k\delta)\in R^-\}=0.$$

If $\alpha\in\Sigma^-$ and $\big(\lambda,w\alpha\big)<0$, then

$$\begin{aligned}\#\{k\in\mathbb{Z}_+\,|\,(\tau(\lambda)w)(\alpha+(k+1)\delta)\in R^-\}&=1-\chi(-w\alpha)-\big(\lambda,w\alpha\big)-1\\&=-\chi(w(-\alpha))+\big(\lambda,w(-\alpha)\big).\end{aligned}$$

If $\alpha\in\Sigma^-$ and $\big(\lambda,w\alpha\big)\ge 0$, then

$$\#\{k\in\mathbb{Z}_+\,|\,(\tau(\lambda)w)(\alpha+(k+1)\delta)\in R^-\}=0.$$

Now adding these contributions, we obtain

$$\begin{aligned} l(\tau(\lambda)w) &= \#\{f \in R^+ \,|\, (\tau(\lambda)w)(f) \in R^-\} \\ &= \sum_{\alpha\in\Sigma^+} |-\big(\lambda, w\alpha\big) + \chi(w\alpha)|, \end{aligned}$$

as desired. □

The following corollary of this length identity will play a crucial role in establishing the commutativity of the Y-operators $Y_1, \ldots, Y_n$, see (3.9).

Corollary 4.3. *We have*

$$l(\tau(\lambda+\mu)) = l(\tau(\lambda)\tau(\mu)) = l(\tau(\lambda)) + l(\tau(\mu)), \qquad \forall \lambda, \mu \in \Lambda_0^+.$$

Proof. Recall that $\lambda \in \Lambda_0$ is a dominant weight iff $\big(\lambda, \alpha\big) \geq 0$ for all $\alpha \in \Sigma^+$. Hence the previous proposition shows that $l(\tau(\lambda)) = \sum_{\alpha\in\Sigma^+}\big(\lambda, \alpha\big)$ for all $\lambda \in \Lambda_0^+$, which immediately implies the desired result. □

Finally, we give here another consequence of proposition 4.2 which is needed in §4.3.

Corollary 4.4. *For all $w \in W$ and $\lambda \in \Lambda_0^+$, we have $l(\tau(\lambda)w) = l(\tau(\lambda)) + l(w)$.*

Proof. We fix $w \in W$ and $\lambda \in \Lambda_0^+$. Observe that if $\chi(w\alpha) = 1$ for $\alpha \in \Sigma^+$, then $\big(\lambda, w\alpha\big) \leq 0$, and if $\chi(w\alpha) = 0$ for $\alpha \in \Sigma^+$, then $\big(\lambda, w\alpha\big) \geq 0$. Hence proposition 4.1**(ii)** and proposition 4.2 yield

$$\begin{aligned} l(\tau(\lambda)w) &= \sum_{\alpha\in\Sigma^+\cap w^{-1}\Sigma^-} \big(1 - (\lambda, w\alpha)\big) + \sum_{\alpha\in\Sigma^+\cap w^{-1}\Sigma^+} \big(\lambda, w\alpha\big) \\ &= l(w) + \sum_{\alpha\in\Sigma^+} |\big(w^{-1}\lambda, \alpha\big)| = l(w) + l(\tau(w^{-1}\lambda)). \end{aligned}$$

So it remains to prove that $l(\tau(w\mu)) = l(\tau(\mu))$ for any $\mu \in \Lambda_0$ and $w \in W$. It suffices to take $w = s_i$ $(i = 1, \ldots, n)$ a simple reflection. Then s_i permutes the set $\Sigma^+ \setminus \{a_i\}$ and maps a_i to $-a_i \in \Sigma^-$. Hence

$$l(\tau(s_i\mu)) = \sum_{\alpha\in\Sigma^+\setminus\{a_i\}} |\big(\mu, \alpha\big)| + |\big(\mu, s_ia_i\big)| = \sum_{\alpha\in\Sigma^+} |\big(\mu, \alpha\big)| = l(\tau(\mu)),$$

as desired. □

4.2. **The affine Hecke algebra of type $\widetilde{C}_n$.** The Hecke algebra of type $\widetilde{C}_n$ is a deformation of the group algebra $\mathbb{C}[\mathcal{W}]$ of the affine Weyl group $\mathcal{W}$. As a vector space over $\mathbb{C}$, the group algebra $\mathbb{C}[\mathcal{W}]$ has the affine Weyl group elements as a linear basis, and the multiplication is defined by extending the group multiplication linearly. By the presentation of the affine Weyl group $\mathcal{W}$ in terms of the involutions s_i $(i = 0, \ldots, n)$, see proposition 2.10, we see that $\mathbb{C}[\mathcal{W}]$ is isomorphic to the unital algebra over $\mathbb{C}$ generated by V_i $(i = 0, \ldots, n)$ satisfying $V_i^2 = 1$ $(i = 0, \ldots, n)$ and satisfying the same braid relations as the simple reflections s_i (see proposition 2.10).

Let now $\mathbf{t} = \{t_f \,|\, f \in R_{nr}\}$ be a multiplicity function of R_{nr}. Its restriction to the reduced root system R is denoted by $\mathbf{t}_R$, so $\mathbf{t}_R = \{t_f \,|\, f \in R\}$. It is isomorphic to a (generic) element in $\big(\mathbb{C}\setminus\{0\}\big)^3$, since $\mathbf{t}_R$ is completely determined by its values $t_0 = t_{a_0}$, $t = t_k = t_{a_k}$ $(k \in \{1, \ldots, n-1\}$ arbitrary), and $t_n = t_{a_n}$. The Hecke algebra H is now defined as the following deformation of the group algebra $\mathbb{C}[\mathcal{W}]$.

Definition 4.5. *The affine Hecke algebra $H = H(\mathcal{W}; \mathbf{t}_R)$ of type $\widetilde{C}_n$ is the unital algebra over $\mathbb{C}$ generated by $V_0, \ldots, V_n$ and satisfying the quadratic relations*

$$(V_i - t_i)(V_i + t_i^{-1}) = 0, \qquad i = 0, \ldots, n,$$

and the $\widetilde{C}_n$-braid relations

$$\begin{aligned} V_iV_{i+1}V_iV_{i+1} &= V_{i+1}V_iV_{i+1}V_i, \qquad i = 0,\, n-1,\\ V_iV_{i+1}V_i &= V_{i+1}V_iV_{i+1}, \qquad i = 1, \ldots, n-2,\\ V_iV_j &= V_jV_i, \qquad |i-j| > 1. \end{aligned}$$

Observe that the quadratic relation for V_i implies that V_i is invertible in H, with inverse given by $V_i^{-1} = V_i - t_i + t_i^{-1}$.

An important property of the affine Hecke algebra H is the existence of a canonical basis $\{V_w \,|\, w \in \mathcal{W}\}$ of H, in analogy with the canonical basis $\mathcal{W}$ of the group algebra $\mathbb{C}[\mathcal{W}]$. It is constructed as follows.

Proposition 4.6. *Let $w \in \mathcal{W}$, and let $w = s_{i_1}s_{i_2}\ldots s_{i_r}$ be a reduced expression. Then*

$$V_w = V_{i_1}V_{i_2}\cdots V_{i_r} \in H$$

is well defined (i.e. independent of the choice of reduced expression $w = s_{i_1}s_{i_2}\cdots s_{i_r}$ for $w \in \mathcal{W}$).

Proof. This is known as Iwahori-Matsumoto's theorem: for any set of elements $(a_0, \ldots, a_n)$ in an algebra A which satisfy the $\widetilde{C}_n$-braid relations, we have that

$$a_w := a_{i_1}a_{i_2}\cdots a_{i_r} \in A$$

is independent of the choice of reduced expression $w = s_{i_1}s_{i_2}\cdots s_{i_r}$ of $w \in \mathcal{W}$. Intuitively, two reduced expressions for the same affine Weyl group element $w \in \mathcal{W}$ can be obtained from each-other by using the braid relations only (and *not* the quadratic relations $s_i^2 = 1$). For a proof, see for instance [24, Thm. 2]. □

Theorem 4.7. *The elements V_w $(w \in \mathcal{W})$ form a $\mathbb{C}$-linear basis of H.*

Proof. See for instance [14, Chapter 7]. □

4.3. **The commutative sub-algebra $\mathcal{A}_Z$ of H.** We are now in a position to generalize the structure $\mathcal{W} = W \ltimes \tau(\Lambda_0)$ of $\mathcal{W}$ as a semi-direct product of the finite Weyl group W and the abelian sub-group $\tau(\Lambda_0)$ to the level of the affine Hecke algebra H.

Let $\mathbf{t}_\Sigma = \{t_\alpha \,|\, \alpha \in \Sigma\}$ be the restriction of the multiplicity function $\mathbf{t}_R$ to the finite root system $\Sigma \subset R$. Then $\mathbf{t}_\Sigma$ is a multiplicity function of Σ, in the sense that it is constant on W-orbits in Σ. In particular, $\mathbf{t}_\Sigma$ is uniquely determined by its two values $(t, t_n) = (t_{a_k}, t_{a_n})$ $(k \in \{1, \ldots, n-1\}$ arbitrary).

Since the Weyl group W is the sub-group of $\mathcal{W}$ generated by the simple reflections s_i $(i = 1, \ldots, n)$, we define $H_0 = H_0(\Sigma; \mathbf{t}_\Sigma) \subset H(\mathcal{W}; \mathbf{t}_R) = H$ to be the unital sub-algebra generated by V_i $(i = 1, \ldots, n)$. It follows from theorem 4.7 that H_0 is finite dimensional with linear basis $\{V_w \,|\, w \in W\}$, and that H_0 (as algebra) only depends on the values of the multiplicity function $\mathbf{t}_R$ at Σ (i.e. on $\mathbf{t}_\Sigma$).

Next we define the analogue of the commutative sub-group $\tau(\Lambda_0)$ in H. We write

$$Z^\lambda = V_{\tau(\lambda)} \in H, \qquad \lambda \in \Lambda_0^+.$$

Lemma 4.8. *For all $\lambda,\mu\in\Lambda_0^+$, we have*

$$Z^\lambda Z^\mu = Z^{\lambda+\mu} = Z^\mu Z^\lambda$$

in H. Furthermore, $Z^0=1\in H$ is the identity element.

Proof. If $w,w'\in\mathcal{W}$ are two elements such that $l(ww')=l(w)+l(w')$, then $V_{ww'}=V_wV_{w'}$. Indeed, this follows from the definition of the basis elements V_u ($u\in\mathcal{W}$) and the observation that $ww'=s_{i_1}\cdots s_{i_r}s_{j_1}\cdots s_{j_t}$ is a reduced expression of $ww'\in\mathcal{W}$ if $w=s_{i_1}\cdots s_{i_r}$ and $w'=s_{j_1}\cdots s_{i_t}$ are reduced expressions of w and w' in $\mathcal{W}$, respectively.

Now apply this observation to $w=\tau(\mu)$, $w'=\tau(\lambda)$ with $\lambda,\mu\in\Lambda_0^+$, using corollary 4.3 and the fact that the translation operators $\tau(\lambda)$ and $\tau(\mu)$ commute in $\mathcal{W}$. □

For $\lambda\in\Lambda_0$, we define

$$Z^\lambda = Z^\mu(Z^\nu)^{-1},\qquad \lambda=\mu-\nu\in\Lambda_0,\ \ \mu,\nu\in\Lambda_0^+.$$

This is independent of the decomposition of $\lambda=\mu-\nu$ as a difference of two dominant weights $\mu,\nu\in\Lambda_0^+$ by the previous lemma. Indeed, if $\lambda=\mu'-\nu'$ is another such decomposition, then $\mu+\nu'=\mu'+\nu\in\Lambda_0^+$, hence by the previous lemma

$$Z^\mu(Z^\nu)^{-1}=Z^{\mu+\nu'}(Z^{\nu+\nu'})^{-1}=Z^{\mu'+\nu}(Z^{\nu+\nu'})^{-1}=Z^{\mu'}(Z^{\nu'})^{-1}.$$

Observe that $Z^\lambda Z^\mu=Z^{\lambda+\mu}=Z^{\mu+\lambda}$ now holds for all $\lambda,\mu\in\Lambda_0$. We write $\mathcal{A}_Z\subset H$ for the commutative sub-algebra generated by the Z^λ ($\lambda\in\Lambda_0$). For any $p\in\mathcal{A}$, say $p=\sum_\lambda d_\lambda x^\lambda$, we set $p(Z)=\sum_\lambda d_\lambda Z^\lambda$. Furthermore, we write $Z_i=Z^{\epsilon_i}$ for $i=1,\ldots,n$. Observe that $Z_i^{\pm1}$ ($i=1,\ldots,n$) generate $\mathcal{A}_Z$ as an algebra.

The following step is to describe the commutation relations between the Z-operators and the elements of the finite Hecke algebra H_0. We start with the following observation.

Lemma 4.9. *Let $\lambda\in\Lambda_0$ and $i\in\{1,\ldots,n\}$.*
(i) *If $\big(\lambda,a_i\big)=0$, then $V_iZ^\lambda=Z^\lambda V_i$ in H.*
(ii) *If $\big(\lambda,a_i\big)=1$, then $Z^\lambda=V_iZ^{s_i\lambda}V_i$ in H.*

Proof. **(i)** Observe that

$$\Lambda_0=\bigoplus_{j=1}^n\mathbb{Z}\omega_j,\qquad \Lambda_0^+=\bigoplus_{j=1}^n\mathbb{Z}_+\omega_j$$

with ω_j the fundamental weight

$$\omega_j=\epsilon_1+\epsilon_2+\cdots+\epsilon_j,\qquad j=1,\ldots,n$$

(or equivalently: ω_j is the unique element in Λ_0 satisfying $\big(\omega_j,a_i^\vee\big)=\delta_{i,j}$ for all $i=1,\ldots,n$). The weights $\lambda\in\Lambda_0$ (respectively dominant weights $\lambda\in\Lambda_0^+$) which are orthogonal to a_i are then given by the $\mathbb{Z}$-span (respectively $\mathbb{Z}_+$-span) of ω_j ($j\neq i$). In particular, if $\big(\lambda,a_i\big)=0$, then there exist $\mu_1,\mu_2\in\Lambda_0^+$ with $\lambda=\mu_1-\mu_2$ and $\langle\mu_j,a_i\rangle=0$ for $j=1,2$, and $Z^\lambda=Z^{\mu_1}(Z^{\mu_2})^{-1}$.

So it suffices to prove **(i)** for $\lambda\in\Lambda_0^+$ orthogonal to a_i. We then have $s_i\lambda=\lambda$, so that $s_i\tau(\lambda)=\tau(\lambda)s_i$, and $l(s_i\tau(\lambda))=l(\tau(\lambda)s_i)=l(\tau(\lambda))+1$ by corollary 4.4. We conclude that

$$V_iZ^\lambda=V_iV_{\tau(\lambda)}=V_{s_i\tau(\lambda)}=V_{\tau(\lambda)s_i}=V_{\tau(\lambda)}V_i=Z^\lambda V_i,$$

as desired.

(ii) If $\big(\lambda, a_i\big) = 1$, then we can write $\lambda = \mu - \nu$ with $\mu, \nu \in \Lambda_0^+$ and $\big(\mu, a_i\big) = 1$, $\big(\nu, a_i\big) = 0$. By **(i)**, we may thus assume without loss of generality that $\nu = 0$, i.e. that $\lambda = \mu \in \Lambda_0^+$.

If $\lambda \in \Lambda_0^+$ and $\big(\lambda, a_i\big) = 1$, then $\lambda + s_i\lambda \in \Lambda_0^+$. Indeed,

$$\big(\lambda + s_i\lambda, a_i\big) = \big(\lambda, s_ia_i + a_i\big) = 0 \tag{4.3}$$

since $s_ia_i = -a_i$, and for $j \in \{1, \ldots, n\} \setminus \{i\}$ we have

$$\big(\lambda + s_i\lambda, a_j\big) = \big(\lambda, s_ia_j + a_j\big) \geq 0$$

since $s_ia_j \in \Sigma^+$. We now set

$$w = \tau(\lambda + s_i\lambda) = \tau(\lambda)\tau(s_i\lambda) = \tau(\lambda)s_i\tau(\lambda)s_i \in \mathcal{W}.$$

We claim that we have the length identity

$$l(w) = l(ws_i) - 1 = l(\tau(\lambda)) + l(s_i\tau(\lambda)) - 1 = 2l(\tau(\lambda)) - 2. \tag{4.4}$$

First we observe that (4.4) implies

$$Z^{\lambda + s_i\lambda} = V_w = V_{ws_i}V_i^{-1} = V_{\tau(\lambda)}V_{s_i\tau(\lambda)}V_i^{-1} = Z^\lambda V_i^{-1}Z^\lambda V_i^{-1},$$

which leads to the desired identity. So it remains to prove (4.4).

The first equality of (4.4) is immediate from corollary 4.4 using the fact that $\lambda + s_i\lambda \in \Lambda_0^+$.

We now use the fact that $\lambda \in \Lambda_0^+$ and the fact that s_i permutes $\Sigma^+ \setminus \{a_i\}$ and maps a_i to $-a_i$, to derive from proposition 4.2 that

$$\begin{aligned} l(s_i\tau(\lambda)) = l(\tau(s_i\lambda)s_i) &= \sum_{\alpha \in \Sigma^+} |-\big(\lambda, \alpha\big) + \chi(s_i\alpha)| \\ &= \sum_{\alpha \in \Sigma^+ \setminus \{a_i\}} \big(\lambda, \alpha\big) = l(\tau(\lambda)) - 1, \end{aligned}$$

where we used $\big(\lambda, a_i\big) = 1$ for the third and fourth equality. This gives the third identity of (4.4). Finally, for the second equality of (4.4), it now suffices to show that $l(w) = 2l(\tau(\lambda)) - 2$. But proposition 4.2, (4.3), $\lambda + s_i\lambda \in \Lambda_0^+$ and the fact that $s_i(\Sigma^+ \setminus \{a_i\}) = \Sigma^+ \setminus \{a_i\}$ imply

$$l(w) = l(\tau(\lambda + s_i\lambda)) = \sum_{\alpha \in \Sigma^+ \setminus \{a_i\}} \big(\lambda + s_i\lambda, \alpha\big) = 2 \sum_{\alpha \in \Sigma^+ \setminus \{a_i\}} \big(\lambda, \alpha\big).$$

Since $\big(\lambda, a_i\big) = 1$ and $\lambda \in \Lambda_0^+$, proposition 4.2 now implies that

$$l(w) = 2 \sum_{\alpha \in \Sigma^+} \big(\lambda, \alpha\big) - 2 = 2l(\tau(\lambda)) - 2,$$

which completes the proof of (4.4). $\square$

Corollary 4.10. *For $i = 1, \ldots, n$ we have*

$$Z_i = V_i \cdots V_{n-1}V_nV_{n-1} \cdots V_0V_1^{-1}V_2^{-1} \cdots V_{i-1}^{-1}$$

in H.

Proof. By proposition 4.2 we have $l(\tau(\epsilon_1))=2n$, so

$$\tau(\epsilon_1)=s_1\cdots s_{n-1}s_ns_{n-1}\cdots s_1s_0$$

is a reduced expression of $\tau(\epsilon_1)$ in $\mathcal{W}$. Hence

$$Z_1=V_{\tau(\epsilon_1)}=V_1\cdots V_{n-1}V_nV_{n-1}\cdots V_1V_0.$$

Now assume that the desired expression for Z_j is valid for $j=1,\ldots,i-1$, with $2\leq i\leq n$ fixed. Observe that $\tau(\epsilon_i)=\tau(s_{i-1}\epsilon_{i-1})=s_{i-1}\tau(\epsilon_{i-1})s_{i-1}$, and that $\big(\epsilon_{i-1},a_{i-1}\big)=1$. It follows from lemma 4.9**(ii)** that $Z_{i-1}=V_{i-1}Z_iV_{i-1}$, which proves the induction step. □

Let $\widetilde{T}_f=T_f(\tilde{\mathbf{t}}|\,q)\in\mathrm{End}_{\mathbb{C}}(\mathcal{A})$ $(f\in R)$ be Noumi's difference-reflection operators with respect to dual parameters, see (3.5). Observe that $\widetilde{T}_\alpha$ $(\alpha\in\Sigma)$ only depends on the multiplicity function $\mathbf{t}_R$.

Proposition 4.11. *For all $i=1,\ldots,n$ and $p\in\mathcal{A}$, we have*

$$V_ip(Z)-\big(\widetilde{T}_{-a_i}p\big)(Z)=\big(s_ip\big)(Z)(V_i-t_i)$$

in H.

Proof. For the moment, we exclude the case $i=n$, so we fix $i\in\{1,\ldots,n-1\}$. Using the explicit expression for Noumi's difference-reflection operators (see (3.5)), we see that the desired commutation relation is equivalent to

$$V_ip(Z)-(s_ip)(Z)V_i=(t-t^{-1})Z^{a_i}\left(\frac{(s_ip)(Z)-p(Z)}{1-Z^{a_i}}\right). \tag{4.5}$$

The left hand side and the right hand side of (4.5) depend linearly on p, so it suffices to prove it for monomials $p(x)=x^\lambda$ $(\lambda\in\Lambda_0)$. Furthermore, it is easy to check that if (4.5) is valid for $p(x)=x^\nu$ with $\nu=\lambda,\mu\in\Lambda_0$, then it is also valid for $p(x)=x^\nu$ with $\nu=-\lambda$ and $\nu=\lambda+\mu$. Hence it suffices to prove (4.5) for $p(x)=x^{\omega_j}$, where $\omega_j=\epsilon_1+\epsilon_2+\cdots+\epsilon_j$ $(j=1,\ldots,n)$ are the fundamental weights. For $j\neq i$, we have $\big(\omega_j,a_i\big)=0$ and $s_i\omega_j=\omega_j$, so (4.5) is then equivalent to $V_iZ^{\omega_j}=Z^{\omega_j}V_i$, which is valid by lemma 4.9**(i)**. If $p(x)=x^{\omega_i}$, then $\big(\omega_i,a_i\big)=1$ and $s_i\omega_i=\omega_i-a_i$. Then (4.5) is equivalent to

$$V_iZ^{\omega_i}-Z^{s_i\omega_i}V_i=(t-t^{-1})Z^{\omega_i},$$

i.e. $V_i^{-1}Z^{\omega_i}=Z^{s_i\omega_i}V_i$. This is indeed valid by lemma 4.9**(ii)**.

Now we consider the case $i=n$. The desired identity is now equivalent to

$$V_np(Z)-\big(s_np\big)(Z)V_n=\big((t_n-t_n^{-1})Z_n^2+(t_0-t_0^{-1})Z_n\big)\left(\frac{\big(s_np\big)(Z)-p(Z)}{1-Z_n^2}\right).$$

By a similar argument as before, it suffices to prove the identity for $p(Z)=Z_i$ $(i=1,\ldots,n)$. For $i=1,\ldots,n-1$, the identity holds true for $p(Z)=Z_i$ by lemma 4.9**(i)**. So it suffices to prove the identity for $p(Z)=Z_n$, in which case we have to show that

$$V_nZ_n-Z_n^{-1}V_n=(t_n-t_n^{-1})Z_n+(t_0-t_0^{-1}).$$

By corollary 4.10 and the identities $V_n^2=(t_n-t_n^{-1})V_n+1$ and $V_0^{-1}=V_0-t_0+t_0^{-1}$, we have

$$\begin{aligned}V_nZ_n&=V_n\bigl(V_nV_{n-1}\cdots V_1V_0V_1^{-1}V_2^{-1}\cdots V_{n-1}^{-1}\bigr)\\&=(t_n-t_n^{-1})Z_n+V_{n-1}\cdots V_2V_1\bigl(V_0^{-1}+t_0-t_0^{-1}\bigr)V_1^{-1}V_2^{-1}\cdots V_{n-1}^{-1}\\&=(t_n-t_n^{-1})Z_n+Z_n^{-1}V_n+(t_0-t_0^{-1}),\end{aligned}$$

as desired. □

Corollary 4.12. *The set* $\{Z^\lambda V_w\,|\,w\in W,\ \lambda\in\Lambda_0\}$ *and the set* $\{V_wZ^\lambda\,|\,w\in W,\ \lambda\in\Lambda_0\}$ *are linear bases of* H. *In particular,* $\mathcal{A}_Z\otimes H_0\simeq H\simeq H_0\otimes\mathcal{A}_Z$ *as vector spaces by multiplication.*

Proof. We first consider the set $\{Z^\lambda V_w\,|\,\lambda\in\Lambda_0,\ w\in W\}$. By proposition 4.10, V_0 can be written as a product of Z_1 and V_i's ($i=1,\ldots,n$). Furthermore, in any product with factors from Z^λ ($\lambda\in\Lambda_0$) and V_w ($w\in W$), we can pull the Z^λ factors to the left of the V_w factors in view of the previous proposition. Hence the elements $Z^\lambda V_w$ ($\lambda\in\Lambda_0$, $w\in W$) span H. For the linear independence, we let

$$\sum_{\lambda,w}d_{\lambda,w}Z^\lambda V_w=0,\qquad d_{\lambda,w}\in\mathbb{C}\tag{4.6}$$

be a finite, vanishing sum in H. There exists a $\mu\in\Lambda_0^+$ such that $\mu+\lambda\in\Lambda_0^+$ for all those $\lambda\in\Lambda_0$ for which $d_{\lambda,w}\neq0$ for some $w\in W$ (since there exists only finitely many such λ). Multiplying the element (4.6) with Z^μ from the left and using corollary 4.4, we see that

$$\sum_{\lambda,w}d_{\lambda,w}V_{\tau(\mu+\lambda)w}=0.$$

By theorem 4.7, we conclude that all coefficients $d_{\lambda,w}$ are zero.

Let now H' be the affine Hecke algebra of type $\widetilde{C}_n$ with respect to the inverse multiplicity function $\mathbf{t}_R^{-1}=(t_0^{-1},t^{-1},t_n^{-1})$, and write V_i' ($i=0,\ldots,n$), V_w' ($w\in\mathcal{W}$) and Z'^λ ($\lambda\in\Lambda_0$) for the elements V_i, V_w and Z^λ in H'. Then there exists a unique anti-algebra homomorphism $\phi:H\to H'$ mapping V_i to $V_i'^{-1}$ for $i=0,\ldots,n$ (indeed, observe that all the defining relations of H are preserved if one formally extends the map $\phi(V_i)=V_i'^{-1}$ ($i=0,\ldots,n$) anti-multiplicatively).

Observe that $\phi(V_w)=V_w'^{-1}$ for all $w\in\mathcal{W}$, hence in particular $\phi(Z^\lambda)=Z'^{-\lambda}$ for all $\lambda\in\Lambda_0$. We conclude now from the first part of this proposition that the elements $\phi(Z^\lambda V_w)=V_w'^{-1}Z'^{-\lambda}$ ($w\in W$ and $\lambda\in\Lambda_0$) form a linear basis of H'. Multiplying from the left by the invertible element V_σ', with $\sigma=-1\in W$ the longest Weyl group element (which maps v to $-v$ for all $v\in V$), and using that $V_\sigma'V_w'^{-1}=V_{\sigma w}'$ for all $w\in W$ since $l(\sigma w)=l(\sigma)-l(w)$ for all $w\in W$, we conclude that the elements $V_w'Z'^\lambda$ ($w\in W$ and $\lambda\in\Lambda_0$) form a linear basis of H'. Now inverting the multiplicity function again, gives the second statement of the corollary. □

It follows from this corollary that the commutative sub-algebra $\mathcal{A}_Z$ of H is naturally isomorphic to the algebra $\mathcal{A}$ of Laurent polynomials in the indeterminates $x_1,\ldots,x_n$ by identifying Z_i with x_i for all $i=1,\ldots,n$.

4.4. **The Noumi representation.** The algebraic structure of the affine Hecke algebra H in terms of the commutative sub-algebra $\mathcal{A}_Z$ and the finite Hecke algebra H_0 allows us to define a crucial representation of H on $\mathcal{A}$, which is known as the Noumi representation. We start with the following observation.

Lemma 4.13. *Noumi's difference-reflection operators $T_i = T_i(\mathbf{t}|\, q) \in End_{\mathbb{C}}(\mathcal{A})$ $(i = 1, \ldots, n)$ defined by (3.5) satisfy the quadratic relations $(T_i - t_i)(T_i + t_i^{-1}) = 0$ $(i = 1 \ldots, n)$ and the C_n-braid relations*

$$\begin{aligned} T_iT_{i+1}T_i &= T_{i+1}T_iT_{i+1}, \qquad i = 1, \ldots, n-2,\\ T_{n-1}T_nT_{n-1}T_n &= T_nT_{n-1}T_nT_{n-1},\\ T_iT_j &= T_jT_i, \qquad |i-j| > 1,\ \ i, j \in \{1, \ldots, n\}. \end{aligned}$$

Proof. Let $\chi : H_0(W; \mathbf{t}_\Sigma) \to \mathbb{C}$ be the character (i.e. one-dimensional algebra homomorphism) defined by $\chi(V_i) = t_i$ for $i = 1, \ldots, n$. Consider the corresponding induced left H-module

$$\mathrm{Ind}_{H_0}^H(\chi) = H \otimes_{H_0} \mathbb{C}.$$

By corollary 4.12, we may identify $\mathrm{Ind}_{H_0}^H(\chi)$ as $\mathbb{C}$-vector space with $\mathcal{A}$. In view of proposition 4.11, we then see that the restriction of the H-action on $\mathcal{A}$ to the sub-algebra $H_0(W; \mathbf{t}_\Sigma)$ is given by

$$V_ip = \widetilde{T}_{-a_i}p, \qquad p \in \mathcal{A}, \quad i = 1, \ldots, n,$$

where $\widetilde{T}_{-a_i}$ is Noumi's difference-reflection operator $T_{-a_i}(\tilde{\mathbf{t}}|\, q)$ with respect to dual parameters. In particular, $\widetilde{T}_{-a_i}$ $(i = 1, \ldots, n)$ satisfy the quadratic relations

$$(\widetilde{T}_{-a_i} - t_i)(\widetilde{T}_{-a_i} + t_i^{-1}) = 0, \qquad i = 1, \ldots, n$$

in $\mathrm{End}_{\mathbb{C}}(\mathcal{A})$, and they satisfy the C_n-braid relations in $\mathrm{End}_{\mathbb{C}}(\mathcal{A})$. The same conclusion is true for the $\widetilde{T}_i$ $(i = 1, \ldots, n)$, since $\widetilde{T}_{-a_i} = \sigma\widetilde{T}_i\sigma$ for all $i = 1, \ldots, n$ by lemma 3.5**(iii)**, where $\sigma = -1 \in W$ is the finite Weyl group element which maps $v \in V$ to $-v$ for all $v \in V$.

We want to think of $\widetilde{T}_i$ as an operator associated with the ith vertex in the extended Dynkin diagram of type $\widetilde{C}_n$ for $i = 1, \ldots, n$. Recalling the definition of the multiplicity function $\mathbf{t}$ and its dual $\tilde{\mathbf{t}}$ (see (3.12)), we see that $\widetilde{T}_i = T_i$ $(i = 1, \ldots, n-1)$ is completely given in terms of the simple root a_i. This is though not the case for $\widetilde{T}_n$ (which depends on the parameters t_0 and t_n), since the parameter t_0 is associated with the zeroth vertex of the extended Dynkin diagram. This can be remedied by the following observation. The $\widetilde{T}_i$ $(i = 1, \ldots, n)$ depend on t_0, t, t_n, while H_0 only depends on t and t_n. So when we regard the $\widetilde{T}_i$ $(i = 1, \ldots, n)$ as operators defining an action of H_0 on $\mathcal{A}$, the parameter t_0 occurs as a "dummy parameter", i.e. an extra degree of freedom. In particular, replacing t_0 by $t_n^\vee$ in the operator $\widetilde{T}_n$ does not change the quadratic relations and braid relations. But interchanging the value t_0 by $t_n^\vee$ results in replacing the dual multiplicity function $\tilde{\mathbf{t}}$ by the multiplicity function $\mathbf{t}$ itself. Hence we conclude that $T_i = T_i(\mathbf{t}|\, q) \in \mathrm{End}_{\mathbb{C}}(\mathcal{A})$ $(i = 1, \ldots, n)$ satisfy the quadratic relations $(T_i - t_i)(T_i + t_i^{-1}) = 0$ $(i = 1, \ldots, n)$, as well as the C_n-braid relations. $\square$

We are now in a position to define the Noumi representation of the affine Hecke algebra H.

Theorem 4.14. *There exists a unique algebra homomorphism $\pi_{\mathbf{t},q} : H(\mathcal{W};\mathbf{t}_R) \to End_{\mathbb{C}}(\mathcal{A})$ satisfying*

$$\pi_{\mathbf{t},q}(V_i) = T_i(\mathbf{t}|\, q) = t_i + t_i^{-1}c_{a_i}(\cdot;\mathbf{t}|q)\big(s_i - 1\big)$$

for $i = 0, \ldots, n$.

Proof. For the proof we need to consider the extended affine Weyl group $\mathcal{W}^e$, which is the sub-group of $\mathrm{End}(\widehat{V})$ generated by $\mathcal{W}$ and the involution

$$\omega = u\tau\big((\epsilon_1 + \epsilon_2 + \cdots + \epsilon_n)/2\big),$$

where $u \in W$ is the Weyl group element which maps the vector $v = (v_1, \ldots, v_n)$ to $(-v_n, \ldots, -v_1)$ for all $v \in V$. It is easy to check that

$$\omega(a_i) = a_{n-i}, \qquad i = 0, \ldots, n$$

hence the extended affine Weyl group $\mathcal{W}^e$ stabilizes the affine root system R. Furthermore, ω stabilizes the lattice $\Lambda = \Lambda_0 + \frac{1}{2}\mathbb{Z}\delta$, so that the canonical action of $\mathcal{W}$ on $\mathcal{A}$ (see lemma 3.1) extends to an action of $\mathcal{W}^e$ on $\mathcal{A}$ by the assignment $w(x^{\mu)} = x^{w\mu}$ for $w \in \mathcal{W}^e$ and $\mu \in \Lambda_0$. In particular,

$$(\omega p)(x) = p(q^{\frac{1}{2}}x_n^{-1}, \ldots, q^{\frac{1}{2}}x_1^{-1}), \qquad p \in \mathcal{A}.$$

Furthermore, the analogue of lemma 3.5**(iii)** for $\omega \in \mathcal{W}^e$ becomes $\omega T_i(\mathbf{t}|\, q)\omega^{-1} = T_{n-i}(\mathbf{t}^\omega|\, q)$ for $i = 0, \ldots, n$, where $\mathbf{t}^\omega$ is the multiplicity function $\mathbf{t}^\omega = \{t_{\omega f} \,|\, f \in R\}$.

Combined with lemma 4.13, we see that the operators $S_i^\omega := T_{n-i}(\mathbf{t}^\omega|\, q)$ $(i = 1, \ldots, n)$ satisfy the quadratic and braid relations of $H_0(W;\mathbf{t}_\Sigma)$, hence the operators

$$S_i := T_{n-i}(\mathbf{t}|\, q), \qquad (i = 1, \ldots, n)$$

satisfy the quadratic and braid relations of $H_0(W;(\mathbf{t}^\omega)_\Sigma)$.

Again using lemma 4.13, we conclude that $(T_i - t_i)(T_i + t_i^{-1}) = 0$ for $i = 0, \ldots, n$, and that the following braid relations are valid in $\mathrm{End}_{\mathbb{C}}(\mathcal{A})$ for $i, j \in \{0, \ldots, n\}$:

$$\begin{aligned} T_iT_{i+1}T_iT_{i+1} &= T_{i_1}T_iT_{i+1}T_i, \qquad i = 0, n-1,\\ T_iT_{i+1}T_1 &= T_{i+1}T_iT_{i+1}, \qquad i = 1, \ldots, n-2,\\ T_iT_j &= T_jT_i, \qquad |i-j| > 1,\ \ (i,j) \notin \{(0,n),(n,0)\}. \end{aligned}$$

So it remains to prove that $T_0T_n = T_nT_0$. But this is a direct consequence of the fact that $\big(a_0, a_n\big) = 0$ (in particular, $s_0(a_n) = a_n$ and $s_n(a_0) = a_0$). □

Remark 4.15. Observe that the representation $\pi_{\mathbf{t},q}$ depends on three extra parameters compared with the affine Hecke algebra $H = H(\mathcal{W};\mathbf{t}_R)$, namely $q, t_0^\vee$ and $t_n^\vee$. The parameter q already entered in the description of the underlying $\mathcal{W}$-action on $\mathcal{A}$: it determines the "shift-length" for the action of the translation part $\tau(\Lambda_0)$. The extra parameters $t_0^\vee, t_n^\vee$ are associated with the extension $R \subset R_{nr}$ of the affine root system R to the non-reduced root system R_{nr}, in which two extra $\mathcal{W}$-orbits are attached to R, one at the zeroth vertex, and one at the nth vertex of the extended Dynkin diagram.

For $t_0^\vee = t_n^\vee = 1$, we are in the setting of Macdonald polynomials. In Cherednik's study of Macdonald polynomials via affine Hecke algebras the associated multiplicity function $\mathbf{t}_R$ is also assumed to be invariant under the action of the extended affine Weyl group $\mathcal{W}^e$ (see the proof of theorem 4.14 for the definition of $\mathcal{W}^e$). In the present setting this would amount to yet another elimination of a degree of freedom since the $\mathcal{W}^e$-invariance of $\mathbf{t}_R$ forces the extra condition $t_0 = t_n$.

In view of corollary 4.10 and (3.9) we can realize the Y-operators $Y_i \in \mathrm{End}_{\mathbb{C}}(\mathcal{A})$ as the image under the Noumi representation $\pi_{\mathbf{t},q}$ of $Z_i \in \mathcal{A}_Z \subset H$:

$$Y_i = \pi_{\mathbf{t},q}(Z_i) \in \mathrm{End}_{\mathbb{C}}(\mathcal{A}), \qquad i = 1, \ldots, n. \tag{4.7}$$

Since the Z_i ($i = 1, \ldots, n$) mutually commute in H, we conclude that the Y_i ($i = 1, \ldots, n$) mutually commute in $\mathrm{End}_{\mathbb{C}}(\mathcal{A})$. This is precisely the content of theorem 3.6.

5. The non-symmetric Koornwinder polynomials

5.1. **Triangularity of the Y-operators.** The lattice Λ_0 contains, besides the cone Λ_0^+ of dominant weights, also the cone $\Lambda_0^>$ consisting of $\mathbb{Z}_+$-linear combinations of the co-roots $\alpha^\vee$ with $\alpha \in \Sigma^+$:

$$\Lambda_0^> = \mathbb{Z}_+ - \mathrm{span}\{\alpha^\vee \mid \alpha \in \Sigma^+\} = \bigoplus_{i=1}^n \mathbb{Z}_+ a_i^\vee.$$

We use $\Lambda_0^>$ to define two partial orders on Λ_0, which both will play an important role in the analysis of the Y-operators. Recall that for arbitrary $\lambda \in \Lambda_0$, we denote $\lambda^+ \in \Lambda_0^+$ for the unique dominant weight within the W-orbit $W\lambda$.

Definition 5.1. *Let $\lambda, \mu \in \Lambda_0$.*
(i) *We write $\lambda \leq \mu$ if $\mu - \lambda \in \Lambda_0^>$ (and $\lambda < \mu$ if $\lambda \leq \mu$ and $\lambda \neq \mu$).*
(ii) *We write $\lambda \preceq \mu$ if $\lambda^+ < \mu^+$, or if $\lambda^+ = \mu^+$ and $\lambda \leq \mu$ (and $\lambda \prec \mu$ if $\lambda \preceq \mu$ and $\lambda \neq \mu$).*

Lemma 5.2. *We have $\mu \leq \mu^+$ for all $\mu \in \Lambda_0$.*

Proof. Let $\mu \in \Lambda_0 \setminus \Lambda_0^+$. We have to show that $\mu < \mu^+$. Recall that the element μ^+ is the unique element in $W\mu$ satisfying $(\mu^+, \alpha) \in \mathbb{Z}_+$ for all $\alpha \in \Sigma^+$. Since $\mu \neq \mu^+$, there thus exists an $\alpha \in \Sigma^+$ with $(\mu, \alpha) \in \mathbb{Z}_{<0}$, so that

$$s_\alpha \mu = \mu - (\mu, \alpha)\alpha^\vee > \mu.$$

If $s_\alpha \mu = \mu^+$, then we are ready. Otherwise, there exists an $\beta \in \Sigma^+$ such that $s_\beta s_\alpha \mu > s_\alpha \mu > \mu$. Continuing this procedure inductively leads to the desired result (indeed, observe that it ends after a finite number of steps since $\#W\mu < \infty$). □

We have now the following technical lemma.

Lemma 5.3. *Let $\mu \in \Lambda_0$ and $\alpha \in \Sigma^+$.*
If $(\mu, \alpha) \geq 2$, then $\mu - r\alpha^\vee \prec \mu$ for $r = 1, \ldots, (\mu, \alpha) - 1$.
If $(\mu, \alpha) \leq -2$, then $\mu + r\alpha^\vee \prec \mu$ for $r = 1, \ldots, -(\mu, \alpha) - 1$.

Proof. We write $m_\alpha = (\mu, \alpha) \in \mathbb{Z}$. Suppose that $m_\alpha \geq 2$ and write $\mu_r = \mu - r\alpha^\vee$ with $r \in \{1, \ldots, m_\alpha - 1\}$. We show that $\mu_r^+ < \mu^+$.

Let $w \in W$ such that $\mu_r^+ = w\mu_r$. Then $w\alpha^\vee \in \Lambda_0^>$ or $w\alpha^\vee \in -\Lambda_0^>$. If $w\alpha^\vee \in \Lambda_0^>$, then $\mu_r^+ = w\mu - rw\alpha^\vee < w\mu \leq \mu^+$, where we have used lemma 5.2 for the last equality. On the other hand, if $w\alpha^\vee \in -\Lambda_0^>$, then $\mu_r^+ = w\mu - rw\alpha^\vee < w\mu - m_\alpha w\alpha^\vee = (ws_\alpha)\mu \leq \mu^+$. This proves the assertion for $m_\alpha \geq 2$. The case $m_\alpha \leq -2$ can be obtained by applying the previous case to $s_\alpha \mu$. □

For $f \in R$ we define

$$\mathcal{R}(f) = t_f s_f + t_f^{-1} c_f(\cdot)\big(1 - s_f\big), \tag{5.1}$$

where $c_f(\cdot)$ is given by (3.3). Observe that $\mathcal{R}(f) = T_f s_f$ for all $f \in R$, where $T_f = T_f(\mathbf{t}|\, q)$ ($f \in R$) are Noumi's difference-reflection operators defined by (3.5). In particular, it follows that $\mathcal{R}(f) \in \mathrm{End}_{\mathbb{C}}(\mathcal{A})$.

Recall that $\mathrm{sgn} : \mathbb{Z} \to \{\pm 1\}$ is the function which maps a non-negative integer to 1 and a strictly negative integer to -1.

Lemma 5.4. *Let $\lambda \in \Lambda_0$. For $f = \alpha + m\delta \in R^+$ with $\alpha \in \Sigma^+$ we have*

$$\mathcal{R}(f)(x^\lambda) = t_f^{sgn((\lambda,f))} x^\lambda + \sum_{\mu \prec \lambda} c_{\lambda,\mu} x^\mu$$

for certain constants $c_{\lambda,\mu} \in \mathbb{C}$.

Proof. Let $f = \alpha + m\delta \in R^+$ with $\alpha \in \Sigma^+$. Then necessarily $m \in \mathbb{Z}_+$. The lemma is clear when $\big(\lambda, f\big) = 0$, since then $s_f(x^\lambda) = x^\lambda$. So we assume that $\big(\lambda, f\big) \neq 0$. We can write

$$\mathcal{R}(f)(x^\lambda) = t_f x^{s_f \lambda} + t_f^{-1}\big(1 - t_f t_{f/2} x^{f/2}\big)\big(1 + t_f t_{f/2}^{-1} x^{f/2}\big) D_f(x^\lambda), \tag{5.2}$$

where $D_f \in \mathrm{End}_{\mathbb{C}}(\mathcal{A})$ is the linear operator defined by (3.6). We distinguish between the two cases $\big(\lambda, f\big) > 0$ and $\big(\lambda, f\big) < 0$.

If $\big(\lambda, f\big) > 0$, then (5.2) and (3.7) show that

$$\mathcal{R}(f)(x^\lambda) = t_f x^\lambda + \sum_{r=1}^{(\lambda,f)} d_r x^{\lambda - r\alpha^\vee}$$

for certain constants d_r. Now by lemma 5.3, and by the fact that $s_\alpha \lambda = \lambda - \big(\lambda, f\big)\alpha^\vee \prec \lambda$, we arrive at $\mathcal{R}(f)(x^\lambda) = t_f x^\lambda +$ lower order terms w.r.t. $\preceq$.

If $\big(\lambda, f\big) < 0$, then the $x^{s_\alpha \lambda}$ term in the expansion of $\mathcal{R}(f)(x^\lambda)$ as linear combination of monomials is zero. Indeed, by (3.7) the contribution to $x^{s_\alpha\lambda}$ in the expansion of

$$t_f^{-1}\big(1 - t_f t_{f/2} x^{f/2}\big)\big(1 + t_f t_{f/2}^{-1} x^{f/2}\big) D_f(x^\lambda)$$

in monomials is given by $-t_f q^{m(\lambda,\alpha^\vee)} x^{s_\alpha \lambda}$, which cancels with the first factor $t_f x^{s_f \lambda}$ in (5.2).

It follows then from (5.2) and (3.7) that

$$\mathcal{R}(f)(x^\lambda) = t_f^{-1} x^\lambda + \sum_{r=1}^{-(\lambda,f)-1} d_r x^{\lambda + r\alpha^\vee}$$

for certain constants d_r, where the sum is empty if $\big(\lambda, f\big) = -1$. By lemma 5.3, we thus see that $\mathcal{R}(f)(x^\lambda) = t_f^{-1} x^\lambda +$ lower order terms w.r.t. $\preceq$, as desired. □

Observe that $\mathcal{R}(wf) = w\mathcal{R}(f)w^{-1}$ for all $w \in \mathcal{W}$ and all $f \in R$. Combined with (3.8) and (3.9), we obtain

$$\begin{aligned} Y_i =& \mathcal{R}(\epsilon_i - \epsilon_{i+1})\mathcal{R}(\epsilon_i - \epsilon_{i+2}) \cdots \mathcal{R}(\epsilon_i - \epsilon_n)\mathcal{R}(2\epsilon_i) \\ &\times \mathcal{R}(\epsilon_i + \epsilon_n) \cdots \mathcal{R}(\epsilon_i + \epsilon_{i+1})\mathcal{R}(\epsilon_i + \epsilon_{i-1}) \cdots \mathcal{R}(\epsilon_i + \epsilon_1) \\ &\times \mathcal{R}(2\epsilon_i + \delta)\tau(\epsilon_i)\mathcal{R}(\epsilon_1 - \epsilon_i)^{-1} \cdots \mathcal{R}(\epsilon_{i-1} - \epsilon_i)^{-1} \end{aligned} \tag{5.3}$$

for $i = 1, \ldots, n$. This leads to the following result.

Proposition 5.5. *For all $i = 1, \ldots, n$ and $\lambda \in \Lambda_0$ we have*

$$Y_i(x^\lambda) = \gamma_{\lambda,i} x^\lambda + \sum_{\mu \prec \lambda} c_{\lambda,\mu} x^\mu$$

for certain constants $c_{\lambda,\mu} \in \mathbb{C}$, where $\gamma_\lambda = (\gamma_{\lambda,1}, \ldots, \gamma_{\lambda,n}) \in \big(\mathbb{C} \setminus \{0\}\big)^n$ is given by (3.11).

Proof. The triangularity of the factors $\mathcal{R}(\cdot)$ in (5.3) (see lemma 5.4) implies the triangularity of Y_i for $i = 1, \ldots, n$. Now it can be shown that the diagonal term is given by $\gamma_{\lambda,i}$ by carefully collecting the leading terms coming from repeated application of lemma 5.4 to the factors $\mathcal{R}(\cdot)$ in (5.3) acting on x^λ, and using $\tau(\epsilon_i)(x^\lambda) = q^{(\lambda,\epsilon_i)} x^\lambda$. □

5.2. **The definition of the non-symmetric Koornwinder polynomials.** Our first objective of this subsection is to prove that the diagonal terms $\{\gamma_\lambda \,|\, \lambda \in \Lambda_0\}$ (see (3.11)) are pair-wise different for generic parameters $\mathbf{t}$ and q. For this we need some standard facts on parabolic sub-groups of W. These facts hold in greater generality, see [14, §1.10, §1.12] for more details.

Let $I \subset \{s_1, \ldots, s_n\}$ be any subset of the simple reflections of W, and $W_I \subset W$ the sub-group generated by I (which is called a parabolic sub-group of W). Then in any coset $wW_I \in W/W_I$, there exists a unique element u of minimal length. The corresponding set of reprentatives W^I of the coset space W/W_I are called the *minimal coset representatives.* Hence any $w \in W$ can be uniquely written as $w = uv$ with $u \in W^I$ and $v \in W_I$. Furthermore, the minimality of the length of u forces the additivity of lengths in this decomposition:

$$l(uv) = l(u) + l(v), \qquad u \in W^I, \;\; v \in W_I. \tag{5.4}$$

There is an alternative description of W^I in terms of root systems as follows. The root sub-system $\Sigma_I \subset \Sigma$ defined by

$$\Sigma_I = \mathbb{R} - \mathrm{span}\{a_i \,|\, i \in \{1, \ldots, n\} : \;\; s_i \in W_I\} \cap \Sigma$$

has $\{a_i \,|\, s_i \in W_I\}$ as a basis, with corresponding positive roots given by $\Sigma_I^+ = \Sigma_I \cap \Sigma^+$. Its Weyl group can be naturally identified with the parabolic sub-group $W_I \subset W$. Then the minimal coset representatives W^I can alternatively be described by

$$W^I = \{w \in W \,|\, w(\Sigma_I^+) \subset \Sigma^+\}. \tag{5.5}$$

Stabilizer sub-groups of dominant weights $\lambda \in \Lambda_0^+$ are examples of parabolic sub-groups, since the stabilizer sub-group $W_\lambda = \{w \in W \,|\, w\lambda = \lambda\}$ is generated by the simple reflections s_i ($i = 1, \ldots, n$) in W_λ, see [14, §1.12]. The coset space W/W_λ is then in one-to-one correspondence with the W-orbit $W\lambda$, the coset wW_λ ($w \in W$) corresponding to the element $w\lambda$ in the orbit $W\lambda$. In other words, for $\lambda \in \Lambda_0^+$ the minimal coset representatives W^λ of W/W_λ are exactly the elements $w_\mu \in W$ ($\mu \in W\lambda$) with $w_\mu \in W$ the unique element of minimal length such that $w_\mu\lambda = \mu$.

Lemma 5.6. *For $\lambda \in \Lambda_0$ and $p \in \mathcal{A}$ we have $p(\gamma_\lambda^{\pm 1}) = (w_\lambda^{-1} p)(\gamma_{\lambda^+}^{\pm 1})$. In particular, if $\lambda \in \Lambda_0$ and $i \in \{1, \ldots, n\}$ are such that $s_i\lambda \neq \lambda$, then $(s_i p)(\gamma_\lambda^{\pm 1}) = p(\gamma_{s_i\lambda}^{\pm 1})$.*

Proof. For the first statement of the lemma, it suffices to prove that

$$w_\lambda \rho_m(\lambda^+) = \rho_m(\lambda), \qquad w_\lambda \rho_l(\lambda^+) = \rho_l(\lambda), \qquad \forall \lambda \in \Lambda_0,$$

see (3.11). We prove the first identity, the second is proved in a similar manner. We rewrite $\rho_m(\lambda)$ as follows:

$$\rho_m(\lambda) = w_\lambda \left(\sum_{\alpha\in\Sigma_m^+} \mathrm{sgn}\big((\lambda^+, w_\lambda^{-1}\alpha)\big)(w_\lambda^{-1}\alpha)^\vee \right).$$

Comparing with the expression $\rho_m(\lambda^+) = \sum_{\alpha\in\Sigma_m^+}\alpha^\vee = \rho_m$ (see (3.24)), it suffices to prove that $\big(\lambda^+, w_\lambda^{-1}\alpha\big) < 0$ for $\alpha\in\Sigma_m^+$ with $w_\lambda^{-1}\alpha\in\Sigma^-$.

We fix an $\alpha\in\Sigma_m^+$ so that $w_\lambda^{-1}\alpha\in\Sigma^-$. Let I be the sub-set of simple reflections in W which stabilize λ^+, so that $W_I = W_{\lambda^+}$. Then $w_\lambda(\Sigma_I^+)\subset\Sigma^+$, and since w_λ preserves lengths of roots, we have $w_\lambda(\Sigma_I\cap\Sigma_m^-)\subset\Sigma_m^-$. It follows that $w_\lambda^{-1}\alpha\in\Sigma_m^-\setminus(\Sigma_m^-\cap\Sigma_I)$, hence $\big(\lambda^+, w_\lambda^{-1}\alpha\big)<0$, as desired.

For the second statement of the lemma it suffices to show that $s_iw_\lambda = w_{s_i\lambda}$ for $i\in\{1,\ldots,n\}$ and $\lambda\in\Lambda_0$ with $s_i\lambda\neq\lambda$. To show this, we remark that $l(s_iw_\lambda) = l(w_\lambda)\pm 1$ iff $w_\lambda^{-1}a_i\in\Sigma^\pm$ by the definition of the length function, since s_i permutes $\Sigma^+\setminus\{a_i\}$ and maps a_i to $-a_i$. Since $s_i\lambda\neq\lambda$ by assumption, we obtain $\big(\lambda, a_i\big) = \big(\lambda^+, w_\lambda^{-1}a_i\big) \gtrless 0$ iff $w_\lambda^{-1}a_i\in\Sigma^\pm$ iff $l(s_iw_\lambda) = l(w_\lambda)\pm 1$. In particular, it suffices to prove $s_iw_\lambda = w_{s_i\lambda}$ when $s_i\lambda\neq\lambda$ and $l(s_iw_\lambda) = l(w_\lambda)-1$, since the case $l(s_iw_\lambda) = l(w_\lambda)+1$ then follows by replacing λ by $s_i\lambda$.

Now suppose that $l(s_iw_\lambda) = l(w_\lambda)-1$ and $s_i\lambda\neq\lambda$. We have $s_iw_\lambda\in w_{s_i\lambda}W_\lambda$, so if $s_iw_\lambda\neq w_{s_i\lambda}$, then $l(w_{s_i\lambda}) < l(s_iw_\lambda) = l(w_\lambda)-1$, hence $l(s_iw_{s_i\lambda}) < l(w_\lambda)$. But $s_iw_{s_i\lambda}\in w_\lambda W_\lambda$, so we arrive at a contradiction with the minimality of the length of w_λ within the coset $w_\lambda W_\lambda$. Hence $s_iw_\lambda = w_{s_i\lambda}$, as desired. □

Now from the explicit expressions (3.25) for the diagonal elements $\gamma_\lambda\in\big(\mathbb{C}\setminus\{0\}\big)^n$ with $\lambda\in\Lambda_0^+$, it is immediate that the elements

$$\big(\gamma_{\lambda,w(1)}^{\xi_1},\ldots,\gamma_{\lambda,w(n)}^{\xi_n}\big)\in\big(\mathbb{C}\setminus\{0\}\big)^n,\qquad \lambda\in\Lambda_0^+,\ w\in S_n,\ \xi_j\in\{\pm1\}$$

are pair-wise different for generic values of $\mathbf{t}$ and q. By the (proof of the) previous lemma, this implies that the diagonal elements $\gamma_\lambda\in\big(\mathbb{C}\setminus\{0\}\big)^n$ ($\lambda\in\Lambda_0$) are pair-wise different for generic values of $\mathbf{t}$ and q.

Theorem 5.7. *There exists a unique basis $\{P_\lambda = P_\lambda(\cdot;\mathbf{t}|\,q)\}_{\lambda\in\Lambda_0}$ of $\mathcal{A}$ such that*

- *$P_\lambda(x) = x^\lambda + \sum_{\mu\prec\lambda} c_{\lambda,\mu}x^\mu$ for certain constants $c_{\lambda,\mu}$,*
- *$p(Y)P_\lambda = p(\gamma_\lambda)P_\lambda$ for all $p\in\mathcal{A}$.*

Proof. Let $\propto$ be a total order on Λ_0 such that $\lambda\prec\mu$ implies $\lambda\propto\mu$.

We fix $\lambda\in\Lambda_0$. Let $\mathcal{A}_\lambda$ be the finite dimensional sub-space of $\mathcal{A}$ spanned by the monomials x^μ with $\mu\preceq\lambda$. Then for all $p\in\mathcal{A}$, $p(Y)$ preserves $\mathcal{A}_\lambda$ by proposition 5.5. In fact, with respect to the basis x^μ ($\mu\preceq\lambda$), ordered along the total ordering $\propto$, $p(Y)$ is represented by a triangular matrix with diagonal terms given by $p(\gamma_\mu)$ ($\mu\preceq\lambda$). We choose now $p\in\mathcal{A}$ such that $p(\gamma_\mu)\neq p(\gamma_\lambda)$ for all $\mu\prec\lambda$, then it follows that $p(Y)|_{\mathcal{A}_\lambda}$ has a non-zero eigenfunction $p_\lambda\in\mathcal{A}_\lambda$ with eigenvalue $p(\gamma_\lambda)$, which is unique up to a non-zero multiplicative constant. Furthermore, the coefficient of x^λ in the expansion of such an eigenfunction $p_\lambda(x)$ in terms of monomials x^μ ($\mu\preceq\lambda$) is non-zero. Hence $p(Y)|_{\mathcal{A}_\lambda}$ has exactly one eigenfunction $P_\lambda\in\mathcal{A}_\lambda$ with eigenvalue $p(\gamma_\lambda)$ and with the coefficient of x^λ in the expansion of $P_\lambda(x)$ in terms of monomials x^μ ($\mu\preceq\lambda$) being equal to one.

For any other $g \in \mathcal{A}$, we now have $g(Y)P_\lambda = g(\gamma_\lambda)x^\lambda + \sum_{\mu \prec \lambda} c_{\lambda,\mu} x^\mu \in \mathcal{A}_\lambda$ for certain constants $c_{\lambda,\mu}$ by proposition 5.5, and

$$p(Y)\big(g(Y)P_\lambda\big) = g(Y)\big(p(Y)P_\lambda\big) = p(\gamma_\lambda)\big(g(Y)P_\lambda\big),$$

hence $g(Y)P_\lambda = g(\gamma_\lambda)P_\lambda$. This concludes the proof of the theorem. □

The Laurent polynomial $P_\lambda \in \mathcal{A}$ is exactly the monic non-symmetric Koornwinder polynomial of degree $\lambda \in \Lambda_0$ as defined in definition 3.8. Indeed, theorem 5.7 implies theorem 3.7 since the diagonal terms γ_λ ($\lambda \in \Lambda_0$) are pair-wise different.

6. The double affine Hecke algebra and duality

6.1. **The double affine Hecke algebra.** The double affine Hecke algebra plays an indispensable role in the understanding of duality for the non-symmetric Koornwinder polynomials. It is defined as follows.

Definition 6.1. *The double affine Hecke algebra $\mathcal{H} = \mathcal{H}(\mathbf{t}|\, q)$ is the sub-algebra of $End_{\mathbb{C}}(\mathcal{A})$ generated by the image $\pi_{\mathbf{t},q}(H(W;\mathbf{t}_R))$ of the affine Hecke algebra $H(W;\mathbf{t}_R)$ under the Noumi representation, and by $\mathcal{A}$ (regarded here as multiplication operators in $End_{\mathbb{C}}(\mathcal{A})$).*

In other words, $\mathcal{H}$ is generated by Noumi's difference-reflection operators $T_i \in \mathrm{End}_{\mathbb{C}}(\mathcal{A})$ ($i = 0, \ldots, n$) and $\mathcal{A}$ (considered as multiplication operators). In order to avoid confusion later on, we write $p(z) \in \mathrm{End}_{\mathbb{C}}(\mathcal{A})$ for the element $p \in \mathcal{A}$ considered as multiplication operator, and z_i for the multiplication operator z^{ϵ_i} ($i = 1, \ldots, n$).

In order to understand the algebraic structure of $\mathcal{H}$, we need to construct an explicit basis of $\mathcal{H}$ first. For this we need a preliminary proposition on the action of the affine Weyl group $\mathcal{W}$ on the field $\mathcal{Q}$ of rational functions in the n indeterminates $x_1, \ldots, x_n$, see proposition 3.1 for the definition of this action.

Proposition 6.2. *The affine Weyl group elements $w \in \mathcal{W}$, considered as endomorphism of $\mathcal{Q}$ via the action defined in lemma 3.1, are $\mathcal{Q}$-linearly independent (where we regard $End_{\mathbb{C}}(\mathcal{Q})$ as a $\mathcal{Q}$-module in the obvious manner, i.e. $\mathcal{Q}$ acts as multiplication operators).*

Proof. Suppose that $\sum_{w,\lambda} c_{w,\lambda}(x) w\tau(\lambda) = 0$ in $\mathrm{End}_{\mathbb{C}}(\mathcal{Q})$ ($w \in W$, $\lambda \in \Lambda_0$), with $c_{w,\lambda} \in \mathcal{Q}$ not all zero (but only finitely many being non-zero). We show that this leads to a contradiction.

Multiplying out the denominators of $c_{w,\lambda}$, we may assume without loss of generality that $c_{w,\lambda} \in \mathcal{A}$. Hence there exist coefficients $c_{w,\lambda,\mu} \in \mathbb{C}$, not all zero (but non zero for only a finite number of triples $(w, \lambda, \mu) \in W \times \Lambda_0^{\times 2}$), so that

$$\sum_{w,\lambda,\mu} c_{w,\lambda,\mu} x^\mu w\tau(\lambda) = 0$$

in $\mathrm{End}_{\mathbb{C}}(\mathcal{Q})$. In other words, there exist Laurent polynomials $p_{w,\mu} \in \mathcal{A}$, not all zero (but non zero for only finitely many $(w, \mu) \in W \times \Lambda_0$), so that

$$\sum_{w,\mu} x^\mu w p_{w,\mu}(\tau(\epsilon_1), \ldots, \tau(\epsilon_n)) = 0$$

in $\mathrm{End}_{\mathbb{C}}(\mathcal{Q})$. Applying this to x^ν with $\nu \in \Lambda_0$ arbitrary, we get

$$\sum_{w,\mu} x^{\mu + w\nu} p_{w,\mu}(q^{\nu_1}, \ldots, q^{\nu_n}) = 0$$

in $\mathcal{A}$ for all $\nu \in \Lambda_0$, where $\nu_i = \big(\nu, \epsilon_i\big)$. Fix now $w \in W$ and $\mu \in \Lambda_0$ with $p_{w,\mu} \neq 0$. It then follows that $p_{w,\mu}(q^{\nu_1}, \ldots, q^{\nu_n}) = 0$ for those $\nu \in \Lambda_0$ satisfying $\mu + w\nu \neq \mu' + w'\nu$ for all pairs $(\mu', w') \neq (\mu, w)$ with $p_{w',\mu'} \neq 0$. But there are only finitely many pairs (μ', w') with $p_{w',\mu'} \neq 0$, and q is assumed to be generic, hence we conclude that $p_{w,\mu} = 0$, which is a contradiction. Hence the automorphisms $w \in \mathcal{W}$ of $\mathcal{Q}$ are $\mathcal{Q}$-linear independent, as desired. $\square$

This leads directly to the following result.

Theorem 6.3. *The sets* $\{z^\lambda T_w \,|\, \lambda \in \Lambda_0,\ w \in \mathcal{W}\}$ *and* $\{z^\lambda T_w Y^\mu \,|\, \lambda, \mu \in \Lambda_0,\ w \in W\}$ *are linear bases of the double affine Hecke algebra* $\mathcal{H}$.

Proof. It suffices to prove the theorem for $\{z^\lambda T_w \,|\, \lambda \in \Lambda_0,\ w \in \mathcal{W}\}$ by (the proof of) corollary 4.12. We need the Bruhat decomposition $\leq$ on $\mathcal{W}$, which can be defined as follows: let $u, w \in \mathcal{W}$, and let $w = s_{i_1} s_{i_2} \cdots s_{i_r}$ be a fixed reduced expression of w. Then $u \leq w$ if there exists a sequence $1 \leq j_1 < j_2 < \ldots j_p \leq r$ such that

$$u = s_{i_{j_1}} s_{i_{j_2}} \cdots s_{i_{j_p}}.$$

This defines a partial order on $\mathcal{W}$ (which is not obvious with the present definition, but see [14, §5,9, §5.10] for more details). From the explicit form of Noumi's difference-reflection operators T_i $(i = 0, \ldots, n)$, it is now obvious that

$$T_w = \sum_{u \leq w} a_{w,u}(x) u \in \mathrm{End}_{\mathbb{C}}(\mathcal{A}) \qquad (w \in \mathcal{W}) \tag{6.1}$$

for certain uniquely defined $a_{w,u} \in \mathcal{Q}$. Furthermore, $a_{w,w} \neq 0$. Suppose now that $\sum_{\lambda,w} c_{\lambda,w} z^\lambda T_w = 0$ on $\mathcal{H}$, with only finitely many constants $c_{\lambda,w} \neq 0$ (but not all zero). Then by (6.1),

$$\sum_{\lambda,w} c_{\lambda,w} \sum_{u \leq w} z^\lambda a_{w,u}(z) u = 0 \quad \text{in} \quad \mathrm{End}_{\mathbb{C}}(\mathcal{A}).$$

Let now w be a maximal element of the finite, non-empty set

$$\{u \in \mathcal{W} \,|\, c_{\lambda,u} \neq 0 \quad \text{for some} \quad \lambda \in \Lambda_0\}$$

with respect to the Bruhat-order. Then the previous proposition implies

$$a_{w,w}(x) \sum_\lambda c_{\lambda,w} x^\lambda = 0 \quad \text{in} \ \mathcal{Q}.$$

But not all $c_{\lambda,w}$ are zero, and $a_{w,w} \in \mathcal{Q}$ is non-zero, hence this leads to the desired contradiction. $\square$

We can now give a characterizing set of commutation relations within the double affine Hecke algebra $\mathcal{H}$, as follows.

Proposition 6.4. *Let* $\mathcal{F} = \mathcal{F}(\mathbf{t}|\, q)$ *be the unital* $\mathbb{C}$*-algebra generated by* H_i $(i = 0, \ldots, n)$ *and* $u_i^{\pm 1}$ $(i = 1, \ldots, n)$*, with relations*

- u_i^{-1} *is the inverse of* u_i*, and the* u_i*'s pair-wise commute for* $i = 1, \ldots, n$ *(we define* $p(u) \in \mathcal{F}$ *for* $p \in \mathcal{A}$ *now in the usual manner);*
- *The quadratic relations* $(H_i - t_i)(H_i + t_i^{-1}) = 0$ *and the* $\widetilde{C}_n$*-braid relations for* $(H_0, \ldots, H_n)$ *;*
- $H_i p(u) - \big(T_i(\mathbf{t}|\, q) p\big)(u) = \big(s_i p\big)(u)\big(H_i - t_i\big)$ *for* $i = 0, \ldots, n$ *and* $p \in \mathcal{A}$*.*

Then $\mathcal{F}(\mathbf{t}|\, q) \simeq \mathcal{H}(\mathbf{t}|\, q)$ *as algebra by identifying* $p(u) \in \mathcal{F}$ *with* $p(z) \in \mathcal{H}$ $(p \in \mathcal{A})$ *and* $H_j \in \mathcal{F}$ *with* $T_j(\mathbf{t}|\, q) \in \mathcal{H}$ $(j = 0, \ldots, n)$*.*

Proof. We first prove the existence of a surjective algebra homomorphism $\phi : \mathcal{F} \to \mathcal{H}$ mapping H_j to T_j for $j = 0, \ldots, n$ and mapping $p(u)$ to $p(z)$ for $p \in \mathcal{A}$. In other words, we have to verify that the defining relations of $\mathcal{F}$ are also valid in $\mathcal{H}$ when the H_j's are replaced by the T_j's and the u_i's are replaced by the z_i's. Only the last of the defining relations of $\mathcal{F}$ then requires proof. But this follows by substituting the explicit expression for Noumi's difference-reflection operator T_i (see (3.5)): we then have for any $p \in \mathcal{A}$ and $i \in \{0, \ldots, n\}$ that

$$\begin{aligned} T_i\, p(z) - (T_i p)(z) &= t_i^{-1} c_{a_i}(z)(s_i\, p(z) - (s_i p)(z)) \\ &= (s_i p)(z) t_i^{-1} c_{a_i}(z)\big(s_i - \mathrm{Id}\big) = (s_i p)(z)\big(T_i - t_i\big) \end{aligned}$$

in $\mathcal{H} \subset \mathrm{End}_{\mathbb{C}}(\mathcal{A})$. Hence the surjective algebra homomorphism $\phi : \mathcal{F} \to \mathcal{H}$ exists.

In order to prove that ϕ is injective, it suffices to prove that $\mathcal{F}$ is spanned by $\{u^\lambda H_w \,|\, \lambda \in \Lambda_0,\, w \in \mathcal{W}\}$ where $H_w = H_{i_1} H_{i_2} \ldots H_{i_r}$ for a reduced expression $w = s_{i_1} s_{i_2} \cdots s_{i_r}$. This is though immediate from the defining relations of $\mathcal{F}$. □

Sometimes it is convenient to have a presentation of $\mathcal{H}$ which is entirely formulated in terms of the generators T_i $(i = 0, \ldots, n)$ and z_j $(j = 1, \ldots, n)$. We give the characterizing commutation relations in the following proposition.

Proposition 6.5. *The characterizing commutation relations of $\mathcal{H}$ in terms of the algebraic generators T_i $(i = 0, \ldots, n)$ and $z_j^{\pm 1}$ $(j = 1, \ldots, n)$ are given by*

- *z_i^{-1} is the inverse of z_i, and the z_i's pair-wise commute $(i = 1, \ldots, n)$;*
- *The $\widetilde{C}_n$-braid relations for $(T_0, \ldots, T_n)$;*
- *The quadratic relations $(T_i - t_i)(T_i + t_i^{-1}) = 0$ for $i = 0, \ldots, n$;*
- *$T_i z_j = z_j T_i$ for $i = 0, \ldots, n$ and $j = 1, \ldots, n$ with $|i - j| > 1$;*
- *$T_i z_{i-1} = z_{i-1} T_i$ for $i = 2, \ldots, n$;*
- *$T_i z_i T_i = z_{i+1}$ for $i = 1, \ldots, n-1$;*
- *$(z_n^{-1} T_n^{-1} - t_n^\vee)(z_n^{-1} T_n^{-1} + t_n^{\vee -1}) = 0$;*
- *$(q^{-1/2} T_0^{-1} z_1 - t_0^\vee)(q^{-1/2} T_0^{-1} z_1 + t_0^{\vee -1}) = 0$.*

Proof. It is easily verified that the generators T_i and z_i of $\mathcal{H}$ satisfy the commutation relations as stated in the proposition (use proposition 6.4). As an example, we give the details on the quadratic relation for $z_n^{-1} T_n^{-1}$. Making use of the commutation relation between z_n and T_n (see proposition 6.4), we compute

$$\begin{aligned} T_n z_n^{-1} &= z_n T_n + \frac{(t_n^{-1} - t_n) + (t_n^{\vee -1} - t_n^\vee) z_n}{1 - z_n^2}(z_n - z_n^{-1}) \\ &= z_n T_n + (t_n - t_n^{-1}) z_n^{-1} + (t_n^\vee - t_n^{\vee -1}). \end{aligned}$$

Combined with $T_n^{-1} = T_n + t_n^{-1} - t_n$ we conclude that $T_n^{-1} z_n^{-1} = z_n T_n + t_n^\vee - t_n^{\vee -1}$, hence

$$(z_n^{-1} T_n^{-1})^2 = z_n^{-1}(z_n T_n + t_n^\vee - t_n^{\vee -1}) T_n^{-1} = 1 + (t_n^\vee - t_n^{\vee -1}) z_n^{-1} T_n^{-1}.$$

This is equivalent to the quadratic relation for $z_n^{-1} T_n^{-1}$. The checks of the other commutation relations are left for the reader.

The list of commutation relations as stated in this proposition is sufficient to be able to commute T_i with an arbitrary Laurent polynomial $p(z)$ in $\mathcal{H}$. Hence, by a similar argument as in the proof of proposition 6.4 and by making use of the basis for $\mathcal{H}$ given in theorem 6.3, it follows that the given list of commutation relations between the T_i's and the z_j's characterize the algebraic structure of $\mathcal{H}$. □

6.2. **Isomorphisms between double affine Hecke algebras.** Observe that the commutation relations between T_i and $p(Y)$ are very similar to the commutation relations between the T_i and $p(Z)$ for $i=1,\ldots,n$ and $p\in\mathcal{A}$. In fact, they can be written as

$$\begin{aligned} T_i^{-1}p(Y)-\big(\widetilde{T}_i'p\big)(Y)&=(s_ip)(Y)(T_i^{-1}-t_i^{-1}),\\ T_i\,p(z)-\big(T_ip\big)(z)&=(s_ip)(z)(T_i-t_i)\end{aligned}\tag{6.2}$$

for $i=1,\ldots,n$ and $p\in\mathcal{A}$, where $\widetilde{T}_i':=T_i(\tilde{\mathbf{t}}^{-1}|\,q^{-1})=T_{-a_i}(\tilde{\mathbf{t}}|\,q)^{-1}$ (for the first equality in (6.2), we have used that $T_i^{-1}-t_i^{-1}=T_i-t_i$). The last equality of (6.2) also holds true for $i=0$. Now if we write $\widetilde{Y}_i',\widetilde{z}_i'$ $(i=1,\ldots,n)$ etc. for the generators of $\widetilde{\mathcal{H}}'=\mathcal{H}(\tilde{\mathbf{t}}^{-1}|\,q^{-1})$, then (6.2) suggests the existence of an algebra homomorphism $\epsilon:\mathcal{H}\to\widetilde{\mathcal{H}}'$ mapping T_i to $\widetilde{T}_i'^{-1}$, mapping z_i to $\widetilde{Y}_i'$ and mapping Y_i to $\widetilde{z}_i'$ $(i=1,\ldots,n)$. Such a map ϵ indeed respects the commutation relations (6.2). There is though more to check, since the relations (6.2), together with the quadratic and C_n-braid relations for T_i $(i=1,\ldots,n)$ are not sufficient to characterize the algebraic structure of $\mathcal{H}$.

In other words, we have to convince ourselves that ϵ preserves the (defining) commutation relations between the z_i and Y_j. These relation are hard to make explicit, so instead we use the presentation of $\mathcal{H}$ in terms of T_i $(i=0,\ldots,n)$ and z_j $(j=1,\ldots,n)$ as given in proposition 6.4. To formulate ϵ in terms of these generators, we first observe that T_0 can be written as

$$T_0=T_1^{-1}T_2^{-1}\cdots T_{n-1}^{-1}T_n^{-1}Y_nT_{n-1}\cdots T_2T_1,$$

see (3.9). We write

$$U_0=T_1T_2\cdots T_{n-1}z_n^{-1}T_n^{-1}T_{n-1}^{-1}\cdots T_2^{-1}T_1^{-1}\in\mathcal{H},\tag{6.3}$$

and $\widetilde{U}_0'\in\widetilde{\mathcal{H}}'$ for the element U_0 with the parameters $(\mathbf{t},q)$ replaced by $(\tilde{\mathbf{t}}^{-1},q^{-1})$. Then, provided the existence of the algebra homomorphism ϵ, the image of T_0 under ϵ would be $\widetilde{U}_0'^{-1}$. Since T_0 satisfies the second commutation relation in (6.2) for $i=0$, the existence of the algebra homomorphism ϵ amounts to checking the following commutation relations.

Proposition 6.6. *The element $U_0\in\mathcal{H}$ satisfies the following commutation relations:*

- $(U_0-t_n^\vee)(U_0+t_n^{\vee-1})=0$*;*
- $U_0T_1U_0T_1=T_1U_0T_1U_0$ *and* $U_0T_i=T_iU_0$ *for* $i=2,\ldots,n$*;*
- $U_0^{-1}p(Y)-\big(\widetilde{T}_0'p\big)(Y)=(s_0'\,p)(Y)(U_0^{-1}-t_n^{\vee-1})$ *for all* $p\in\mathcal{A}$*, where* s_0' *acts on* $\mathcal{A}$ *by* $(s_0'\,p)(x)=p(q^{-1}x_1^{-1},x_2,\ldots,x_n)$*.*

Proof. The proof of the proposition is rather tedious, and is therefore postponed to §9. □

This proposition, combined with proposition 6.4, implies that the algebra homomorphism ϵ exists. More precisely, it leads to the following theorem.

Theorem 6.7. *There exists a unique algebra isomorphism $\epsilon=\epsilon_{\mathbf{t},q}:\mathcal{H}\to\widetilde{\mathcal{H}}'$ mapping Y_i to $\widetilde{z}_i'$, mapping z_i to $\widetilde{Y}_i'$ and mapping T_i to $\widetilde{T}_i'^{-1}$ for $i=1,\ldots,n$. Furthermore, ϵ maps T_0 to $\widetilde{U}_0'^{-1}$. The inverse of ϵ is given by $\widetilde{\epsilon}':=\epsilon_{\tilde{\mathbf{t}}^{-1},q^{-1}}$.*

Proof. We define $\epsilon(T_0) = \widetilde{U}_0'^{-1}$, $\epsilon(T_i) = \widetilde{T}_i'^{-1}$, $\epsilon(z_i) = \widetilde{Y}_i'$ for $i = 1, \ldots, n$. By proposition 6.6 and (6.2), it is easy to check that ϵ preserves the characterizing commutation relations of $\mathcal{H}$ as given in proposition 6.4, hence ϵ extends uniquely to an algebra homomorphism $\epsilon : \mathcal{H} \to \widetilde{\mathcal{H}}'$.

We show now that $\epsilon(Y_i) = \widetilde{z}_i'$ for $i = 1, \ldots, n$. By proposition 6.5 we have $z_{i+1} = T_i z_i T_i$ for $i = 1, \ldots, n-1$. Furthermore,

$$z_n = T_n^{-1} \cdots T_2^{-1} T_1^{-1} U_0^{-1} T_1 T_2 \cdots T_{n-1}$$

by the definition of U_0, so that

$$z_i = T_i^{-1} T_{i+1}^{-1} \cdots T_{n-1}^{-1} T_n^{-1} T_{n-1}^{-1} \cdots T_2^{-1} T_1^{-1} U_0^{-1} T_1 T_2 \cdots T_{i-1}, \qquad i = 1, \ldots, n.$$

Combined with (3.9) and the definition of ϵ, we conclude that $\epsilon(Y_i) = \widetilde{z}_i'$ for $i = 1, \ldots, n$.

Consider now the algebra homomorphism $\widetilde{\epsilon}' \circ \epsilon : \mathcal{H} \to \mathcal{H}$. This algebra homomorphism acts as the identity on the generators T_i, Y_i and z_i ($i = 1, \ldots, n$) of $\mathcal{H}$, hence $\widetilde{\epsilon}' \circ \epsilon = \mathrm{Id}_{\mathcal{H}}$. Similarly, we see that $\epsilon \circ \widetilde{\epsilon}' = \mathrm{Id}_{\widetilde{\mathcal{H}}'}$. Hence ϵ is an algebra isomorphism with inverse $\widetilde{\epsilon}'$, as desired. □

Using theorem 6.7 and proposition 6.4, we obtain the following stronger version of proposition 6.6. It can be seen as the the counterpart of proposition 6.4, in which the role of the z-operators are replaced by the role of the Y-operators.

Proposition 6.8. *The elements U_0, T_i and Y_i ($i = 1, \ldots, n$) generate $\mathcal{H}$ as an algebra. The characterizing commutation relations of $\mathcal{H}$ with respect to these generators are given by*

- $(U_0 - t_n^\vee)(U_0 + t_n^{\vee -1}) = 0$ *and* $(T_i - t_i)(T_i + t_i^{-1}) = 0$ *for* $i = 1, \ldots, n$*;*
- *The $\widetilde{C}_n$-braid relations for $(U_0, T_1, \ldots, T_n)$;*
- *Y_i^{-1} is the inverse of Y_i and the Y_i pair-wise commute ($i = 1, \ldots, n$);*
- *The Lusztig-type commutation relations*

$$\begin{aligned} U_0^{-1} p(Y) - \big(\widetilde{T}_0' p\big)(Y) &= (s_0' p)(Y)(U_0^{-1} - t_n^{\vee -1}), \\ T_i^{-1} p(Y) - \big(\widetilde{T}_i' p\big)(Y) &= (s_i p)(Y)(T_i^{-1} - t_i^{-1}) \end{aligned}$$

for $i = 1, \ldots, n$ and $p \in \mathcal{A}$.

The so-called duality (anti-)isomorphism, is the composition of ϵ with the following elementary (anti-)isomorphism.

Lemma 6.9. *There exists a unique algebra isomorphism $\dagger = \dagger_{\mathbf{t},q} : \mathcal{H} \to \mathcal{H}'$ (respectively anti-algebra isomorphism $\ddagger = \ddagger_{\mathbf{t},q} : \mathcal{H} \to \mathcal{H}'$) satisfying $T_i \mapsto T_i'^{-1}$ ($i = 0, \ldots, n$) and $z_j \mapsto z_j'^{-1}$ ($j = 1, \ldots, n$).*

Proof. This follows by verifying that the characterizing algebraic relations of $\mathcal{H}$ in terms of the generators z_i ($i = 1, \ldots, n$) and T_j ($j = 0, \ldots, n$) are respected by $\dagger$ (respectively $\ddagger$) when $\dagger$ (respectively $\ddagger$) is formally extended as algebra homomorphism (respectively anti-algebra homomorphism). The actual verification is easy and is left to the reader. □

We use the notation $X^\ddagger$ and $X^\dagger$ for the image of $X \in \mathcal{H}$ under $\ddagger$ and $\dagger$, respectively. Furthermore, we write $\widetilde{\dagger}'$ (respectively $\widetilde{\ddagger}'$) for $\dagger$ (respectively $\ddagger$) with respect to the parameters $(\tilde{\mathbf{t}}^{-1}, q^{-1})$.

Definition 6.10. **(i)** *We call the algebra isomorphism* $\Phi = \Phi_{\mathbf{t},q} = \widetilde{\dagger}' \circ \epsilon : \mathcal{H} \to \widetilde{\mathcal{H}}$ *the duality isomorphism of* $\mathcal{H}$.
(ii) *We call the anti-algebra isomorphism* $\Psi = \Psi_{\mathbf{t},q} = \widetilde{\ddagger}' \circ \epsilon : \mathcal{H} \to \widetilde{\mathcal{H}}$ *the duality anti-isomorphism of* $\mathcal{H}$.

Observe that Φ (respectively Ψ) is uniquely characterized as the (anti-)algebra homomorphism $\mathcal{H} \to \widetilde{\mathcal{H}}$ which maps U_0 to $\widetilde{T}_0$, T_i to $\widetilde{T}_i$ and Y_i to $\widetilde{z}_i^{-1}$ for $i = 1, \ldots, n$, where we denote $\widetilde{T}_i, \widetilde{z}_i$ etc. for the generators T_i, z_i etc. in the double affine Hecke algebra $\widetilde{\mathcal{H}} = \mathcal{H}(\tilde{\mathbf{t}}|\, q)$ with respect to dual parameters.

Corollary 6.11. Ψ *maps* z_i *to* $\widetilde{Y}_i^{-1}$ *for* $i = 1, \ldots, n$*. In particular, the inverse of the duality anti-isomorphism* $\Psi = \Psi_{\mathbf{t},q}$ *is given by* $\widetilde{\Psi} := \Psi_{\tilde{\mathbf{t}},q}$.

Proof. We compute $(Y^\lambda)^\ddagger$ ($\lambda \in \Lambda_0$), i.e. the image of the Y-operator $Y^\lambda \in \mathcal{H}$ under the anti-isomorphism $\ddagger$. We first assume that $\lambda \in \Lambda_0^+$. Let $\tau(\lambda) = s_{i_1} s_{i_2} \cdots s_{i_r}$ be a reduced expression of $\tau(\lambda)$ in $\mathcal{W}$. Then by the definition of $\ddagger$, we see that

$$(Y^\lambda)^\ddagger = (T_{\tau(\lambda)})^\ddagger = (T_{i_1} T_{i_2} \cdots T_{i_r})^\ddagger = (T'_{i_1} T'_{i_2} \cdots T'_{i_r})^{-1} = Y'^{-\lambda}.$$

It follows that $(Y^\lambda)^\ddagger = Y'^{-\lambda}$ for all $\lambda \in \Lambda_0$, hence $Y_i^\ddagger = Y_i'^{-1}$ for $i = 1, \ldots, n$. So $\Psi = \widetilde{\ddagger}' \circ \epsilon$ maps z_i to $\widetilde{Y}_i^{-1}$ for $i = 1, \ldots, n$.

It follows now that the algebra homomorphism $\widetilde{\Psi} \circ \Psi$ acts as the identity on the algebraic generators z_i, T_i and Y_i ($i = 1, \ldots, n$) of $\mathcal{H}$, hence $\widetilde{\Psi} \circ \Psi = \mathrm{Id}_{\mathcal{H}}$. In a similar fashion, we see that $\Psi \circ \widetilde{\Psi} = \mathrm{Id}_{\widetilde{\mathcal{H}}}$. This completes the proof of the corollary. □

6.3. **Duality of the non-symmetric Koornwinder polynomials.** We define evaluation mappings $\mathrm{Ev} : \mathcal{H} \to \mathbb{C}$ and $\widetilde{\mathrm{Ev}} : \widetilde{\mathcal{H}} \to \mathbb{C}$ by

$$\mathrm{Ev}(X) = \big(X(1)\big)(x_0^{-1}), \qquad \widetilde{\mathrm{Ev}}(\widetilde{X}) = \big(\widetilde{X}(1)\big)(\gamma_0^{-1})$$

for $X \in \mathcal{H}$ and $\widetilde{X} \in \widetilde{\mathcal{H}}$, where $1 \in \mathcal{A}$ is the Laurent polynomial identically equal to one. Observe that the renormalized non-symmetric Koornwinder polynomial $E(\gamma_\lambda; \cdot)$ (see definition 3.10) is exactly the constant multiple of the monic Koornwinder polynomial P_λ for which the associated multiplication operator $E(\gamma_\lambda; z)$ in $\mathcal{H}$ is mapped to one under the evaluation map Ev.

The evaluation mappings Ev and $\widetilde{\mathrm{Ev}}$ are compatible with respect to the duality anti-isomorphism $\Psi : \mathcal{H} \to \widetilde{\mathcal{H}}$ in the following sense.

Lemma 6.12. *For all* $X \in \mathcal{H}$*, we have* $\widetilde{\mathrm{Ev}}\big(\Psi(X)\big) = \mathrm{Ev}(X)$.

Proof. By linearity, it suffices to prove the equality for $X = z^\lambda T_w Y^\mu$, where $\lambda, \mu \in \Lambda_0$ and $w \in W$. Observe that

$$\Psi(X) = \widetilde{z}^{-\mu} \widetilde{T}_{w^{-1}} \widetilde{Y}^{-\lambda}.$$

Now $Y^\mu(1) = \gamma_0^\mu 1$ and $\widetilde{Y}^{-\lambda}(1) = x_0^{-\lambda} 1$ since $1 \in \mathcal{A}$ is the non-symmetric Koornwinder polynomial of degree zero, so that

$$\mathrm{Ev}(X) = x_0^{-\lambda} \gamma_0^\mu \,\mathrm{Ev}(T_w), \qquad \widetilde{\mathrm{Ev}}(\Psi(X)) = \gamma_0^\mu x_0^{-\lambda}\, \widetilde{\mathrm{Ev}}(\widetilde{T}_{w^{-1}}).$$

Now $T_i(1) = t_i\, 1$ and $\widetilde{T}_i(1) = \tilde{t}_i\, 1 = t_i\, 1$ for $i = 1, \ldots, n$ by the definition of Noumi's difference-reflection operators T_i (see (3.5)), hence $\mathrm{Ev}(T_w) = t_w$, where $t_w = t_{i_1} t_{i_2} \cdots t_{i_r}$ if $w = s_{i_1} s_{i_2} \cdots s_{i_r}$ is a reduced expression of $w \in W$. Similarly,

$\widetilde{\mathrm{Ev}}(\widetilde{T}_{w^{-1}}) = \tilde{t}_{w^{-1}} = t_{w^{-1}} = t_w$ for all $w \in W$. This now immediately yields the desired identity $\widetilde{\mathrm{Ev}}\big(\Psi(X)\big) = \mathrm{Ev}(X)$. □

We define now two pairings $B : \mathcal{H} \times \widetilde{\mathcal{H}} \to \mathbb{C}$ and $\widetilde{B} : \widetilde{\mathcal{H}} \times \mathcal{H} \to \mathbb{C}$ by $B(X, \widetilde{X}) = \mathrm{Ev}\big(\widetilde{\Psi}(\widetilde{X})X\big)$ and $\widetilde{B}(\widetilde{X}, X) = \widetilde{\mathrm{Ev}}\big(\Psi(X)\widetilde{X}\big)$ for $X \in \mathcal{H}$ and $\widetilde{X} \in \widetilde{\mathcal{H}}$. Then lemma 6.12 shows that

$$B(X, \widetilde{X}) = \widetilde{B}(\widetilde{X}, X), \qquad X \in \mathcal{H},\ \widetilde{X} \in \widetilde{\mathcal{H}}. \tag{6.4}$$

In the following lemma we collect some elementary identities for these bilinear forms.

Lemma 6.13. *Let $p \in \mathcal{A}$. Let $X, X_1, X_2 \in \mathcal{H}$ and $\widetilde{X}, \widetilde{X}_1, \widetilde{X}_2 \in \widetilde{\mathcal{H}}$.*

(i) $B\big(X_1X_2, \widetilde{X}\big) = B\big(X_2, \Psi(X_1)\widetilde{X}\big)$ *and* $B(X, \widetilde{X}_1\widetilde{X}_2) = B(\widetilde{\Psi}(\widetilde{X}_1)X, \widetilde{X}_2)$.

(ii) $B\big(XT_i, \widetilde{X}\big) = t_iB\big(X, \widetilde{X}\big)$ *for* $i = 0, \ldots, n$.

(iii) $B\big((X(p))(z), \widetilde{X}\big) = B(Xp(z), \widetilde{X})$ *and* $B\big(X, (\widetilde{X}(p))(\widetilde{z})\big) = B\big(X, \widetilde{X}p(\widetilde{z})\big)$, *where $(X(p))(z)$ is the multiplication operator in $\mathcal{H}$ corresponding to the Laurent polynomial $X(p) \in \mathcal{A}$, and $Xp(z)$ is the product of the elements X and $p(z)$ in $\mathcal{H}$.*

Proof. **(i)** Recall that Ψ is an anti-algebra isomorphism with inverse $\widetilde{\Psi}$. Hence

$$B(X_1X_2, \widetilde{X}) = \mathrm{Ev}(\widetilde{\Psi}(\widetilde{X})X_1X_2) = \mathrm{Ev}(\widetilde{\Psi}(\Psi(X_1)\widetilde{X})X_2) = B(X_2, \Psi(X_1)\widetilde{X}).$$

Similarly one proves the second identity.

(ii) By the definition of Noumi's difference-reflection operators $T_i \in \mathcal{H}$ (see (3.5)) we have $T_i(1) = t_i\,1$ for $i = 0, \ldots, n$, where $1 \in \mathcal{A}$ is the Laurent polynomial identically equal to one. Hence

$$B(XT_i, \widetilde{X}) = \big(\widetilde{\Psi}(\widetilde{X})XT_i(1)\big)(x_0^{-1}) = t_i\big(\widetilde{\Psi}(\widetilde{X})X(1)\big)(x_0^{-1}) = t_iB(X, \widetilde{X}),$$

as desired.

The first equality of **(iii)** is a direct consequence of the identity $(X(p))(z)(1) = X(p) = (X\,p(z))(1)$ in $\mathcal{A}$. The second equality of **(iii)** follows from the first by applying (6.4). □

By lemma 6.13 we have for all $p \in \mathcal{A}$ and all $\lambda \in \Lambda_0$,

$$\begin{aligned}\widetilde{B}\big(p(\widetilde{z}), E(\gamma_\lambda; z)\big) &= \widetilde{B}\big(1, (p(Y^{-1})E(\gamma_\lambda; \cdot))(z)\big)\\ &= p(\gamma_\lambda^{-1})\widetilde{B}\big(1, E(\gamma_\lambda; z)\big) = p(\gamma_\lambda^{-1})\end{aligned} \tag{6.5}$$

where we used for the last equality that

$$\widetilde{B}\big(1, E(\gamma_\lambda; z)\big) = B\big(E(\gamma_\lambda; z), 1\big) = E(\gamma_\lambda; x_0^{-1}) = 1$$

by (6.4). In a similar manner, one shows that

$$p(x_\mu^{-1}) = B\big(p(z), \widetilde{E}(x_\mu; \widetilde{z})\big), \qquad p \in \mathcal{A},\ \mu \in \Lambda_0. \tag{6.6}$$

Taking $p = \widetilde{E}(x_\mu; \cdot)$ in (6.5) and $p = E(\gamma_\lambda; \cdot)$ in (6.6) and using the duality (6.4) for the pairing, we arrive at

$$\begin{aligned}E(\gamma_\lambda; x_\mu^{-1}) &= B\big(E(\gamma_\lambda; z), \widetilde{E}(x_\mu; \widetilde{z})\big)\\ &= \widetilde{B}\big(\widetilde{E}(x_\mu; \widetilde{z}), E(\gamma_\lambda; z)\big) = \widetilde{E}(x_\mu; \gamma_\lambda^{-1}), \qquad \forall\, \lambda, \mu \in \Lambda_0\end{aligned} \tag{6.7}$$

which is the duality of the renormalized non-symmetric Koornwinder polynomials, see theorem 3.11.

6.4. **Spectral difference-reflection operators.** In this subsection we apply the duality of the non-symmetric Koornwinder polynomials to rewrite the action of the difference-reflection operators U_0 and T_i ($i = 1, \ldots, n$) on the non-symmetric Koornwinder polynomials $E(\gamma_\lambda; \cdot)$ in terms of operators acting on the spectral parameter $\lambda \in \Lambda_0$. For this, we need to extend the W-action on Λ_0 to an action of the affine Weyl group $\mathcal{W}$ on Λ_0 in the following way.

Lemma 6.14. *Define $s_0 \cdot \lambda = (-1-\lambda_1, \lambda_2, \ldots, \lambda_n)$ for $\lambda = (\lambda_1, \ldots, \lambda_n) \in \Lambda_0$. This uniquely extends to an action of $\mathcal{W}$ on Λ_0 (denoted by $w \cdot \lambda$ for $w \in \mathcal{W}$ and $\lambda \in \Lambda_0$) such that the restriction to W gives the action of $W \simeq S_n \ltimes (\pm 1)^n$ on Λ_0 by permutations and sign changes of the basis elements ϵ_i ($i = 1, \ldots, n$).*

Proof. In terms of the simple generators s_i ($i = 0, \ldots, n$), the action of $\mathcal{W}$ on Λ_0 reads as

$$\begin{aligned}
s_0 \cdot \lambda &= (-1-\lambda_1, \lambda_2, \ldots, \lambda_n), \\
s_i \cdot \lambda &= s_i\lambda = (\lambda_1, \ldots \lambda_{i-1}, \lambda_{i+1}, \lambda_i, \lambda_{i+2}, \ldots, \lambda_n), \\
s_n \cdot \lambda &= s_n\lambda = (\lambda_1, \ldots, \lambda_{n-1}, -\lambda_n)
\end{aligned}$$

for $i = 1, \ldots, n-1$ and $\lambda \in \Lambda_0$. It is clear that these operators on Λ_0 satisfy $s_i^2 = 1$ for $i = 0, \ldots, n$ and that they satisfy the $\widetilde{C}_n$-braid relations. Hence these operations on Λ_0 induce an unique action of $\mathcal{W}$ on Λ_0. □

We call the action defined in lemma 6.14 the *dot-action* of $\mathcal{W}$ on Λ_0. The following property of the dot-action is very useful.

Lemma 6.15. *For $\lambda \in \Lambda_0$ and $p \in \mathcal{A}$, we have $(s_0p)(\gamma_\lambda^{-1}) = p(\gamma_{s_0\cdot\lambda}^{-1})$.*

Proof. By the definition of the action of $\mathcal{W}$ on $\mathcal{A}$ (see lemma 3.1), and the definition of the diagonal terms γ_λ (see (3.11)), we have for all $\mu, \lambda \in \Lambda_0$ that

$$\big(s_0(x^\mu)\big)(\gamma_\lambda^{-1}) = (t_0t_n)^{-(\rho_l(\lambda), s_{\epsilon_1}\mu)}\, t^{-(\rho_m(\lambda), s_{\epsilon_1}\mu)}\, q^{\mu_1 - (\lambda, s_{\epsilon_1}\mu)},$$

where $\mu_1 = \big(\mu, \epsilon_1\big)$. On the other hand, by the definition of the dot-action (see lemma 6.14),

$$(\gamma_{s_0\cdot\lambda}^{-1})^\mu = (t_0t_n)^{-(\rho_l(s_0\cdot\lambda),\mu)}\, t^{-(\rho_m(s_0\cdot\lambda),\mu)}\, q^{\mu_1 - (\lambda, s_{\epsilon_1}\mu)}.$$

Comparing the two outcomes, and using the fact that $\rho_m(\nu) = w_\nu\rho_m$ for all $\nu \in \Lambda_0$ and similarly for ρ_l (see (3.24) and §5.2 for the notations), we see that the lemma will be a direct consequence of the identity

$$w_{s_0\cdot\lambda} = s_{\epsilon_1}w_\lambda, \qquad \lambda \in \Lambda_0 \tag{6.8}$$

in the finite Weyl group W. We give a proof of (6.8) using an explicit description of w_λ^{-1} in terms of its action on $\lambda \in \Lambda_0$. So fix $\lambda = (\lambda_1, \ldots, \lambda_n) \in \Lambda_0$, where $\lambda_i = \big(\lambda, \epsilon_i\big)$. For $j \in \{1, \ldots, n-1\}$ we set $\sigma_j = s_{n-1}s_{n-2}\cdots s_j$, and we set $\sigma_n = e$ the identity element in W. Let $j \in \{1, \ldots, n\}$ be the largest such that $\lambda_j < 0$. If $j \leq n-1$, then

$$s_n\sigma_j\lambda > \sigma_j\lambda > s_{n-2}s_{n-3}\cdots s_j\lambda > \cdots > s_j\lambda > \lambda$$

with respect to the dominance order $\leq$ on Λ_0, and if $j = n$, then $s_n\lambda > \lambda$. Now recall that $s_iw_\lambda = w_{s_i\lambda}$ for $i \in \{1, \ldots, n\}$ and $\lambda \in \Lambda_0$ such that $s_i\lambda \neq \lambda$ by the proof of lemma 5.6, hence $s_n\sigma_jw_\lambda = w_{s_n\sigma_j\lambda}$. On the other hand, $s_n\sigma_j\lambda$ has now one negative coefficient less than the original element λ. So we can iterate this process to make all coefficients of λ positive.

To be precise, let $1 \leq j_1 < j_2 < \cdots < j_r \leq n$ be the indices j for which the coefficient λ_j is strictly negative. Set $\mu = \sigma_{j_1}\sigma_{j_2}\cdots\sigma_{j_r}\lambda \in \Lambda_0$. Then μ is obtained from λ by moving the strictly negative coordinates to the far right, interchanging their order, and then taking the absolute values of these coefficients (i.e. the last r terms of μ are given by $\mu = (\cdots, |\lambda_{j_r}|, \ldots, |\lambda_{j_2}|, |\lambda_{j_1}|)$). Furthermore,

$$w_\mu = s_n\sigma_{j_1}s_n\sigma_{j_2}\cdots s_n\sigma_{j_r}w_\lambda = v_\lambda u_\lambda w_\lambda$$

where $u_\lambda = s_{\epsilon_{j_1}}s_{\epsilon_{j_2}}\cdots s_{\epsilon_{j_r}} \in (\pm 1)^n \subset W$ and $v_\lambda = \sigma_{j_1}\sigma_{j_2}\cdots\sigma_{j_r} \in S_n \subset W$. The new element μ does not have to be in Λ_0^+ yet. If $\mu \notin \Lambda_0^+$, then there exists an $i \in \{1, \ldots, n-1\}$ such that $\mu_i \lneq \mu_{i+1}$, so that $s_i\mu < \mu$ with respect to the dominance order and $w_{s_i\mu} = s_i w_\mu$. Continuing this way, we conclude that $w_\lambda = u_\lambda \pi_\lambda^{-1}$, where $u_\lambda = u_\lambda^{-1} \in (\pm 1)^n \subset W$ with coefficient -1 iff the corresponding coefficient of λ is strictly negative, and $\pi_\lambda \in S_n \subset W$ defined as the composition $\pi_\lambda = \nu_\lambda v_\lambda$ with v_λ as before, and ν_λ the permutation which turns $\mu = v_\lambda u_\lambda \lambda$ into a partition in such a way that the order between equal coefficients of μ are preserved.

Let us now return to the proof of (6.8). We first observe that we may assume $\lambda_1 = \big(\lambda, \epsilon_1\big) \geq 0$ without loss of generality. Indeed, if (6.8) is true for such λ, and $\mu \in \Lambda_0$ satisfies $\mu_1 < 0$, then $s_0 \cdot \mu$ satisfies $(s_0 \cdot \mu)_1 \geq 0$. Thus by the assumption $w_\mu = w_{s_0 \cdot s_0 \cdot \mu} = s_{\epsilon_1} w_{s_0 \cdot \mu}$, i.e. $w_{s_0 \cdot \mu} = s_{\epsilon_1} w_\mu$.

So let $\lambda \in \Lambda_0$ such that $\lambda_1 \geq 0$, and write $|\lambda| = (|\lambda_1|, |\lambda_2|, \ldots, |\lambda_n|) = u_\lambda \lambda \in \Lambda_0$. Let $i \in \{1, \ldots, n\}$ such that $\pi_\lambda^{-1}(i) = 1$. Since $\lambda_1 \geq 0$, we have by the explicit combinatorial description of $w_\lambda = u_\lambda \pi_\lambda^{-1}$ that

$$\lambda^+ = \pi_\lambda |\lambda| = \big(|\lambda_{\pi_\lambda^{-1}(1)}|, \ldots, |\lambda_{\pi_\lambda^{-1}(n)}|\big)$$

with

$$|\lambda_{\pi_\lambda^{-1}(i-1)}| \gneq |\lambda_{\pi_\lambda^{-1}(i)}| = \lambda_1, \qquad \text{if } i \geq 2. \tag{6.9}$$

Combined with $s_0 \cdot \lambda = (-1 - \lambda_1, \lambda_2, \ldots, \lambda_n)$, we see that

$$\begin{aligned}(s_0 \cdot \lambda)^+ &= \pi_{s_0 \cdot \lambda} u_{s_0 \cdot \lambda}(s_0 \cdot \lambda) \\ &= \big(|\lambda_{\pi_\lambda^{-1}(1)}|, \ldots, |\lambda_{\pi_\lambda^{-1}(i-1)}|, |\lambda_{\pi_\lambda^{-1}(i)}| + 1, |\lambda_{\pi_\lambda^{-1}(i+1)}|, \ldots, |\lambda_{\pi_\lambda^{-1}(n)}|\big).\end{aligned}$$

So we see that $w_{s_0 \cdot \lambda} = u_{s_0 \cdot \lambda} \pi_{s_0 \cdot \lambda}^{-1}$ with $u_{s_0 \cdot \lambda} = s_{\epsilon_1} u_\lambda$, and by the combinatorial rule for $\pi_{s_0 \cdot \lambda} \in S_n$ and (6.9), $\pi_{s_0 \cdot \lambda} = \pi_\lambda$. Hence (6.8) is verified for $\lambda_1 \geq 0$, which completes the proof of the lemma.

□

We are now in the position to rewrite the action of U_0 and T_i $(i = 1, \ldots, n)$ on the renormalized Koornwinder polynomials $E(\gamma; \cdot)$ in terms of linear operators acting on the spectral parameter $\gamma \in \mathrm{Spec}(Y) = \{\gamma_\lambda \mid \lambda \in \Lambda_0\}$. We define an action of $\mathcal{W}$ on $\mathrm{Spec}(Y)$ by $w\gamma_\lambda = \gamma_{w \cdot \lambda}$ $(\lambda \in \Lambda_0, w \in \mathcal{W})$.

Proposition 6.16. **(i)** *For $\gamma \in Spec(Y)$ we have*

$$\big(U_0 E(\gamma; \cdot)\big)(x) = \tilde{t}_0 E(\gamma; x) + \tilde{t}_0^{-1} c_{a_0}(\gamma^{-1}; \tilde{\mathbf{t}}; q)\big(E(s_0\gamma; x) - E(\gamma; x)\big).$$

(ii) *For $i = 1, \ldots, n$ and $\gamma \in Spec(Y)$ we have*

$$\big(T_i E(\gamma; \cdot)\big)(x) = \tilde{t}_i E(\gamma; x) + \tilde{t}_i^{-1} c_{a_i}(\gamma^{-1}; \tilde{\mathbf{t}}; q)\big(E(s_i\gamma; x) - E(\gamma; x)\big).$$

Proof. By (6.6) and lemma 6.13 we have

$$B\big(E(\gamma_\lambda;z),\widetilde{T}_i\,\widetilde{E}(x_\mu;\widetilde{z})\big)=\begin{cases}\big(U_0E(\gamma_\lambda;\cdot)\big)(x_\mu^{-1}) & \text{if } i=0\\ \big(T_iE(\gamma_\lambda;\cdot)\big)(x_\mu^{-1}) & \text{if } i=1,\ldots,n\end{cases}$$

for all $\lambda,\mu\in\Lambda_0$. So it suffices to prove that

$$\begin{aligned}B\big(E(\gamma_\lambda;z),\widetilde{T}_i\,\widetilde{E}(x_\mu;\widetilde{z})\big)&=\tilde{t}_iE(\gamma_\lambda;x_\mu^{-1})\\&\quad+\tilde{t}_i^{-1}c_{a_i}(\gamma_\lambda^{-1};\tilde{\mathbf{t}};q)\big(E(\gamma_{s_i\cdot\lambda};x_\mu^{-1})-E(\gamma_\lambda;x_\mu^{-1})\big)\end{aligned}\tag{6.10}$$

for all $\lambda,\mu\in\Lambda_0$ and all $i=0,\ldots,n$.

By lemma 6.13**(iii)**, (6.4) and (6.5) we have

$$\begin{aligned}B\big(E(\gamma_\lambda;z),\widetilde{T}_i\widetilde{E}(x_\mu;\widetilde{z})\big)&=B\big(E(\gamma_\lambda;z),\big(\widetilde{T}_i\widetilde{E}(x_\mu;\cdot)\big)(\widetilde{z})\big)\\&=\big(\widetilde{T}_i\widetilde{E}(x_\mu;\cdot)\big)(\gamma_\lambda^{-1}).\end{aligned}\tag{6.11}$$

Now we plug in the explicit expression for Noumi's difference-reflection operator $\widetilde{T}_i$. We have to consider two cases.

If $s_i\cdot\lambda=\lambda$ (hence in particular $i\neq 0$), then we claim that $\gamma_\lambda^{a_i}=\gamma_0^{a_i}$ ($=t^2$ if $i<n$ and $=t_0^2t_n^2$ if $i=n$), so that $c_{a_i}(\gamma_\lambda^{-1};\tilde{\mathbf{t}}|\,q)=0$. If the claim is valid, then substitution of the explicit expression for Noumi's difference-reflection operator $\widetilde{T}_i$ in (6.11) shows that

$$B\big(E(\gamma_\lambda;z),\widetilde{T}_i\widetilde{E}(x_\mu;\widetilde{z})\big)=\tilde{t}_i\widetilde{E}\big(x_\mu;\gamma_\lambda^{-1}\big)=\tilde{t}_iE(\gamma_\lambda;x_\mu^{-1})$$

by (6.7), which is in accordance with (6.10).

We prove the claim for $i\in\{1,\ldots,n-1\}$ with $s_i\cdot\lambda=s_i\lambda=\lambda$, the case $i=n$ is proved in a similar manner. By the explicit expression (3.11) for γ_λ, it then suffices to show that $\big(\rho_m(\lambda),a_i\big)=2$ and $\big(\rho_l(\lambda),a_i\big)=0$. But, since $s_i\lambda=\lambda$, $s_i\big(\Sigma_m^+\setminus\{a_i\}\big)=\Sigma_m^+\setminus\{a_i\}$ and $s_i(\Sigma_l^+)=\Sigma_l^+$, we have $\big(\rho_l(\lambda),a_i\big)=\frac{1}{2}\sum_{\alpha\in\Sigma_l^+}\mathrm{sgn}((\lambda,\alpha))(\alpha,a_i)$ and

$$\big(\rho_m(\lambda),a_i\big)=\sum_{\alpha\in\Sigma_m^+}\mathrm{sgn}((\lambda,\alpha))(\alpha,a_i)=2+\sum_{\alpha\in\Sigma_m^+\setminus\{a_i\}}\mathrm{sgn}((\lambda,\alpha))(\alpha,a_i).$$

The claim then follows from the fact that for $I=\Sigma_l^+$ and $I=\Sigma_m^+\setminus\{a_i\}$,

$$\begin{aligned}\sum_{\alpha\in I}\mathrm{sgn}((\lambda,\alpha))(\alpha,a_i)&=\sum_{\alpha\in I}\mathrm{sgn}((\lambda,s_i\alpha))(s_i\alpha,a_i)\\&=\sum_{\alpha\in I}\mathrm{sgn}((s_i\lambda,\alpha))(\alpha,s_ia_i)\\&=-\sum_{\alpha\in I}\mathrm{sgn}((\lambda,\alpha))(\alpha,a_i)=0.\end{aligned}$$

If $s_i\cdot\lambda\neq\lambda$, then $(s_ip)(\gamma_\lambda^{-1})=p(\gamma_{s_i\cdot\lambda}^{-1})$ for all $p\in\mathcal{A}$ by lemma 5.6 and lemma 6.15. Hence substitution of the explicit expression for Noumi's difference-reflection operator $\widetilde{T}_i$ in (6.11) and applying the duality (6.7) then immediately proves (6.10). This completes the proof of the proposition. □

7. Bi-orthogonality relations and diagonal terms

7.1. **Bi-orthogonality.** From now on, we assume that q has modulus < 1 in order to ensure convergence of the weight function. Furthermore, we assume that for the reparametrization of the multiplicity function $\mathbf{t}$ in terms of the parameters $\{a, b, c, d\}$ and t (see (3.18)), we have that the moduli of a, b, c, d and t are also < 1. We use freely the notations as introduced in §3.6 and §3.7.

We start with the observation that the bilinear form $\langle \cdot, \cdot \rangle = \langle \cdot, \cdot \rangle_{\mathbf{t},q}$ is non-degenerate for generic parameter values.

Lemma 7.1. *The bilinear form $\langle \cdot, \cdot \rangle$ is non-degenerate in both factors. In other words, $\langle p_1, p_2 \rangle = 0$ for all $p_1 \in \mathcal{A}$ implies $p_2 = 0$, and similarly for the other factor.*

Proof. By analytic continuation, it suffices to show that $\langle \cdot, \cdot \rangle$ is non-degenerate when a, b, c, d, q and t are real. But then the weight function $\Delta(\cdot)$ is of the form $\Delta(x) = \mathcal{C}(x)\Delta_+(x)$, with $\mathcal{C} \in \mathcal{Q}$ and $\Delta_+(x)$ a *positive* weight function on the torus $\mathbb{T}^n$. Let $p \in \mathcal{A}$ so that $p(x)\mathcal{C}(x)$ is positive for $x \in \mathbb{T}^n$ (which can be done since $\mathcal{C} \in \mathcal{Q}$). For $0 \neq p_2(x) = \sum_\lambda c_\lambda x^\lambda \in \mathcal{A}$, we now set

$$p_1(x) = p(x) \sum_\lambda \overline{c_\lambda} x^\lambda \in \mathcal{A},$$

then it follows from the definition of $\langle \cdot, \cdot \rangle$ that

$$\langle p_1, p_2 \rangle = \frac{1}{(2\pi i)^n} \iint\limits_{x \in \mathbb{T}^n} |p_2(x^{-1})|^2 p(x)\mathcal{C}(x)\Delta_+(x) \frac{dx}{x} > 0,$$

which shows the non-degeneracy in the second factor of $\langle \cdot, \cdot \rangle$. The non-degeneracy in the first factor is proved in a similar manner. □

We now show that the anti-algebra isomorphism $\ddagger$ of $\mathcal{H}$ (see lemma 6.9) corresponds to taking the adjoint with respect to the non-degenerate bilinear form $\langle \cdot, \cdot \rangle$.

Proposition 7.2. *For $X \in \mathcal{H}$ we have*

$$\langle X(p_1), p_2 \rangle = \langle p_1, X^\ddagger(p_2) \rangle, \qquad p_1, p_2 \in \mathcal{A}. \tag{7.1}$$

Proof. It suffices to prove (7.1) for a set of algebraic generators for $\mathcal{H}$, since $\ddagger$ is an anti-algebra homomorphism. Indeed, if the proposition is correct for $X_i \in \mathcal{H}$ $(i = 1, 2)$, then for all $p_1, p_2 \in \mathcal{A}$,

$$\begin{aligned} \langle (X_1X_2)p_1, p_2 \rangle &= \langle X_2 p_1, X_1^\ddagger p_2 \rangle \\ &= \langle p_1, X_2^\ddagger X_1^\ddagger p_2 \rangle = \langle p_1, (X_1X_2)^\ddagger p_2 \rangle, \end{aligned}$$

so (7.1) is also valid for $X_1X_2 \in \mathcal{H}$. We will now verify (7.1) for the algebraic generators $p(z)$ $(p \in \mathcal{A})$ and T_i $(i = 0, \ldots, n)$ of $\mathcal{H}$.

For $X = p(z)$ $(p \in \mathcal{A})$ we have

$$\begin{aligned} \langle p(z)p_1, p_2 \rangle &= \frac{1}{(2\pi i)^n} \iint\limits_{\mathbb{T}^n} p_1(x)p(x)p_2(x^{-1})\Delta(x) \frac{dx}{x} \\ &= \langle p_1, p(z'^{-1})p_2 \rangle = \langle p_1, p(z)^\ddagger\, p_2 \rangle. \end{aligned}$$

So it remains to prove (7.1) for $X = T_i$ $(i = 0, \ldots, n)$. Let $p_1, p_2 \in \mathcal{A}$. We write $\sigma = -1 \in W$ for the Weyl group element which maps v to $-v$ for all $v \in V$.

Observe that $(\sigma\, p)(x) = p(x^{-1})$ for all $p \in \mathcal{A}$. It follows by direct computations using the explicit expression for Noumi's difference-reflection operator T_i that

$$\big(T_i p_1\big)(x)\big(\sigma p_2\big)(x) - p_1(x)\big(\sigma\big(T_i'^{-1}p_2\big)\big)(x) = t_i^{-1}h_i(x)c_{a_i}(x;\mathbf{t};q) \qquad (7.2)$$

for $i = 0, \ldots, n$, with

$$h_i(x) = \big(s_i p_1\big)(x)\big(\sigma p_2\big)(x) - p_1(x)\big(s_i(\sigma p_2)\big)(x)$$

and with the action of s_i as defined in lemma 3.1. Now observe that h_i is s_i-alternating, i.e. $s_i h_i = -h_i$ for $i = 0, \ldots, n$. On the other hand,

$$c_{a_i}(x)\Delta(x) = \prod_{f \in R^+\setminus\{a_i\}} \frac{1}{c_f(x)} \qquad (7.3)$$

is invariant under the action of s_i for $i = 0, \ldots, n$, where the action of s_i is extended from $\mathcal{A}$ to (suitably nice) functions in the n variables $x = (x_1, \ldots, x_n)$ via the formulas (3.17). The invariance of the function (7.3) under the action of s_i is an immediate consequence of the fact that s_i permutes the roots $R^+ \setminus \{a_i\}$. Hence $\langle T_i p_1, p_2\rangle - \langle p_1, T_i'^{-1}p_2\rangle$ can be rewritten as an integral over $\big(\mathbb{T}^n, \frac{dx}{x}\big)$ with s_i-alternating integrand for all $i \in \{0, \ldots, n\}$.

Now $\langle T_i p_1, p_2\rangle - \langle p_1, T_i'^{-1}p_2\rangle = 0$ for $i = 1, \ldots, n$ follows from the fact that the measure $\big(\mathbb{T}^n, \frac{dx}{x}\big)$ is W-invariant: $\int_{\mathbb{T}^n} g(x)dx/x = \int_{\mathbb{T}^n}(wg)(x)dx/x$ for all $w \in W$, and for sufficiently nice functions g.

The case $i = 0$ is more subtle. The behaviour of the measure $\big(\mathbb{T}^n, \frac{dx}{x}\big)$ under the action of s_0 is given by

$$\iint_{x\in\mathbb{T}^n} (s_0 h)(x)\frac{dx}{x} = \int_{y_1\in q\mathbb{T}}\iint_{y\in\mathbb{T}^{n-1}} h(y_1, y)\frac{dy_1}{y_1}\frac{dy}{y},$$

which implies that

$$\begin{aligned}&\langle T_0 p_1, p_2\rangle - \langle p_1, T_0'^{-1}p_2\rangle = \\ &\quad = \frac{1}{2(2\pi i)^n}\int_{y_1\in\mathbb{T}-q\mathbb{T}}\iint_{y\in\mathbb{T}^{n-1}} t_0^{-1}h_0(y_1, y)c_{a_0}(y_1, y)\Delta(y_1, y)\frac{dy_1}{y_1}\frac{dy}{y}.\end{aligned} \qquad (7.4)$$

For fixed $y \in \mathbb{T}^{n-1}$, the integrand in the right-hand side of (7.4) depends analytically on $y_1 \in \{c \in \mathbb{C}\,|\, q \le |c| \le 1\}$. Indeed, by the expression of $\Delta_+(x)$ in terms of q-shifted factorials (see lemma 3.12), we see that the y_1-dependent factor of $c_{a_0}(y_1, y)\Delta(y_1, y)$ is given by

$$\begin{aligned}&\frac{\big(y_1^2, q^2y_1^{-2}; q\big)_\infty}{\big(ay_1, by_1, cy_1, dy_1, qay_1^{-1}, qby_1^{-1}, qcy_1^{-1}, qdy_1^{-1}; q\big)_\infty} \\ &\qquad\qquad \cdot\prod_{j=2}^n \frac{\big(y_1y_j, y_1y_j^{-1}, qy_1^{-1}y_j, qy_1^{-1}y_j^{-1}; q\big)_\infty}{\big(t^2y_1y_j, t^2y_1y_j^{-1}, qt^2y_1^{-1}y_j, qt^2y_1^{-1}y_j^{-1}; q\big)_\infty},\end{aligned}$$

which has the desired analytic behaviour due to the conditions on the parameters a, b, c, d, q and t. Thus by Cauchy's theorem we conclude that $\langle T_0 p_1, p_2\rangle - \langle p_1, T_0'^{-1}p_2\rangle = 0$. This completes the proof of the proposition. $\square$

Recall that $E'(\gamma_\lambda^{-1};\cdot)$ ($\lambda \in \Lambda_0$) is the renormalized non-symmetric Koornwinder polynomial of degree λ with respect to inverse parameters $(\mathbf{t}^{-1}, q^{-1})$. Fix now $\lambda, \mu \in \Lambda_0$ such that $\lambda \neq \mu$, hence $\gamma_\lambda \neq \gamma_\mu$, see §5.2. So there exists a Laurent polynomial $p \in \mathcal{A}$ with $p(\gamma_\lambda) \neq p(\gamma_\mu)$. Combined with proposition 7.2 and $p(Y)^\ddagger = p(Y'^{-1})$ (see the proof of corollary 6.11), we obtain

$$\begin{aligned} p(\gamma_\lambda)\,\langle E(\gamma_\lambda;\cdot), E'(\gamma_\mu^{-1};\cdot)\rangle &= \langle p(Y)E(\gamma_\lambda;\cdot), E'(\gamma_\mu^{-1};\cdot)\rangle \\ &= \langle E(\gamma_\lambda;\cdot), p(Y'^{-1})E'(\gamma_\mu^{-1};\cdot)\rangle \\ &= p(\gamma_\mu)\,\langle E(\gamma_\lambda;\cdot), E'(\gamma_\mu^{-1};\cdot)\rangle. \end{aligned}$$

Since $p(\gamma_\lambda) \neq p(\gamma_\mu)$, we conclude that

$$\langle E(\gamma_\lambda;\cdot), E'(\gamma_\mu^{-1};\cdot)\rangle = 0.$$

This proves the bi-orthogonality relations for the non-symmetric Koornwinder polynomials with respect to $\langle\cdot,\cdot\rangle$, see theorem 3.14.

7.2. **Diagonal terms.** The main objective of this subsection is to evaluate the diagonal terms $\langle E(\gamma_\lambda;\cdot), E'(\gamma_\lambda^{-1};\cdot)\rangle$ ($\lambda \in \Lambda_0$) in terms of residues of the weight function $\Delta(\cdot)$, see theorem 3.16. Observe that by lemma 7.1 and theorem 3.14, we know that the diagonal terms are non-zero for generic values of the parameters. We again use freely the notation of §3.6 and §3.7.

Let

$$F = \{g : \mathrm{Spec}(Y') \to \mathbb{C} \,|\, \#\mathrm{supp}(g) < \infty\},$$

where $\mathrm{Spec}(Y') = \{\gamma_\lambda^{-1} \,|\, \lambda \in \Lambda_0\}$ is the spectrum of the Y'-operators. Let $\mathcal{F} = \mathcal{F}_{\mathbf{t},q} : \mathcal{A} \to F$ be the linear map defined by

$$\mathcal{F}(p)(\gamma) = \langle p, E'(\gamma;\cdot)\rangle, \qquad p \in \mathcal{A},\ \gamma \in \mathrm{Spec}(Y'). \tag{7.5}$$

We call $\mathcal{F}$ the *non-symmetric Koornwinder transform*. Observe that $\mathcal{F}$ is injective since $\langle\cdot,\cdot\rangle$ is non-degenerate, and that $\mathcal{F}$ is surjective by theorem 3.14.

Remark 7.3. Recall that for $\mathbf{t} = \mathbf{1}$ the multiplicity function identically equal to one, $E(\gamma_\lambda; x; \mathbf{1}|\, q)$ is equal to the monomial x^λ for $\lambda \in \Lambda_0$, and the weight function $\Delta(x; \mathbf{1}|\, q)$ is identically equal to one. Hence $\mathcal{F}_{\mathbf{1},q}$ relates to the classical Fourier transform on the torus $\mathbb{T}^n$.

In the next proposition we determine the intertwining properties of $\mathcal{F}$ with respect to the action of the double affine Hecke algebra $\mathcal{H}$ on $\mathcal{A}$. Recall the action of $\mathcal{W}$ on $\mathrm{Spec}(Y')$, defined at the end of §6.4: $w\gamma_\lambda^{-1} = \gamma_{w\cdot\lambda}^{-1}$ for $\lambda \in \Lambda_0$ and $w \in \mathcal{W}$. This induces an action of $\mathcal{W}$ on F by

$$(wg)(\gamma_\lambda^{-1}) = g(\gamma_{w^{-1}\cdot\lambda}^{-1}), \qquad \lambda \in \Lambda_0,\ w \in \mathcal{W},\ g \in F.$$

Proposition 7.4. *The maps*

$$\begin{aligned} (\widetilde{T}_i g)(\gamma) &= \tilde{t}_i g(\gamma) + \tilde{t}_i^{-1} c_{a_i}(\gamma; \tilde{\mathbf{t}}; q)\big((s_i g)(\gamma) - g(\gamma)\big), \qquad i \in \{0,\ldots,n\}, \\ \big(p(\widetilde{z})g\big)(\gamma) &= p(\gamma)g(\gamma), \qquad p \in \mathcal{A} \end{aligned} \tag{7.6}$$

where $g \in F$ and $\gamma \in$ Spec(Y'), uniquely extend to an action of the double affine Hecke algebra $\widetilde{\mathcal{H}} = \mathcal{H}(\tilde{\mathbf{t}}|\, q)$ on F. Furthermore,

$$\mathcal{F}(X(p)) = \Phi(X)\mathcal{F}(p), \qquad X \in \mathcal{H},\ p \in \mathcal{A}, \tag{7.7}$$

where Φ is the duality isomorphism.

Proof. We first prove the intertwining property (7.7) for the algebraic generators U_0, T_i and $p(Y)$ ($i = 1, \ldots, n$ and $p \in \mathcal{A}$) of $\mathcal{H}$. Now for any $p, p_1 \in \mathcal{A}$ and $\gamma \in \operatorname{Spec}(Y')$ we have by the definition of the non-symmetric Koornwinder polynomial (see theorem 3.7) and by proposition 7.2,

$$\mathcal{F}(p(Y)p_1)(\gamma) = \langle p(Y)p_1, E'(\gamma;\cdot)\rangle = \langle p_1, p(Y'^{-1})E'(\gamma;\cdot)\rangle = p(\gamma^{-1})\mathcal{F}(p_1)(\gamma)$$

since $p(Y)^{\ddagger} = p(Y'^{-1})$ for $p \in \mathcal{A}$ by the proof of corollary 6.11. Hence

$$\mathcal{F}(p(Y)p_1) = p(\widetilde{z}^{-1})(\mathcal{F}(p_1)) = \Phi(p(Y))\mathcal{F}(p_1),$$

which proves (7.7) for $X = p(Y)$ ($p \in \mathcal{A}$).

For $X = U_0$, we observe that $U_0^{\ddagger} = U_0'^{-1} = U_0' + \tilde{t}_0 - \tilde{t}_0^{-1} \in \mathcal{H}'$. Hence for $p \in \mathcal{A}$ and $\gamma \in \operatorname{Spec}(Y')$, we derive from proposition 6.16 and proposition 7.2 that

$$\begin{aligned}
\mathcal{F}(U_0\, p)(\gamma) &= \langle U_0\, p, E'(\gamma;\cdot)\rangle = \langle p, U_0'^{-1}E'(\gamma;\cdot)\rangle \\
&= \tilde{t}_0\langle p, E'(\gamma;\cdot)\rangle + \tilde{t}_0 c_{a_0}(\gamma^{-1};\tilde{\mathbf{t}}^{-1};q^{-1})\big(\langle p, E'(s_0\gamma;\cdot)\rangle - \langle p, E'(\gamma;\cdot)\rangle\big) \\
&= \tilde{t}_0\mathcal{F}(p)(\gamma) + \tilde{t}_0^{-1}c_{a_0}(\gamma;\tilde{\mathbf{t}};q)\big(\mathcal{F}(p)(s_0\gamma) - \mathcal{F}(p)(\gamma)\big) \\
&= \big(\widetilde{T}_0\mathcal{F}(p)\big)(\gamma),
\end{aligned}$$

where we used that $c_f(\gamma^{-1};\mathbf{t}^{-1};q^{-1}) = t_f^{-2}c_f(\gamma;\mathbf{t};q)$ for $f \in R$ in the fourth equality. Since $\Phi(U_0) = \widetilde{T}_0$, we see that (7.7) is valid for $X = U_0$. The case $X = T_i$ ($i = 1, \ldots, n$) of (7.7) is proved in exactly the same manner as for $X = U_0$. We leave the verification to the reader.

Using these intertwining properties, we can immediately conclude that the formulas (7.6) uniquely extend to an action of $\widetilde{\mathcal{H}}$ on F and that (7.7) holds for all $X \in \mathcal{H}$ in view of the bijectivity of the non-symmetric Koornwinder transform $\mathcal{F}$ and the fact that $\Phi : \mathcal{H} \to \widetilde{\mathcal{H}}$ is an algebra isomorphism. $\square$

Next we determine the inverse of the non-symmetric Koornwinder transform $\mathcal{F}$. We let $\mathcal{G} = \mathcal{G}_{\mathbf{t},q} : F \to \mathcal{A}$ be the linear endomorphism defined by

$$(\mathcal{G}g)(x) = \sum_{\lambda\in\Lambda_0} g(\gamma_\lambda^{-1})E(\gamma_\lambda;x;\mathbf{t};q)\widetilde{w}(\gamma_\lambda^{-1}), \qquad g \in F, \tag{7.8}$$

where the non-zero discrete weight $\widetilde{w}(\gamma_\lambda^{-1}) = w(\gamma_\lambda^{-1};\tilde{\mathbf{t}};q)$ is defined by (3.27) and (3.30).

Proposition 7.5. *We have*

$$\mathcal{G}(\widetilde{X}g) = \Phi^{-1}(\widetilde{X})\mathcal{G}(g), \qquad \widetilde{X} \in \widetilde{\mathcal{H}},\ g \in F.$$

Proof. It suffices to check the intertwining property for $\widetilde{X} = p(\widetilde{z})$ ($p \in \mathcal{A}$) and $\widetilde{X} = \widetilde{T}_i$ ($i = 0, \ldots, n$), compare with the proof of proposition 7.4.

For $\widetilde{X} = p(\widetilde{z})$ ($p \in \mathcal{A}$), we have for $g \in F$,

$$\begin{aligned}
\mathcal{G}(p(\widetilde{z})\, g)(x) &= \sum_{\lambda\in\Lambda_0} p(\gamma_\lambda^{-1})g(\gamma_\lambda^{-1})E(\gamma_\lambda;x)\widetilde{w}(\gamma_\lambda^{-1}) \\
&= \sum_{\lambda\in\Lambda_0} g(\gamma_\lambda^{-1})\big(p(Y^{-1})E(\gamma_\lambda;\cdot)\big)(x)\widetilde{w}(\gamma_\lambda^{-1}) \\
&= \big(p(Y^{-1})\mathcal{G}(g)\big)(x) = \big(\Phi^{-1}(p(\widetilde{z}))\mathcal{G}(g)\big)(x).
\end{aligned}$$

So it remains to check the intertwining property for $\widetilde{X} = \widetilde{T}_i$ ($i = 0, \ldots, n$). We use the short-hand notation $\widetilde{c}_f(\gamma) = c_f(\gamma;\tilde{\mathbf{t}};q)$ for all $f \in R$. Let $g \in F$ and

$i \in \{0,\ldots,n\}$. Since $\Phi^{-1}(\widetilde{T}_0)=U_0$ and $\Phi^{-1}(\widetilde{T}_i)=T_i$ for $i=1,\ldots,n$, we have by proposition 6.16,

$$\begin{aligned}&\big(\Phi^{-1}(\widetilde{T}_i)\big(\mathcal{G}g\big)\big)(x)=\\&=\sum_{\lambda\in\Lambda_0} g(\gamma_\lambda^{-1})\big(\tilde{t}_iE(\gamma_\lambda;x)+\tilde{t}_i^{-1}\widetilde{c}_{a_i}(\gamma_\lambda^{-1})\big(E(\gamma_{s_i\cdot\lambda};x)-E(\gamma_\lambda;x)\big)\big)\widetilde{w}(\gamma_\lambda^{-1}).\end{aligned}$$

Combined with the definition of the action of $\widetilde{\mathcal{H}}$ on F, see proposition 7.4, we obtain

$$\mathcal{G}\big(\widetilde{T}_ig\big)-\Phi^{-1}(\widetilde{T}_i)\big(\mathcal{G}g\big)=\tilde{t}_i^{-1}\sum_{\lambda\in\Lambda_0}h_i(\gamma_\lambda;\cdot)\widetilde{c}_{a_i}(\gamma_\lambda^{-1})\widetilde{w}(\gamma_\lambda^{-1})$$

with $h_i(\gamma_\lambda;\cdot)\in\mathcal{A}$ given by

$$h_i(\gamma_\lambda;x)=g(\gamma_{s_i\cdot\lambda}^{-1})E(\gamma_\lambda;x)-g(\gamma_\lambda^{-1})E(\gamma_{s_i\cdot\lambda};x).$$

Since $h_i(\gamma_{s_i\cdot\lambda};x)=-h_i(\gamma_\lambda;x)$ for $i=0,\ldots,n$ and $\lambda\in\Lambda_0$, it thus suffices to prove that

$$\widetilde{c}_{a_i}(\gamma_\lambda^{-1})\widetilde{w}(\gamma_\lambda^{-1})=\widetilde{w}_+(\gamma_{\lambda^+}^{-1})\prod_{\alpha\in\Sigma^-\cup\{a_i\}}\widetilde{c}_\alpha(\gamma_\lambda^{-1}) \tag{7.9}$$

is invariant under replacement of $\lambda\in\Lambda_0$ by $s_i\cdot\lambda$ for all $i\in\{0,\ldots,n\}$ and all $\lambda\in\Lambda_0$. For $i\in\{1,\ldots,n\}$ this is immediate by lemma 5.6.

As in the proof of proposition 7.2, the proof for $i=0$ is more subtle. We begin by rewriting $\widetilde{w}(\gamma_\lambda^{-1})$ as a (kind of) multiple residue of $\widetilde{\Delta}(x)\frac{dx}{x}=\Delta(x;\tilde{\mathbf{t}};q)\frac{dx}{x}$ at $x=\gamma_\lambda^{-1}$. This can be done using the W-invariance of the weight function $\widetilde{\Delta}_+(\cdot)=\Delta_+(\cdot;\tilde{\mathbf{t}};q)$, together with the combinatorial structure of the Weyl group elements $w_\lambda\in W$, see the proof of lemma 6.15. The result is as follows.

Write $w_\lambda=u_\lambda v_\lambda$ with $v_\lambda\in S_n$ and $u_\lambda\in(\pm1)^n$ with respect to the natural identification $W\simeq S_n\ltimes(\pm1)^n$, and let $n_\lambda=\#\{i\in\{1,\ldots,n\}\,|\,\lambda_i<0\}$. By the precise combinatorial description of w_λ, see the proof of lemma 6.15, we have that n_λ is also equal to the number of -1 components of $u_\lambda\in(\pm1)^n$. Now we use the W-invariance of $\widetilde{\Delta}_+(\cdot)$ and the fact that

$$\operatorname*{Res}_{y=y_0}\left(\frac{g(y)}{y}\right)=-\operatorname*{Res}_{y=y_0^{-1}}\left(\frac{g(y)}{y}\right)$$

for a one variable function $g(y)$ having a simple pole at $y=y_0$ and satisfying $g(y)=g(y^{-1})$. Then we obtain from the original definition (3.30) of $\widetilde{w}(\gamma_\lambda^{-1})$ ($\lambda\in\Lambda_0$), together with lemma 5.6, that

$$\widetilde{w}\big(\gamma_\lambda^{-1}\big)=\operatorname*{\mathbf{Res}}_{x=\gamma_\lambda^{-1}}\left(\frac{\widetilde{\Delta}(x)}{x_1\cdots x_n}\right) \tag{7.10}$$

for all $\lambda\in\Lambda_0$, where the multiple residue at $x=\gamma_\lambda^{-1}$ is defined by

$$\operatorname*{\mathbf{Res}}_{x=\gamma_\lambda^{-1}}=(-1)^{n_\lambda}\operatorname*{Res}_{x_{v_\lambda(1)}=\gamma_\lambda^{-\epsilon_{v_\lambda(1)}}}\Big(\operatorname*{Res}_{x_{v_\lambda(2)}=\gamma_\lambda^{-\epsilon_{v_\lambda(2)}}}\Big(\cdots\operatorname*{Res}_{x_{v_\lambda(n)}=\gamma_\lambda^{-\epsilon_{v_\lambda(n)}}}\Big(\cdot\Big)\cdots\Big)\Big).$$

In particular, we obtain

$$\widetilde{c}_{a_0}(\gamma_\lambda^{-1})\widetilde{w}(\gamma_\lambda^{-1})=\operatorname*{\mathbf{Res}}_{x=\gamma_\lambda^{-1}}\left(\frac{\widetilde{c}_{a_0}(x)\widetilde{\Delta}(x)}{x_1\cdots x_n}\right) \tag{7.11}$$

for all $\lambda \in \Lambda_0$. Now we consider (7.11) with λ replaced by $s_0 \cdot \lambda$. We first consider the changes in the multiple residue. By (6.8), we have $w_{s_0\cdot\lambda} = s_{\epsilon_1} w_\lambda$ for all $\lambda \in \Lambda$, i.e. $n_{s_0\cdot\lambda} = n_\lambda \pm 1$ and $v_{s_0\cdot\lambda} = v_\lambda$. Furthermore,

$$(\gamma_{s_0\cdot\lambda}^{-1})^{\epsilon_i} = \begin{cases} q\gamma_\lambda^{\epsilon_1} & \text{if } i = 1, \\ \gamma_\lambda^{-\epsilon_i} & \text{if } i = 2, \ldots, n \end{cases}$$

by lemma 6.15. We conclude that if we replace the residue at $x_1 = \gamma_\lambda^{-\epsilon_1}$ by the residue at $x_1 = q\gamma_\lambda^{\epsilon_1}$ in the definition of the multiple residue at $x = \gamma_\lambda^{-1}$, then we obtain minus the multiple residue at $x = \gamma_{s_0\cdot\lambda}^{-1}$. On the other hand, we know by the proof of proposition 7.2 that $\widetilde{c}_{a_0}(x)\widetilde{\Delta}(x)$ is invariant under the action of s_0. Hence the invariance of (7.11) under replacement of λ by $s_0 \cdot \lambda$ follows from the simple observation that

$$\operatorname*{Res}_{y=y_0} \left(\frac{g(y)}{y} \right) = - \operatorname*{Res}_{y=qy_0^{-1}} \left(\frac{g(y)}{y} \right)$$

when $g(y)$ is a function depending on a single variable y, having a simple pole at $y = y_0$, and satisfying the invariance condition $g(qy^{-1}) = g(y)$. □

In the following theorem we combine proposition 7.4 and proposition 7.5 to show that $\mathcal{G}$ is, up to a constant, the inverse of the Koornwinder transform $\mathcal{F}$..

Theorem 7.6. *We have $\mathcal{G} \circ \mathcal{F} = c\, Id_{\mathcal{A}}$ and $\mathcal{F} \circ \mathcal{G} = c\, Id_F$ with $c = c_{\mathbf{t},q} = w(\gamma_0^{-1}; \tilde{\mathbf{t}}; q)\, \langle 1, 1\rangle_{\mathbf{t},q}$.*

Proof. By proposition 7.4 and proposition 7.5 we have

$$\mathcal{G}(\mathcal{F}(p)) = \mathcal{G}(\mathcal{F}(p(z)1)) = p(z)\mathcal{G}(\mathcal{F}(1)), \qquad \forall p \in \mathcal{A}, \tag{7.12}$$

where $1 \in \mathcal{A}$ is the Laurent polynomial identically equal to one. Furthermore, it follows from theorem 3.14 that

$$\mathcal{G}\big(\mathcal{F}(E(\gamma; \cdot))\big) = \langle E(\gamma; \cdot), E'(\gamma^{-1}; \cdot)\rangle_{\mathbf{t},q}\, w(\gamma^{-1}; \tilde{\mathbf{t}}; q)\, E(\gamma; \cdot) \tag{7.13}$$

for $\gamma \in \mathrm{Spec}(Y)$. Formula (7.13) reduces to $\mathcal{G}(\mathcal{F}(1)) = c\,1$ when $\gamma = \gamma_0$, with the constant c as given in the statement of the theorem. Combined with (7.12) it follows that $\mathcal{G} \circ \mathcal{F} = c\,\mathrm{Id}_{\mathcal{A}}$. Since $\mathcal{F}$ is bijective, we then also have $\mathcal{F} \circ \mathcal{G} = c\,\mathrm{Id}_F$. □

Now we fix $\gamma = \gamma_\lambda \in \mathrm{Spec}(Y)$, $\lambda \in \Lambda_0$. Since $\mathcal{G} \circ \mathcal{F} = c\,\mathrm{Id}_{\mathcal{A}}$ by the previous theorem, we have $\mathcal{G}(\mathcal{F}(E(\gamma; \cdot))) = c\, E(\gamma; \cdot)$. Comparing this outcome with the right-hand side of (7.13), we obtain

$$\langle E(\gamma; \cdot), E'(\gamma^{-1}; \cdot)\rangle_{\mathbf{t},q}\, w(\gamma^{-1}; \tilde{\mathbf{t}}; q) = c_{\mathbf{t},q} = \langle 1, 1\rangle_{\mathbf{t},q}\, w(\gamma_0^{-1}; \tilde{\mathbf{t}}; q).$$

Hence we obtain the expressions for the diagonal terms $\langle E(\gamma_\lambda; \cdot), E'(\gamma_\lambda^{-1}; \cdot)\rangle$ in terms of multiple residue $\widetilde{w}(\gamma_\lambda^{-1})$ as stated in theorem 3.16.

8. Symmetric Koornwinder polynomials

8.1. **Symmetric Koornwinder polynomials.** Recall that $\mathcal{A}^W \subset \mathcal{A}$ is the sub-algebra consisting of Laurent polynomials $p \in \mathcal{A}$ which are W-invariant (i.e. $wp = p$ for all $w \in W$), where the action is as given in lemma 3.1.

Similarly we write $\mathcal{A}_Y^W$ for the sub-algebra of $\mathcal{A}_Y$ consisting of W-invariant elements, where the action is given by $w(Y^\lambda) = Y^{w\lambda}$ for $w \in W$ and $\lambda \in \Lambda_0$. A linear basis of $\mathcal{A}^W$ and $\mathcal{A}_Y^W$ is given by the monomials $m_\lambda(x) = \sum_{\mu \in W\lambda} x^\mu$ and $m_\lambda(Y) = \sum_{\mu \in W\lambda} Y^\mu$ ($\lambda \in \Lambda_0^+$), respectively.

Lemma 8.1. *Let $p \in \mathcal{A}$. Then $p \in \mathcal{A}^W$ iff $T_i p = t_i p$ $(i = 1, \ldots, n)$.*

Proof. Let $p \in \mathcal{A}^W$. By the explicit expression for Noumi's difference-reflection operator T_i, see (3.5), we have $T_i p = t_i p$ for all $i = 1, \ldots, n$. On the other hand, if $p \in \mathcal{A}$ satisfies $T_i p = t_i p$ for all $i = 1, \ldots, n$, then again by (3.5), $s_i p = p$ for $i = 1, \ldots, n$, hence $p \in \mathcal{A}^W$. □

It follows from lemma 8.1 and the commutation relation (6.2) that $(T_i^{-1} - t_i^{-1})p(Y) = p(Y)(T_i^{-1} - t_i^{-1})$ for $i = 1, \ldots, n$ and $p \in \mathcal{A}^W$, hence Noumi's difference-reflection operators T_i $(i = 1, \ldots, n)$ commute with any $p(Y) \in \mathcal{A}_Y^W$. This leads to the following lemma.

Lemma 8.2. *The action of $\mathcal{A}_Y^W$ on $\mathcal{A}$ preserves $\mathcal{A}^W$, i.e. $p(Y)|_{\mathcal{A}^W} \in End_{\mathbb{C}}(\mathcal{A}^W)$.*

Proof. Fix $p_1 \in \mathcal{A}^W$ and let $p(Y) \in \mathcal{A}_Y^W$. Then we have

$$T_i p(Y)\, p_1 = p(Y) T_i\, p = t_i p(Y) p_1, \qquad i = 1, \ldots, n,$$

hence $p(Y)p_1 \in \mathcal{A}^W$ by lemma 8.1. □

Let $\mathcal{Q}[\mathcal{W}] \subset \mathrm{End}_{\mathbb{C}}(\mathcal{Q})$ be the subalgebra generated by $\mathcal{Q}$ (acting as multiplication operators) and by the automorphisms $\mathcal{W}$ (see lemma 3.1). Observe that $\mathcal{H} \subset \mathcal{Q}[\mathcal{W}]$, and that

$$\mathcal{Q}[\mathcal{W}] = \bigoplus_{w \in \mathcal{W}} \mathcal{Q} w = \bigoplus_{w \in W, \lambda \in \Lambda_0} \mathcal{Q} \tau(\lambda) w$$

as a $\mathcal{Q}$-submodule of $\mathrm{End}_{\mathbb{C}}(\mathcal{Q})$ by proposition 6.2. Furthermore, $\mathcal{Q}[\tau(\Lambda_0)] = \bigoplus_{\lambda \in \Lambda_0} \mathcal{Q}\, \tau(\lambda)$ is the subalgebra of $\mathcal{Q}[\mathcal{W}]$ consisting of q-difference operators with coefficients in $\mathcal{Q}$.

With $D \in \mathcal{Q}[\mathcal{W}]$, say

$$D = \sum_{w \in W} D(x, w) w, \qquad D(x, w) \in \mathcal{Q}[\tau(\Lambda_0)],$$

we associate a q-difference operator by

$$D_{sym} = \sum_{w \in W} D(x, w) \in \mathcal{Q}[\tau(\Lambda_0)].$$

Observe that $Df = D_{sym} f$ if $f \in \mathcal{Q}$ is W-invariant. Now lemma 5.2, proposition 5.5 and lemma 5.6 imply

$$p(Y)_{sym}\, m_\lambda = p(\gamma_\lambda) m_\lambda + \sum_{\mu \in \Lambda^+ : \mu < \lambda} c_{\lambda,\mu} m_\mu, \qquad p \in \mathcal{A}^W, \ \lambda \in \Lambda_0^+$$

for certain constants $c_{\lambda,\mu} \in \mathbb{C}$. Hence $p(Y)_{sym}|_{\mathcal{A}^W} = p(Y)|_{\mathcal{A}^W}$ is a triangular endomorphism of $\mathcal{A}^W$ with respect to the basis of monomials m_λ $(\lambda \in \Lambda_0^+)$ and with respect to the partial order $\leq$, with diagonal terms given by $p(\gamma_\lambda)$ $(\lambda \in \Lambda_0^+)$. Since the W-orbits of the spectral points $\{\gamma_\lambda \,|\, \lambda \in \Lambda_0^+\}$ (where $W = S_n \ltimes (\pm 1)^n$ acts by permutations and inversions on $\gamma \in (\mathbb{C} \setminus \{0\})^n$) are pair-wise different for generic parameters, we arrive at the following symmetric analogue of theorem 5.7.

Theorem 8.3. *There exists a unique basis $\{P_\lambda^+\}_{\lambda \in \Lambda^+}$ of $\mathcal{A}^W$ such that*

- *$P_\lambda^+(x) = m_\lambda(x) + \sum_{\mu \in \Lambda^+ : \mu < \lambda} c_{\lambda,\mu} m_\mu(x)$ for certain constants $c_{\lambda,\mu}$,*
- *$p(Y)_{sym}\, P_\lambda^+ = p(\gamma_\lambda) P_\lambda^+$ for all $p(Y) \in \mathcal{A}_Y^W$,*

for all $\lambda \in \Lambda_0^+$.

Theorem 8.3 implies theorem 3.17. In particular, the polynomial $P_\lambda^+(x) = P_\lambda^+(x;\mathbf{t}|\,q)$ defined in theorem 8.3 is the monic, symmetric Koornwinder polynomial of degree $\lambda \in \Lambda_0^+$ as defined in definition 3.18.

8.2. **The second order q-difference operator.** In this subsection we show that the q-difference operator

$$L = \big(m_{\epsilon_1}(Y) - m_{\epsilon_1}(\gamma_0)\big)_{sym} \in \mathcal{Q}[\tau(\Lambda_0)] \tag{8.1}$$

coincides with Koornwinder's second order q-difference operator, see theorem 3.19. We use the expression (5.3) for the Y-operator Y_i as starting point for the computation of L, where $\mathcal{R}(f) = T_f s_f$ ($f \in R$) is given by (5.1). Using (5.3) and the fact that $\mathcal{R}(f)^{-1} = \mathcal{R}(-f) + (t_f^{-1} - t_f)s_f$, we see that $(Y_i^{\pm 1})_{sym}$ can be written as

$$\begin{aligned}(Y_i)_{sym} &= t^{1-i}\big(\mathcal{R}(\epsilon_i - \epsilon_{i+1})\mathcal{R}(\epsilon_i - \epsilon_{i+2})\cdots\mathcal{R}(\epsilon_i - \epsilon_n)\mathcal{R}(2\epsilon_i)\\ &\times\mathcal{R}(\epsilon_i+\epsilon_n)\cdots\mathcal{R}(\epsilon_i+\epsilon_{i+1})\mathcal{R}(\epsilon_i+\epsilon_{i-1})\cdots\mathcal{R}(\epsilon_i+\epsilon_1)\mathcal{R}(2\epsilon_i+\delta)\tau(\epsilon_i)\big)_{sym}\end{aligned} \tag{8.2}$$

and

$$(Y_i^{-1})_{sym} = t^{1+i-2n}t_n^{-1}\big(\mathcal{R}(\epsilon_{i-1}-\epsilon_i)\cdots\mathcal{R}(\epsilon_1-\epsilon_i)\tau(-\epsilon_i)\mathcal{R}(2\epsilon_i+\delta)^{-1}\big)_{sym}. \tag{8.3}$$

Since $\mathcal{R}(f) \in \mathcal{Q} \oplus \mathcal{Q}\,s_f \subset \mathcal{Q}[\mathcal{W}]$ for $f \in R$, and since the factor $\mathcal{R}(2\epsilon_i+\delta)\tau(\epsilon_i)$ and $\tau(-\epsilon_i)\mathcal{R}(2\epsilon_i+\delta)^{-1}$in the expression (8.2) and (8.3) can be rewritten as

$$\begin{aligned}\mathcal{R}(2\epsilon_i+\delta)\tau(\epsilon_i) &= t_0 s_{\epsilon_i} + t_0^{-1}c_{2\epsilon_i+\delta}(\cdot)\big(\tau(\epsilon_i) - s_{\epsilon_i}\big),\\ \tau(-\epsilon_i)\mathcal{R}(2\epsilon_i+\delta)^{-1} &= t_0^{-1}s_{\epsilon_i} + t_0^{-1}c_{-2\epsilon_i-\delta}(\cdot)\big(\tau(-\epsilon_i) - s_{\epsilon_i}\big)\end{aligned} \tag{8.4}$$

since $s_{2\epsilon_i+\delta} = s_{\epsilon_i}\tau(-\epsilon_i) = \tau(\epsilon_i)s_{\epsilon_i}$, it immediately follows that L is of the form

$$L = \phi(x) + \sum_{j=1}^n\big(\phi_j^+(x)\tau(\epsilon_j) + \phi_j^-(x)\tau(-\epsilon_j)\big) \tag{8.5}$$

for (unique) coefficients $\phi, \phi_j^\pm \in \mathcal{Q}$. Since $L(1) = 0$, where $1 \in \mathcal{A}$ is the Laurent polynomial identically equal to one, we have $\phi(x) = -\sum_{j=1}^n(\phi_j^+(x) + \phi_j^-(x))$, so that

$$L = \sum_{j=1}^n\big(\phi_j^+(x)(\tau(\epsilon_j)-1) + \phi_j^-(x)(\tau(-\epsilon_j)-1)\big). \tag{8.6}$$

We will reduce the computation of the coefficients $\phi_j^\pm$ to the computation of ϕ_1^+, using the following easy lemma.

Lemma 8.4. *The coefficients $\phi_j^\pm \in \mathcal{Q}$ of a second order q-difference operator $L \in \mathcal{Q}[\tau(\Lambda_0)]$ of the form* (8.6) *are uniquely determined from the action of L on the sub-algebra $\mathcal{A}^W$ of W-invariant Laurent polynomials.*

Proof. Suppose $L|_{\mathcal{A}^W} = L'|_{\mathcal{A}^W}$ with L' of the form (8.6) with coefficients $\phi_j'^{\pm} \in \mathcal{Q}$. We have to show that $\phi_j^\pm = \phi_j'^{\pm}$ for $j = 1,\ldots,n$. Dividing out the common denominators, we may assume that $\phi_j^\pm, \phi_j'^{\pm} \in \mathcal{A}$ for $j = 1,\ldots,n$.

We now take $p_k(x) = m_{k\epsilon_1}(x) \in \mathcal{A}^W$ with $k \in \mathbb{Z}_+$, then

$$(Lp_k)(x) = (q^k-1)\sum_{j=1}^n\big(x_j^k(\phi_j^+(x) - q^{-k}\phi_j^-(x)) + x_j^{-k}(\phi_j^-(x) - q^{-k}\phi_j^+(x))\big)$$

and similarly for L'. Since q is generic (in particular, not a root of unity), $L|_{\mathcal{A}^W} = L'|_{\mathcal{A}^W}$ leads to

$$\phi_j^{\pm}(x) - q^{-k}\phi_j^{\mp}(x) = \phi_j'^{\pm}(x) - q^{-k}\phi_j'^{\mp}(x), \qquad k \gg 0$$

by comparison of powers of $x_j^{\pm 1}$, and hence $\phi_j^{\pm} = \phi_j'^{\pm}$ for $j = 1, \ldots, n$, as desired. $\square$

It follows from this lemma and lemma 8.2 that L is W-invariant, i.e. $w \circ L \circ w^{-1} = L$ in $\mathrm{End}_{\mathbb{C}}(\mathcal{A})$ for all $w \in W$. By the W-invariance of L, and since $\{\pm\epsilon_j\}_{j=1}^n$ is exactly the W-orbit $W\epsilon_1$, the explicit expressions for $\phi_j^{\pm}$ will immediately follow from the explicit expression for the coefficient ϕ_1^+ of L.

In order to evaluate ϕ_1^+, we compute the $\tau(\epsilon_1)$-contribution of $(Y_i^{\pm 1})_{sym} \in \mathcal{Q}[\tau(\Lambda_0)]$ for $i = 1, \ldots, n$. By (8.2), (8.4) and the fact that $\mathcal{R}(f) \in \mathcal{Q} \oplus \mathcal{Q}\, s_f$, the $\tau(\epsilon_1)$ contribution of $(Y_i)_{sym}$ can only be non-zero when there exists an ordered sub-word $w \in W$ of the word

$$w_i := s_{\epsilon_i - \epsilon_{i+1}} \cdots s_{\epsilon_i - \epsilon_n} s_{\epsilon_i} s_{\epsilon_i + \epsilon_n} \cdots s_{\epsilon_i + \epsilon_{i+1}} s_{\epsilon_i + \epsilon_{i-1}} \cdots s_{\epsilon_i + \epsilon_1} \in W \tag{8.7}$$

which maps ϵ_i to ϵ_1. Similarly, by (8.3) and (8.4) we see that the $\tau(\epsilon_1)$ contribution of $(Y_i^{-1})_{sym}$ can only be non-zero when there exists an ordered sub-word $v \in W$ of the word

$$v_i := s_{\epsilon_{i-1} - \epsilon_i} \cdots s_{\epsilon_2 - \epsilon_i} s_{\epsilon_1 - \epsilon_i} \in W \tag{8.8}$$

which maps $-\epsilon_i$ to ϵ_1. The ordered sub-words which satisfy these properties are easy to determine. The result is as follows.

Lemma 8.5. **(i)** *For $i \in \{2, \ldots, n\}$, there exists no ordered sub-word w of w_i which maps ϵ_i to ϵ_1.*

(ii) *The unit element $e \in W$ is the only sub-word of w_1 which maps ϵ_1 to itself.*

(iii) *For $i \in \{1, \ldots, n\}$, there exists no ordered sub-word v of v_i which maps $-\epsilon_i$ to ϵ_1.*

Proof. The proof is left to the reader. $\square$

So for the $\tau(\epsilon_1)$ component of L, it suffices to pick up the $\tau(\epsilon_1)$ contribution of $(Y_1)_{sym}$ associated with the sub-word $e \in W$ of w_1 in (8.2). This is given by

$$\begin{aligned}
\phi_1^+(x) &= t^{-1}c_{\epsilon_1-\epsilon_2}(x)t^{-1}c_{\epsilon_1-\epsilon_3}(x)\cdots t^{-1}c_{\epsilon_1-\epsilon_n}(x)t_n^{-1}c_{2\epsilon_1}(x) \\
&\quad \times t^{-1}c_{\epsilon_1+\epsilon_n}(x)t^{-1}c_{\epsilon_1+\epsilon_{n-1}}(x)\cdots t^{-1}c_{\epsilon_1+\epsilon_2}(x)t_0^{-1}c_{2\epsilon_1+\delta}(x) \\
&= (t_0t_n)^{-1}t^{2(1-n)}c_{2\epsilon_1}(x)c_{2\epsilon_1+\delta}(x)\prod_{i=2}^n c_{\epsilon_1-\epsilon_i}(x)c_{\epsilon_1+\epsilon_i}(x).
\end{aligned}$$

Now substitution of the expressions (3.19) for the c_f's, and making use of the W-invariance of L, we arrive at the explicit expression of L as given in theorem 3.19.

8.3. **Duality of the symmetric Koornwinder polynomials.** The duality of the renormalized symmetric Koornwinder polynomials $E^+(\gamma_\lambda;\cdot)$ ($\lambda \in \Lambda_0^+$), see definition 3.21, can be established in a similar fashion as the duality of the renormalized non-symmetric Koornwinder polynomials, see §6.3. We freely use the notations of §6.3.

Similar arguments as for the proof of (6.5) and (6.6), we have for all $p \in \mathcal{A}^W$ and all $\lambda, \mu \in \Lambda_0^+$ the identity

$$p(\gamma_\lambda) = \widetilde{B}\big(p(\widetilde{z}), E^+(\gamma_\lambda; z)\big), \qquad p(x_\mu) = B\big(p(z), \widetilde{E}^+(x_\mu; \widetilde{z})\big). \tag{8.9}$$

Now we substitute $p = \widetilde{E}^+(x_\mu; \cdot)$ in the first equality of (8.9) and $p = E^+(\gamma_\lambda; \cdot)$ in the second equality of (8.9), and we use the duality (6.4) of the bilinear form B to arrive at

$$E^+(\gamma_\lambda; x_\mu) = \widetilde{E}^+(x_\mu; \gamma_\lambda), \qquad \lambda, \mu \in \Lambda_0^+,$$

which proves the duality of the symmetric Koornwinder polynomials, see theorem 3.22.

8.4. **Macdonald's generalization of the Poincaré series.** The correction term of the W-invariant part $\Delta_+(\cdot)$ of the complex weight function $\Delta(\cdot)$ is given by the rational function $\mathcal{C} \in \mathcal{Q}$, see (3.16). The relation between the non-symmetric theory and the symmetric theory of the Koornwinder polynomials strongly depends on the symmetrizing properties of $\mathcal{C}$. These properties were investigated by Macdonald as generalizations of Poincaré series of Weyl groups. In this subsection we discuss these properties in detail for $\mathcal{C}$.

We define $\mathcal{A}_- \subset \mathcal{A}$ as the sub-space of $\mathcal{A}$ consisting of Laurent polynomial $p \in \mathcal{A}$ satisfying $wp = \sigma(w)p$, where $\sigma : W \to \{\pm 1\}$ is the character of W given by $\sigma(w) = (-1)^{l(w)}$ for all $w \in W$ (see lemma 3.1 for the action of W on $\mathcal{A}$).

We denote ρ by the half-sum of positive roots, so

$$\rho = \frac{1}{2} \sum_{\alpha \in \Sigma^+} \alpha = \omega_1 + \cdots + \omega_n = \sum_{i=1}^{n} (n - i + 1)\epsilon_i \in \Lambda_0^+,$$

where (recall) $\omega_i = \epsilon_1 + \cdots + \epsilon_i$ $(i = 1, \ldots, n)$ are the fundamental weights. Now we set $\Lambda_0^{++} = \Lambda_0^+ + \rho$, so

$$\Lambda_0^{++} = \{\lambda = (\lambda_1, \ldots, \lambda_n) \in \Lambda_0 \,|\, \lambda_1 > \lambda_2 > \cdots > \lambda_n > 0\}.$$

The elements in Λ_0^{++} are called *regular dominant weights*, since Λ_0^{++} is exactly the subset of dominant weights $\lambda \in \Lambda_0^+$ for which the stabilizer sub-group $W_\lambda \subset W$ only consists of the identity element.

Since the intersection of $W\lambda$ with Λ_0^+ consists of one point $\lambda^+ \in \Lambda_0^+$ for all $\lambda \in \Lambda_0$, we see that the "anti-symmetric" monomials

$$m_\lambda^-(x) = \sum_{w \in W} \sigma(w) x^{w\lambda} \in \mathcal{A}_-, \qquad \lambda \in \Lambda_0^+$$

span $\mathcal{A}_-$. If $\lambda \in \Lambda_0^+ \setminus \Lambda_0^{++}$, then there exists a simple reflection $s_i \in W$ stabilizing λ, since the stabilizer sub-group W_λ is a parabolic sub-group of W. It follows that $m_\lambda^-(x) = 0$ iff $\lambda \in \Lambda_0^+ \setminus \Lambda_0^{++}$, and that $\{m_\lambda^- \,|\, \lambda \in \Lambda_0^{++}\}$ is a linear basis of $\mathcal{A}_-$.

Lemma 8.6. *We have*

$$m_\rho^-(x) = x^\rho \prod_{\alpha \in \Sigma^-} (1 - x^\alpha)$$

in $\mathcal{A}$.

Proof. We write $p(x) = x^\rho \prod_{\alpha \in \Sigma^-}(1 - x^\alpha)$. For $i = 1, \ldots, n$ we have $s_i(\rho) = \rho - a_i$ since $\rho = \omega_1 + \cdots + \omega_n$ and $\big(\omega_i, a_j^\vee\big) = \delta_{i,j}$. Since s_i permutes $\Sigma^+ \setminus \{a_i\}$ and maps

a_i to $-a_i$, we see that $s_i\,p = -p = \sigma(s_i)p$ for all $i = 1, \ldots, n$. Hence $p \in \mathcal{A}_-$. We now use the dominance order $\leq$ on Λ_0, see definition 5.1 and lemma 5.2. Then

$$m_\rho^-(x) - p(x) = \sum_{\lambda < \rho} c_\lambda x^\lambda$$

for some constants c_λ. On the other hand, the left hand side of this identity lies in $\mathcal{A}_-$. So if it is non-zero, then its expansion in monomials has a contribution for the monomial x^λ for some $\lambda \in \Lambda_0^{++}$. But there are no regular dominant weights $\lambda \in \Lambda_0^{++}$ which are strictly smaller than ρ since $\Lambda_0^+ \subset \Lambda_0^>$, hence $m_\rho^-(x) = p(x)$, as desired. □

Proposition 8.7.

$$\sum_{w \in W} w\mathcal{C}(\cdot; \mathbf{t}; q) = C_{\mathbf{t},q} \tag{8.10}$$

lies in the base field $\mathbb{C}$ *of* $\mathcal{Q}$. *In particular,*

$$\langle p_1, p_2 \rangle_{\mathbf{t},q} = \frac{C_{\mathbf{t},q}}{|W|} \langle p_1, p_2 \rangle_{+,\mathbf{t},q}, \qquad \forall p_1, p_2 \in \mathcal{A}^W \tag{8.11}$$

where $|W| = 2^n n!$ *is the cardinality of the finite Weyl group* W.

Proof. The identity (8.11) follows directly from (8.10) using (3.15), the definitions of $\langle \cdot, \cdot \rangle$ and $\langle \cdot, \cdot \rangle_+$, and the invariance of the measure $(\mathbb{T}^n, \frac{dx}{x})$ under the action of W.

For (8.10), we observe that lemma 8.6 and the definition of $\mathcal{C} \in \mathcal{Q}$ (see (3.16)) implies that

$$\begin{aligned} m_\rho^-(x) \sum_{w \in W} (w\mathcal{C})(x) &= \\ = \sum_{w \in W} \sigma(w) w &\left(x^\rho \prod_{\alpha \in \Sigma^-} (1 - t_\alpha t_{\alpha/2} x^{\alpha/2})(1 + t_\alpha t_{\alpha/2}^{-1} x^{\alpha/2}) \right), \end{aligned} \tag{8.12}$$

which thus lies in $\mathcal{A}$. On the other hand, $\sum_{w \in W} w\mathcal{C} \in \mathcal{Q}$ is W-invariant, and $m_\rho^- \in \mathcal{A}_-$, so that the expression (8.12) actually lies in $\mathcal{A}_-$. We have to show that (8.12) is a constant multiple of $m_\rho^-(x)$. To show this, it is sufficient to prove that if the monomial x^μ ($\mu \in \Lambda_0$) occurs in the expansion of (8.12) as linear combination of monomials with non-zero coefficient, then $\mu \leq \rho$, cf. the proof of lemma 8.6. But every monomial contribution within the large brackets of (8.12) is of the form x^μ with $\mu = \frac{1}{2} \sum_{\alpha \in \Sigma^+} \xi_\alpha \alpha$, $\xi_\alpha \in \{-1, 0, 1\}$ and $\xi_\alpha \neq 0$ when $\alpha \in \Sigma_m^+$. Clearly, such a μ is smaller than (or equal to) ρ with respect to the dominance order. Since ρ is a dominant weight, we then also have $w\mu \leq \rho$ for all $w \in W$ and for any such μ. Hence (8.12) is a constant multiple of $m_\rho^-(x)$, which completes the proof of the proposition. □

A product form of the constant $C_{\mathbf{t},q}$ can be obtained by specializing the left hand side of (8.10) at x_0^{-1}. In fact, we have the following more general result (see §5.2 for the notations concerning parabolic sub-groups).

Lemma 8.8. *Let* $\lambda \in \Lambda_0^+$, *then* $C_{\mathbf{t};q} = \sum_{w \in W^\lambda} \mathcal{C}(x_{w\lambda}^{-1}; \mathbf{t}; q)$. *In particular,* $C_{\mathbf{t},q} = \mathcal{C}(x_0^{-1}; \mathbf{t}; q)$.

Proof. Let $\lambda \in \Lambda_0^+$. By the definition (8.10) of $C_{\mathbf{t},q}$ we have

$$C_{\mathbf{t},q} = \sum_{u \in W^\lambda, w \in W_\lambda} (w^{-1}u^{-1}\mathcal{C})(x_\lambda^{-1}; \mathbf{t}; q) \tag{8.13}$$

(indeed, observe that $w\mathcal{C}(x) \in \mathcal{Q}$ is regular at $x = x_\lambda^{-1}$ for all $w \in W$, so that we may specialize $\sum_{w\in W} w\mathcal{C}(x) \in \mathcal{Q}$ at $x = x_\lambda^{-1}$). We consider a term $(w^{-1}u^{-1}\mathcal{C})(x_\lambda^{-1})$ in the sum (8.13) with $w \neq 1$. Then there exists a simple root a_i ($i \in \{1, \ldots, n\}$) which is orthogonal to λ (i.e. $\big(\lambda, a_i\big) = 0$), and such that $\alpha = (uw)(a_i) \in \Sigma^-$ (we have used (5.5) here). Now the factor $c_{w^{-1}u^{-1}\alpha}(x_\lambda^{-1}) = c_{a_i}(x_\lambda^{-1})$ of $(w^{-1}u^{-1}\mathcal{C})(x_\lambda^{-1})$ is zero, since $x_\lambda^{a_i} = x_0^{a_i} (= t^2$ if $i \in \{1, \ldots, n-1\}$ and $= t_n^{\vee 2} t_n^2$ if $i = n)$, see the proof of proposition 6.16. Hence the contribution in the sum (8.13) is zero unless $w = 1$. The lemma follows now from lemma 5.6. □

8.5. **The link between the symmetric and non-symmetric theory.** In this subsection we expand the renormalized symmetric Koornwinder polynomials as linear combinations of the non-symmetric Koornwinder polynomials. We first show that $E^+(\gamma_\lambda; \cdot)$ ($\lambda \in \Lambda_0^+$) can be obtained by letting the trivial idempotent of H_0 act on any non-symmetric Koornwinder polynomial $E(\gamma_\mu; \cdot)$ ($\mu \in W\lambda$) under the Noumi representation $\pi_{\mathbf{t},q}$.

So we first have to introduce the trivial idempotent of H_0 and establish some of its elementary properties. We work directly in the image of the Noumi representation $\pi_{\mathbf{t},q}$. By Iwahori-Matsumoto's theorem (cf. the proof of proposition 4.6), we may write $t_w = t_{i_1} t_{i_2} \cdots t_{i_r}$ for a reduced expression $w = s_{i_1} s_{i_2} \cdots s_{i_r} \in W$. We then set

$$C_+ = \frac{1}{\sum_{w\in W} t_w^2} \sum_{w \in W} t_w T_w \in \mathcal{H}, \tag{8.14}$$

which satisfies the following elementary properties.

Lemma 8.9. **(i)** $(T_i - t_i)C_+ = 0$ *for* $i = 1, \ldots, n$.

(ii) $C_+^2 = C_+$.

(iii) $C_+ \in \mathcal{H} \subset End_{\mathbb{C}}(\mathcal{A})$ *is a projection onto* $\mathcal{A}^W$. *In particular, we have* $\mathcal{A}^W = \{p \in \mathcal{A} \,|\, C_+ p = p\}$.

(iv) $(C^+)^{\ddagger} = C'^+$, *where*

$$C'^+ = \frac{1}{\sum_{w\in W} t_w^{-2}} \sum_{w \in W} t_w^{-1} T'_w \in \mathcal{H}'$$

is the trivial idempotent with respect to inverse parameters, and $\ddagger$ *is the anti-isomorphism of* $\mathcal{H}$ *defined in lemma 6.9.*

Proof. **(ii)** is immediate from **(i)** and **(iii)** is a direct consequence of **(i)** and lemma 8.1. So it remains to prove **(i)** and **(iv)**.

For **(i)**, we fix $i \in \{1, \ldots, n\}$ and decompose $W = W_i^+ \cup W_i^-$ (disjoint union), where $W_i^{\pm}$ consists of the Weyl group elements $w \in W$ such that $l(s_i w) = l(w) \pm 1$.

Observe that $W_i^+ = s_i\, W_i^-$. Then we compute

$$\begin{aligned} T_i\Big(\sum_{w\in W} t_w T_w\Big) &= \sum_{w\in W_i^+}\big(t_w T_i T_w + t_{s_iw} T_i T_{s_iw}\big)\\ &= \sum_{w\in W_i^+} t_w\big(T_i + t_i T_i^2\big)T_w\\ &= \sum_{w\in W_i^+} t_w\big(t_i + t_i^2 T_i\big)T_w = t_i \sum_{w\in W} t_w T_w, \end{aligned}$$

where we have used the quadratic relations $(T_i - t_i)(T_i + t_i^{-1}) = 0$ for the third equality.

For **(iv)**, we observe that

$$(C_+)^{\ddagger} = \frac{1}{\sum_{w\in W} t_w^2}\sum_{w\in W} t_w {T'_w}^{-1} \in \mathcal{H}'.$$

Hence it suffices to show that $C_+ \in \mathcal{H}$ can also be written as

$$C_+ = \frac{1}{\sum_{w\in W} t_w^{-2}}\sum_{w\in W} t_w^{-1} T_w^{-1}.$$

We show this by changing the summation variable in (8.14) to $u = \sigma w$, where $\sigma = -1 \in W$ is the longest Weyl group element (which maps v to $-v$ for all $v \in V$). Then $l(\sigma w) = l(\sigma) - l(w)$ for all $w \in W$, so that $t_{\sigma w} = t_\sigma t_w^{-1}$ and $T_{\sigma w} = T_\sigma T_w^{-1}$ for all $w \in W$. In particular,

$$\sum_{w\in W} t_w^2 = t_\sigma^2 \sum_{w\in W} t_w^{-2}, \qquad \sum_{w\in W} t_w T_w = t_\sigma T_\sigma \sum_{w\in W} t_w^{-1} T_w^{-1}.$$

Substituting these expressions into (8.14) and using **(i)** then shows

$$C_+ = t_\sigma^{-1} T_\sigma \frac{1}{\sum_{w\in W} t_w^{-2}}\sum_{w\in W} t_w^{-1}T_w^{-1} = \frac{1}{\sum_{w\in W} t_w^{-2}}\sum_{w\in W} t_w^{-1}T_w^{-1},$$

as desired. □

Proposition 8.10. *Let $\lambda \in \Lambda_0^+$ and $\mu \in W\lambda$, then*

$$E^+(\gamma_\lambda;\cdot) = C_+ E(\gamma_\mu;\cdot).$$

Proof. Fix $\lambda \in \Lambda_0^+$ and $\mu \in W\lambda$. Let $p(Y) \in \mathcal{A}_Y^W$. Since the T_i's commute with $p(Y)$ for $i = 1,\ldots,n$, we have that $p(Y)C_+ = C_+ p(Y)$ in $\mathcal{H}$. Hence

$$p(Y)\big(C_+\, E(\gamma_\mu;\cdot)\big) = C_+\big(p(Y)\, E(\gamma_\mu;\cdot)\big) = p(\gamma_\lambda) C_+ E(\gamma_\mu;\cdot),$$

since $p(\gamma_\mu) = p(\gamma_\lambda)$ by lemma 5.6 and by the W-invariance of p. Since $p(Y) \in \mathcal{A}_Y^W$ is arbitrary and $C_+ E(\gamma_\mu;\cdot) \in \mathcal{A}^W$ by lemma 8.9**(iii)**, it follows that $C_+ E(\gamma_\mu;\cdot)$ is a constant multiple of $E^+(\gamma_\lambda;\cdot)$. To show that the constant multiple is one, it suffices to show that

$$\big(C_+ E(\gamma_\mu;\cdot)\big)(x_0^{-1}) = 1.$$

Now $c_{a_i}(x_0^{-1};\mathbf{t}|\,q) = 0$ for all $i = 1,\ldots,n$, so that $(T_i\, p)(x_0^{-1}) = t_i p(x_0^{-1})$ for $i = 1,\ldots,n$. In particular,

$$\big(T_w E(\gamma_\mu;\cdot)\big)(x_0^{-1}) = t_w E(\gamma_\mu; x_0^{-1}) = t_w, \qquad \forall w \in W.$$

Hence

$$\big(C_+E(\gamma_\mu;\cdot)\big)(x_0^{-1})=\frac{1}{\sum_{w\in W}t_w^2}\sum_{w\in W}t_w\big(T_wE(\gamma_\mu;\cdot)\big)(x_0^{-1})=1,$$

as desired. □

We are now in a position to derive the explicit expansion of the symmetric Koornwinder polynomial in terms of non-symmetric Koornwinder polynomials, see theorem 3.27. By proposition 6.16 and proposition 8.10 we have an expansion of the form

$$E^+(\gamma_\lambda;x)=\sum_{\mu\in W\lambda}c_\mu^\lambda E(\gamma_\mu;x),\qquad \lambda\in\Lambda_0^+$$

for unique coefficients $c_\mu^\lambda\in\mathbb{C}$. We first show that $c_\mu^\lambda=K_\lambda\,\widetilde{\mathcal{C}}(\gamma_\mu^{-1})$ for all $\mu\in W\lambda$, with the constant K_λ independent of $\mu\in W\lambda$. By theorem 3.14 and theorem 3.16, we have

$$\langle E^+(\gamma_\lambda;\cdot),E'(\gamma_\mu^{-1};\cdot)\rangle=c_\mu^\lambda\langle E(\gamma_\mu;\cdot),E'(\gamma_\mu^{-1};\cdot)\rangle=\frac{c_\mu^\lambda}{\widetilde{\mathcal{C}}(\gamma_\mu^{-1})}\frac{\langle 1,1\rangle\,\widetilde{w}(\gamma_0^{-1})}{\widetilde{w}_+(\gamma_\lambda^{-1})}\tag{8.15}$$

for all $\lambda\in\Lambda_0^+$ and $\mu\in W\lambda$. On the other hand, by lemma 8.9, proposition 8.10, proposition 7.2 and remark 3.26,

$$\begin{aligned}\langle E^+(\gamma_\lambda;\cdot),E'(\gamma_\mu^{-1};\cdot)\rangle&=\langle C_+E^+(\gamma_\lambda;\cdot),E'(\gamma_\mu^{-1};\cdot)\rangle\\&=\langle E^+(\gamma_\lambda;\cdot),C_+^\ddagger E'(\gamma_\mu^{-1};\cdot)\rangle\\&=\langle E^+(\gamma_\lambda;\cdot),C_+'E'(\gamma_\mu^{-1};\cdot)\rangle\\&=\langle E^+(\gamma_\lambda;\cdot),E'^+(\gamma_\lambda^{-1};\cdot)\rangle=\langle E^+(\gamma_\lambda;\cdot),E^+(\gamma_\lambda;\cdot)\rangle\end{aligned}\tag{8.16}$$

for all $\lambda\in\Lambda_0^+$ and $\mu\in W\lambda$, where $E'^+(\gamma_\lambda^{-1};\cdot)$ is the renormalized symmetric Koornwinder polynomial of degree λ with respect to inverse parameters. We conclude that the expression (8.15) is independent of $\mu\in W\lambda$, hence

$$c_\mu^\lambda=K_\lambda\widetilde{\mathcal{C}}(\gamma_\mu^{-1}),\qquad \mu\in W\lambda,\ \ \lambda\in\Lambda_0^+$$

for some constant K_λ independent of $\mu\in W\lambda$ and

$$E^+(\gamma_\lambda;x)=K_\lambda\sum_{\mu\in W\lambda}\widetilde{\mathcal{C}}(\gamma_\mu^{-1})E(\gamma_\mu;x),\qquad \lambda\in\Lambda_0^+.$$

Now we evaluate this expression at $x=x_0^{-1}$ and we use lemma 8.8, to see that $K_\lambda=\widetilde{\mathcal{C}}(\gamma_0^{-1})^{-1}$ (independent of $\lambda\in\Lambda_0^+$). This completes the proof of theorem 3.27.

8.6. **Orthogonality relations and quadratic norms.** In this subsection we establish the orthogonality relations and quadratic norms of the renormalized symmetric Koornwinder polynomials with respect to the non-degenerate bilinear form $\langle\cdot,\cdot\rangle_+$ on $\mathcal{A}^W$.

Recall from proposition 8.7 that $\langle\cdot,\cdot\rangle_+$ is the restriction of the bilinear form $\langle\cdot,\cdot\rangle$ to the sub-algebra $\mathcal{A}^W$, up to an explicit constant. Hence the bi-orthogonality relations of the non-symmetric Koornwinder polynomials with respect to $\langle\cdot,\cdot\rangle$ (see theorem 3.14), remark 3.26 and theorem 3.27, imply that

$$\langle E^+(\gamma_\lambda;\cdot),E^+(\gamma_\mu;\cdot)\rangle_+=0,\qquad \lambda,\mu\in\Lambda_0^+,\ \ \lambda\neq\mu,$$

which are the orthogonality relations of the symmetric Koornwinder polynomials as stated in theorem 3.25. For the diagonal terms, we observe that (8.15) and (8.16) lead to

$$\langle E^+(\gamma_\lambda;\cdot), E^+(\gamma_\lambda;\cdot)\rangle = \frac{\langle 1,1\rangle\, \widetilde{w}(\gamma_0^{-1})}{\widetilde{\mathcal{C}}(\gamma_0^{-1})\widetilde{w}_+(\gamma_\lambda^{-1})}, \qquad \lambda\in\Lambda_0^+ \tag{8.17}$$

since we have seen in the previous subsection that $c_\mu^\lambda = \widetilde{\mathcal{C}}(\gamma_\mu^{-1})/\widetilde{\mathcal{C}}(\gamma_0^{-1})$. Now by proposition 8.7 applied twice, we obtain from (8.17) that

$$\langle E^+(\gamma_\lambda;\cdot), E^+(\gamma_\lambda;\cdot)\rangle_+ = \frac{\langle 1,1\rangle_+\, \widetilde{w}(\gamma_0^{-1})}{\widetilde{\mathcal{C}}(\gamma_0^{-1})\widetilde{w}_+(\gamma_\lambda^{-1})}.$$

But $\widetilde{w}(\gamma_0^{-1}) = \widetilde{\mathcal{C}}(\gamma_0^{-1})\widetilde{w}_+(\gamma_0^{-1})$, so that

$$\frac{\langle E^+(\gamma_\lambda;\cdot), E^+(\gamma_\lambda;\cdot)\rangle_+}{\langle 1,1\rangle_+} = \frac{\widetilde{w}_+(\gamma_0^{-1})}{\widetilde{w}_+(\gamma_\lambda^{-1})}, \qquad \lambda\in\Lambda_0^+$$

which establishes the evaluation of the diagonal terms for the symmetric Koornwinder polynomials, see theorem 3.25.

9. Appendix

In this section we prove the commutation relations stated in proposition 6.6. We recall here that U_0 and Y_i are given by the expressions

$$\begin{aligned} U_0 &= T_1T_2\cdots T_{n-1}z_n^{-1}T_n^{-1}T_{n-1}^{-1}\cdots T_1^{-1},\\ Y_i &= T_i\cdots T_{n-1}T_nT_{n-1}\cdots T_1T_0T_1^{-1}T_2^{-1}\cdots T_{i-1}^{-1}. \end{aligned} \tag{9.1}$$

For the actual verification of the Lusztig type commutation relation between U_0 and $p(Y)$ ($p\in\mathcal{A}$) in proposition 6.6, we observe that it suffices to prove it for $p(Y)=Y_i$ ($i=1,\ldots,n$), cf. the proof of proposition 4.11. If we furthermore substitute the explicit expression for the difference-reflection operator T_0, we see that it suffices to prove the following commutation relations in $\mathcal{H}$:

- **(a)** $(U_0-t_n^\vee)(U_0+t_n^{\vee-1})=0$;
- **(b)** $U_0T_1U_0T_1 = T_1U_0T_1U_0$;
- **(c)** $U_0T_i = T_iU_0$ for $i=2,\ldots,n$;
- **(d)** $U_0^{-1}Y_1 = q^{-1}Y_1^{-1}U_0^{-1} + (t_n^{\vee-1}-t_n^\vee)Y_1 + q^{-1/2}(t_0^{\vee-1}-t_0^\vee)$;
- **(e)** $U_0Y_i = Y_iU_0$ for $i=2,\ldots,n$.

We first check the easy commutation relations **(a)** and **(c)**. The relation **(a)** is immediate from the quadratic relation for $z_n^{-1}T_n^{-1}$ (see proposition 6.5) and (9.1), which shows that U_0 is conjugate to $z_n^{-1}T_n^{-1}$ in $\mathcal{H}$. For **(c)** we note that by the commutation relations $T_iz_iT_i = z_{i+1}$ ($i=1,\ldots,n-1$) (see proposition 6.5) and (9.1), we have

$$U_0 = z_1^{-1}T_1^{-1}\cdots T_{n-1}^{-1}T_n^{-1}T_{n-1}^{-1}\cdots T_1^{-1} = z_1^{-1}T_0Y_1^{-1}. \tag{9.2}$$

If we write $[X_1,X_2]=X_1X_2-X_2X_1$ for the commutator of two elements $X_1,X_2\in\mathcal{H}$, then by proposition 6.5, $[T_j,z_1^{-1}]=0$ and $[T_j,T_0]=0$ for $j=2,\ldots,n$. Furthermore, by Lusztig's commutation relation between the T_j's and $p(Y)$'s (see (6.2)), we also have $[T_j,Y_1^{-1}]=0$ for $j=2,\ldots,n$. Combined with (9.2), this proves **(c)**.

We next prove **(e)** by showing that $U_0^{-1}Y_j = Y_jU_0^{-1}$ for $j=2,\ldots,n$. We fix $j\in\{2,\ldots,n\}$. Using the commutativity of the Y_j's and using (9.2), we see that $U_0^{-1}Y_j = Y_jU_0^{-1}$ is equivalent to $[T_0^{-1}z_1,Y_j]=0$. Now using the expression (9.1) for Y_j, together with the commutation relations between the T_i's and z_k's (see proposition 6.5), we compute

$$\begin{aligned} z_1Y_j &= T_j\cdots T_{n-1}T_nT_{n-1}\cdots T_2z_1T_1T_0T_1^{-1}\cdots T_{j-1}^{-1}\\ &= T_j\cdots T_{n-1}T_nT_{n-1}\cdots T_2T_1^{-1}T_0z_2T_1^{-1}\cdots T_{j-1}^{-1}\\ &= T_j\cdots T_{n-1}T_nT_{n-1}\cdots T_2T_1^{-1}T_0T_1T_2^{-1}\cdots T_{j-1}^{-1}z_1. \end{aligned}$$

Applying the $\widetilde{C}_n$-braid relations for the T_i's, we obtain

$$\begin{aligned} T_0^{-1}z_1Y_j &= T_j\cdots T_{n-1}T_nT_{n-1}\cdots T_2T_0^{-1}T_1^{-1}T_0T_1T_2^{-1}\cdots T_{j-1}^{-1}z_1\\ &= T_j\cdots T_{n-1}T_nT_{n-1}\cdots T_2T_1T_0T_1^{-1}T_0^{-1}T_2^{-1}\cdots T_{j-1}^{-1}z_1\\ &= T_j\cdots T_{n-1}T_nT_{n-1}\cdots T_2T_1T_0T_1^{-1}T_2^{-1}\cdots T_{j-1}^{-1}T_0^{-1}z_1 = Y_jT_0^{-1}z_1 \end{aligned}$$

(here we have used that $T_0^{-1}T_1^{-1}T_0T_1 = T_1T_0T_1^{-1}T_0^{-1}$, which is a direct consequence of the braid relations). This completes the proof of **(e)**.

Next we consider the commutation relation **(d)**. By (9.2) and proposition 6.5, we have

$$\begin{aligned} q^{1/2}U_0Y_1 = q^{1/2}z_1^{-1}T_0 &= q^{-1/2}T_0^{-1}z_1 + t_0^{\vee-1} - t_0^{\vee}\\ &= q^{-1/2}Y_1^{-1}U_0^{-1} + t_0^{\vee-1} - t_0^{\vee}. \end{aligned} \tag{9.3}$$

Using the quadratic relation **(a)** for U_0 (which we already proved), we immediately see that **(d)** is implied by (9.3).

Instead of proving the commutation relation **(b)**, we prove the following equivalent commutation relation in $\mathcal{H}$:

$$U_0^{-1}T_1^{-1}U_0^{-1}T_1^{-1} = T_1^{-1}U_0^{-1}T_1^{-1}U_0^{-1}. \tag{9.4}$$

We set $\Xi = T_2\cdots T_{n-1}T_nT_{n-1}\cdots T_2 \in \mathcal{H}$. By proposition 6.5, we have $\Xi\, z_1 = z_1\Xi$, and $U_0^{-1} = T_1\Xi\, T_1z_1$ by (9.1) and (9.2). Combined with the commutation relation $T_1z_1 = z_2T_1^{-1}$ (see proposition 6.5), we see that

$$\begin{aligned} T_1^{-1}U_0^{-1}T_1^{-1}U_0^{-1} &= \Xi\, T_1z_1\Xi\, T_1z_1\\ &= \Xi\, T_1\Xi\, z_1T_1z_1 = \Xi\, T_1\Xi\, z_1z_2T_1^{-1} \end{aligned} \tag{9.5}$$

on the one hand, and

$$\begin{aligned} U_0^{-1}T_1^{-1}U_0^{-1}T_1^{-1} &= T_1\Xi\, T_1z_1\Xi\, T_1z_1T_1^{-1}\\ &= T_1\Xi\, T_1\Xi\, z_1T_1z_1T_1^{-1} = T_1\Xi\, T_1\Xi\, T_1^{-1}z_1z_2T_1^{-1} \end{aligned} \tag{9.6}$$

on the other hand. Now multiplying the expressions (9.5) and (9.6) on the right by the invertible element $T_1z_1^{-1}z_2^{-1}T_1 \in \mathcal{H}$, we see that it suffices to prove that

$$\Xi\, T_1\Xi\, T_1 = T_1\Xi\, T_1\Xi \tag{9.7}$$

in $\mathcal{H}$. Set $\Xi_i = T_i\cdots T_{n-1}T_nT_{n-1}\cdots T_i$ for $i = 2,\ldots,n$, then we claim that the following commutation relations are valid in $\mathcal{H}$:

$$\Xi_iT_{i-1}\Xi_iT_{i-1} = T_{i-1}\Xi_iT_{i-1}\Xi_i, \qquad i = 2,\ldots,n. \tag{9.8}$$

Observe that (9.7) is the special case $i = 2$ of (9.8). We prove now (9.8) by downward induction on i. For $i = n$, we have $\Xi_n = T_n$, hence (9.8) is valid by the braid relation $T_nT_{n-1}T_nT_{n-1} = T_{n-1}T_nT_{n-1}T_n$ in $\mathcal{H}$. For the induction step, we assume that (9.8) is valid for $i = k+1$, with $k \in \{2,\ldots,n-1\}$. The $\widetilde{C}_n$-braid relations for the T_i's imply $T_{k-1}\Xi_{k+1} = \Xi_{k+1}T_{k-1}$. Combined with the braid relation $T_kT_{k-1}T_k = T_{k-1}T_kT_{k-1}$ and the identity $\Xi_k = T_k\Xi_{k+1}T_k$, we see that the expression in the left hand side of (9.8) with $i = k$ can be rewritten as

$$\begin{aligned}\Xi_kT_{k-1}\Xi_kT_{k-1} &= T_k\Xi_{k+1}T_kT_{k-1}T_k\Xi_{k+1}T_kT_{k-1}\\ &= T_k\Xi_{k+1}T_{k-1}T_kT_{k-1}\Xi_{k+1}T_kT_{k-1}\\ &= T_kT_{k-1}\Xi_{k+1}T_k\Xi_{k+1}T_{k-1}T_kT_{k-1}\\ &= T_kT_{k-1}(\Xi_{k+1}T_k\Xi_{k+1}T_k)T_{k-1}T_k,\end{aligned}$$

while the expression in the right hand side of (9.8) with $i = k$ can be rewritten as

$$\begin{aligned}T_{k-1}\Xi_kT_{k-1}\Xi_k &= T_{k-1}T_k\Xi_{k+1}T_kT_{k-1}T_k\Xi_{k+1}T_k\\ &= T_{k-1}T_k\Xi_{k+1}T_{k-1}T_kT_{k-1}\Xi_{k+1}T_k\\ &= T_{k-1}T_kT_{k-1}\Xi_{k+1}T_k\Xi_{k+1}T_{k-1}T_k\\ &= T_kT_{k-1}(T_k\Xi_{k+1}T_k\Xi_{k+1})T_{k-1}T_k,\end{aligned}$$

hence (9.8) is also valid for $i = k$ by the induction hypothesis. This completes the proof of (9.8), and hence also the proof of the commutation relation **(b)**.

With these considerations, the proof of proposition 6.6 is now complete.

10. Precise references to the literature

§2 Macdonald [20] derived the basic properties of affine root systems and affine Weyl groups, and classified the irreducible root systems. In particular, in [20] the non-reduced affine root system $C^\vee C$ was introduced. The derivations in §2 are mainly ad hoc. For a more systematic treatment, see Humphreys' book [14].

§3.2–3.3 The difference-reflection operators associated with the non-reduced root system of type $C^\vee C$, as well as the associated Y-operators, were written down by Noumi in [26]. They generalize the difference-reflection operators and Y-operators for classical, reduced affine root systems of Cherednik, see [5]–[9]. The commutativity of the Y-operators (theorem 3.6) was proven by Noumi [26].

§3.4–3.5 The non-symmetric Koornwinder polynomials (definition 3.10) were defined by Sahi [29]. The duality of the non-symmetric Koornwinder polynomials (theorem 3.11) was also proven in [29].

§3.6 The bi-orthogonality of the non-symmetric Koornwinder polynomials (theorem 3.14) was proven by Sahi [30] for discrete values of the parameters, and by Stokman [32] for continuous values of the parameters.

§3.7 The evaluation of the diagonal terms for the non-symmetric Koornwinder polynomials (theorem 3.16) was derived in [32].

§3.8 Symmetric Koornwinder polynomials were defined by Koornwinder [17] as a generalization of the symmetric Macdonald polynomials associated with the root system BC (see [23]), as well as a multivariable generalization of the Askey-Wilson polynomials (see [2]). Koornwinder [17] proved that they are eigenfunctions of the second order q-difference operator L as defined in theorem 3.19, and proved their orthogonality relations (see theorem 3.25). Koornwinder and Macdonald conjectured further properties of these polynomials in an unpublished note, such as the duality (theorem 3.22) and the explicit expression for the quadratic norms (theorem 3.25). These conjectures were proven by Van Diejen [10] for a sub-family of the Koornwinder polynomials.

Noumi [26] clarified the important role of affine Hecke algebras in the theory of symmetric Koornwinder polynomials (in the same spirit as Cherednik's affine Hecke algebra approach to Macdonald polynomials).This was also announced by Macdonald [22]. The explicit relation between Koornwinder's second order q-difference operator and the Y-operators (see theorem 3.19) was proven in [26]. The proof of duality for the symmetric Koornwinder polynomials (see theorem 3.22) for the complete family of symmetric Koornwinder polynomials was proven by Sahi [29], using the analogue of Cherednik's [6] double affine Hecke algebra in the Koornwinder setting. By Van Diejen's [10] results, this in turn implied the quadratic norm evaluations for the symmetric Koornwinder polynomials (see theorem 3.25) for all parameters values. A proof of the quadratic norm evaluations using affine Hecke algebras, which in particular leads to the expressions in terms of multiple residues in a natural way, was derived in [32]. The precise link between the non-symmetric Koornwinder polynomials and the symmetric Koornwinder polynomials (theorem 3.27) follows easily from results in [32].

§4.1–4.3 The treatment of the affine Hecke algebra as given in these three subsections follows the paper of Lusztig [19], who partly attributes the results to Bernstein and Zelevinski. We focused our presentation to the case of affine Hecke algebras of type $\widetilde{C}_n$, which is exactly the awkward case in Lusztig [19]. The awkwardness lies in the fact that $\langle \Lambda_0, a_n \rangle = 2\mathbb{Z}$ instead of $\mathbb{Z}$, so that lemma 4.9 does not suffice to prove proposition 4.11. The computation of the commutation relation of proposition 4.11 for $i = n$ was derived in Lusztig from (quite elaborate) computations in the associated extended braid group. The shortcut of this result as presented in the proof of proposition 4.11 seems to be new. Observe that $\langle \Lambda_0, a_n \rangle = 2\mathbb{Z}$ (where Λ_0 should be considered here as the co-root lattice of Σ) is exactly the crucial property of the reduced root system Σ which allows one to squeeze in two extra parameters into the theory (see §2.3 on the level of affine root systems), leading eventually to the affine Hecke algebra interpretation of the complete family of Koornwinder polynomials.

§4.4 The Noumi representation was defined in [26], following the approach of Cherednik [5] in case of reduced root systems.

§5.1 The triangularity of the Y-operators (see proposition 5.4) was derived in [32].

§5.2 The definition of the non-symmetric Koornwinder polynomials was given by Sahi [30]. The presentation given here, which makes essential use of the triangularity of the Y-operators, follows [32]. The advantage is that the triangularity of the Koornwinder polynomials is automatically incorporated in their definition (see theorem 5.7). In Sahi's approach this requires proof, see [30].

§6.1–6.3 The introduction of the double affine Hecke algebra, the derivation of its basic algebraic structure and its application to the duality theorem of the non-symmetric Koornwinder polynomial (theorem 3.11), follow closely the paper of Sahi [29].

§6.4 The application of the duality in obtaining spectral difference-reflection operators is taken from [32].

§7.1 The results in this sub-section were derived in an algebraic manner in [30] for a discrete set of parameter values. This sub-section follows the analytic approach of [32] (which in particular leads to the results for continuous parameter values).

§7.2 The derivation of the diagonal terms of the non-symmetric Koornwinder polynomials in terms of multiple residues of the weight function follows [32]. The approach is motivated by Cherednik's paper [8], in which it became apparent that (non-symmetric) Harish-Chandra transforms can be well understood by computing their intertwining properties under the action of the (degenerate) double affine Hecke algebra.

§8.1–8.2 The results in these sections linking symmetric Koornwinder polynomials to the affine Hecke algebra follow Noumi [26].

§8.3 The proof of the duality for the symmetric Koornwinder polynomials follows Sahi [29].

§8.4 The generalization of the Poincaré serie of type C_n was derived by Macdonald [21].

§8.5 The precise expansion of the renormalized symmetric Koornwinder polynomials as linear combination of the renormalized non-symmetric Koornwinder polynomials can be easily derived from [32]. In [32] the "monic version", i.e. the explicit expansion of the monic symmetric Koornwinder polynomial P_λ^+ ($\lambda \in \Lambda_0^+$) in terms of the monic non-symmetric Koornwinder polynomials P_μ ($\mu \in \Lambda_0$) was proven, and explicit expressions for $P_\mu(x_0^{-1})$ and $P_\lambda^+(x_0)$ were derived (the so-called "evaluation formulas"). Combined they lead to theorem 3.27. The proof of the expansion theorem (theorem 3.27) as given in §8.5 is new.

§8.6 The derivation of the quadratic norms of the symmetric Koornwinder polynomials in terms of residues of the weight function as given in §8.6 follows [32].

§9 The proof of commutation relations between specific elements of the double affine Hecke algebra, which is needed for the duality of the Koornwinder polynomials in §6.2, follows Sahi [29].

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Jasper V. Stokman, KdV Institute for Mathematics, Universiteit van Amsterdam, Plantage Muidergracht 24, 1018 TV Amsterdam, The Netherlands.

E-mail address: `jstokman@science.uva.nl`