9

Macdonald–Koornwinder polynomials

Jasper V. Stokman

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9.1 Introduction

In this chapter symmetric and nonsymmetric Macdonald–Koornwinder polynomials are introduced and their basic properties are discussed. These include (bi-)orthogonality relations, norm formulas, q-difference(-reflection) equations, duality and evaluation formulas. We develop the theory in such a way that it naturally encompasses all known cases, as well as a new rank two case. See the first paragraph of §9.3 about the precise meaning of our terminology of Macdonald polynomials, Koornwinder polynomials and Macdonald–Koornwinder polynomials.

Symmetric Macdonald–Koornwinder polynomials are multivariate orthogonal Laurent polynomials. Their rank one cases are the Askey–Wilson polynomials, the continuous q-Jacobi polynomials and the continuous q-ultraspherical polynomials [1], which are three families of classical one-variable q-orthogonal polynomials from the q-Askey scheme [58]. The symmetric Macdonald–Koornwinder polynomials are q-deformations of the symmetric Jacobi polynomials associated with root systems, also known as symmetric Heckman–Opdam polynomials (Chapter 8). They are defined either by orthogonality with respect to an explicit orthogonality measure or as common eigenfunctions of linear, triangular q-difference operators. In general they do not have an explicit expression in terms of products of one-variable basic hypergeometric series, in contrast to the Tratnik [101] type multivariate q-orthogonal polynomials from, e.g., [37, 88]. Instead, the symmetric Macdonald–Koornwinder polynomials associated with classical root systems admit explicit expansion formulas in interpolation polynomials, see [80, 81].

The parallels between symmetric Macdonald–Koornwinder polynomials and symmetric Jacobi polynomials associated with root systems are quite strong. From the point of view of applications for instance, symmetric Jacobi polynomials associated with root systems generalize spherical functions on compact symmetric spaces and provide eigenstates for quan-

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tum trigonometric Calogero–Moser systems (Chapter 8), while the symmetric Macdonald–Koornwinder polynomials have an interpretation as spherical functions on quantum compact symmetric spaces [75, 77, 78, 64] and give rise to eigenstates for Ruijsenaars' [89] relativistic analogs of quantum trigonometric Calogero–Moser systems [9, 18, 55].

Important special cases of the symmetric Macdonald–Koornwinder polynomials are the GL_n symmetric Macdonald polynomials and the symmetric Koornwinder polynomials. The underlying finite root systems are of type A and type BC, respectively. The symmetric GL_n Macdonald polynomials were introduced by Macdonald [70, Ch. VI] as a two-parameter family of multivariate orthogonal polynomials in n variables having both the Jack polynomials and the Hall–Littlewood polynomials as limit cases. They have a wealth of applications in combinatorics, algebraic geometry, topology and representation theory (Chapter 10). In an important preprint from 1987 (it appeared in print in 2000, see [72]) Macdonald introduced root system generalizations of the GL_n Macdonald polynomials. The resulting symmetric Macdonald polynomials are multivariate orthogonal polynomials labelled by so-called admissible pairs of reduced root systems. Recasting the initial data in terms of affine root systems (cf. [18, 73]) it is natural to speak of an untwisted and a twisted theory of symmetric Macdonald polynomials associated with root systems. An important further extension for nonreduced root systems was constructed by Koornwinder [61] in 1991. They are nowadays known as symmetric Koornwinder polynomials. The symmetric Koornwinder polynomials are the only Macdonald–Koornwinder polynomials for which elliptic analogs exist to date, see [85, 86] and Chapter 6.

A crucial subsequent development was the definition, due to Macdonald [71], Cherednik [14] and Sahi [91], of nonsymmetric versions of the Macdonald–Koornwinder polynomials. It was inspired by Heckman's definition of the nonsymmetric variants of the Heckman–Opdam polynomials, see [83] and §8.3. A symmetrization procedure turns the nonsymmetric Macdonald–Koornwinder polynomials into the symmetric ones. It is within the nonsymmetric theory that Cherednik's [9, 12] double affine Hecke algebra appears as the fundamental algebraic structure underlying the Macdonald–Koornwinder polynomials. The double affine Hecke algebra has been instrumental in obtaining the norm and evaluation formulas for Macdonald–Koornwinder polynomials, which were conjectured in the symmetric case by Macdonald in his 1987 preprint [72]. Many of these ideas and techniques were developed first for the Jacobi polynomials associated with root systems. See Chapter 8 for a detailed account and references.

Cherednik's [18] approach to Macdonald–Koornwinder polynomials using double affine Hecke algebras has been developed for the above mentioned four different cases (the GL_n case, the untwisted case, the twisted case and the Koornwinder (or $\mathrm{C}^\vee\mathrm{C}$) case. Cherednik [18] treats the first three cases separately. Macdonald's [73] exposition covers the last three cases, but various steps still need case by case analysis. Haiman [41] developed a general framework that naturally encompasses the above four cases of the Macdonald–Koornwinder theory. We slightly adjust Haiman's [41] setup and use it to give a uniform treatment of the Macdonald–Koornwinder theory. This theory, besides its four known subclasses mentioned above, includes a new class of rank two Macdonald–Koornwinder type polynomials (see §9.3.9).

Before giving a detailed description of the content of the chapter we will first introduce the symmetric GL_n Macdonald polynomials and the symmetric Koornwinder polynomials in the next two subsections, to give the reader a flavour of the type of multivariate orthogonal Laurent polynomials we are dealing with. We end the introductory section by listing various topics on Macdonald–Koornwinder polynomials that we will not be able to treat in this chapter.

9.1.1 Symmetric GL_n Macdonald polynomials

Macdonald [68], [70, Ch. VI] introduced in 1988 a two-parameter family of symmetric orthogonal polynomials in n variables $t_1, \ldots, t_n$, nowadays often referred to as the Macdonald polynomials. In the general theory of Macdonald–Koornwinder polynomials associated to root data as developed in [72, 73, 41] and in the present chapter, the Macdonald polynomials relate to the symmetric GL_n Macdonald polynomials, which are the symmetric Macdonald–Koornwinder polynomials associated to the GL_n root datum.

The symmetric GL_n Macdonald polynomials P_λ^+ ($\lambda \in \Lambda^+$) form a distinguished two parameter family of complex linear bases of the algebra $\mathbb{C}[t_1^{\pm 1}, \ldots, t_n^{\pm 1}]^{S_n}$ of S_n-invariant Laurent polynomials in n variables $t_1, \ldots, t_n$. They are labeled by the set Λ^+ of n tuples $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{Z}^n$ satisfying $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$. The P_λ^+ for $\lambda \in \Lambda^+$ with $\lambda_n \geq 0$ are the Macdonald polynomials from [70, Ch. VI] (in particular, they form a linear basis of the algebra $\mathbb{C}[t_1, \ldots, t_n]^{S_n}$ of symmetric polynomials in $t_1, \ldots, t_n$).

The symmetric GL_n Macdonald polynomials can be defined as common eigenfunctions of q-difference operators or in terms of orthogonality relations, with the inner product either defined analytically or combinatorially. We consider here the analytic approach (see [70, Ch. VI, §4] and §10.3, for the combinatorial approach).

Consider the set Λ^+ of n tuples $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{Z}^n$ satisfying $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$. Write $t = (t_1, \ldots, t_n)$ and $\mathbb{C}[t^{\pm 1}]^{S_n} = \mathbb{C}[t_1^{\pm 1}, \ldots, t_n^{\pm 1}]^{S_n}$. The *symmetric monomials*

$$m_\lambda(t) := \sum_{\mu \in S_n \lambda} t_1^{\mu_1} \cdots t_n^{\mu_n}, \qquad \lambda \in \Lambda^+,$$

with the symmetric group S_n acting on $\mathbb{Z}^n$ by permuting the entries, form a linear basis of $\mathbb{C}[t^{\pm 1}]^{S_n}$. The set Λ^+ is partially ordered by the *dominance order*:

$$\mu \leq \lambda \text{ if } \mu_1 + \cdots + \mu_i \leq \lambda_1 + \cdots + \lambda_i \text{ for } i = 1, \ldots, n-1 \text{ and } \textstyle\sum_{j=1}^n \mu_j = \sum_{j=1}^n \lambda_j.$$

The space $\mathbb{C}[t^{\pm 1}]^{S_n}$ is a dense subspace of the Hilbert space $L^2(T_u, v_+(t)d_u t)^{S_n}$ of S_n-invariant L^2-functions on the compact torus $T_u := \{t \in \mathbb{C}^n \mid |t_i| = 1\}$ with $d_u t$ the normalized Haar measure of T_u and with the two-parameter family of weight functions $v_+(t) = v_+(t; \kappa, q)$ ($0 < q, \kappa < 1$) given by

$$v_+(t; \kappa, q) = \prod_{1 \leq i \neq j \leq n} \frac{(t_i/t_j; q)_\infty}{(\kappa^2 t_i/t_j; q)_\infty}$$

(see (9.3.9) for the definition of the q-shifted factorial). Denote $\langle \cdot, \cdot \rangle$ for the associated inner product.

Definition 9.1.1 With Λ^+, $\leq$, m_λ defined as above, the monic *symmetric* GL_n *Macdonald polynomial* $P_\lambda^+(t) = P_\lambda^+(t;\kappa,q)$ of degree $\lambda \in \Lambda^+$ is the unique S_n-invariant Laurent polynomial in the variables $t_1,\ldots,t_n$ satisfying

1. $P_\lambda^+(t) = m_\lambda(t) + \sum_{\mu\in\Lambda^+:\mu<\lambda} d_{\lambda,\mu} m_\mu(t)$ for certain $d_{\lambda,\mu}\in\mathbb{C}$,
2. $\langle P_\lambda^+, m_\mu\rangle = 0$ if $\mu\in\Lambda^+$ and $\mu<\lambda$.

The polynomials P_λ^+ are orthogonal with respect to $\langle\cdot,\cdot\rangle$,

$$\langle P_\lambda^+, P_\mu^+\rangle = 0 \quad \text{ if } \lambda\neq\mu.$$

This is clear from the definition only if λ and μ are compatible with respect to the dominance order. The proof in general uses the commuting trigonometric Ruijsenaars–Macdonald q-difference operators [89], [70, Ch. VI, §3]

$$(D_j f)(t) := \sum_{\substack{I\subseteq\{1,\ldots,n\}\\ \#I=j}} \left(\prod_{r\in I, s\notin I} \frac{\kappa^{-1}t_r - \kappa t_s}{t_r - t_s}\right) f(q^{-\sum_{r\in I}\epsilon_r} t) \qquad 1\leq j\leq n,$$

where $q^{-\sum_{r\in I}\epsilon_r}t$ is the n-vector with entry $q^{-1}t_i$ at $i\in I$ and entry t_j at $j\notin I$. They act on $\mathbb{C}[t^{\pm1}]^{S_n}$ as triangular linear operators with respect to the partially ordered linear basis $\{m_\lambda(t)\}_{\lambda\in\Lambda^+}$ of symmetric monomials, with the order inherited from the dominance order $\leq$ on the index set Λ^+. The operators D_j are self-adjoint with respect to the inner product $\langle\cdot,\cdot\rangle$. These properties imply that the P_λ^+ is a common eigenfunction of the operators D_j (see §9.3.7 for more details). For the full orthogonality of the polynomials $P_\lambda^+(t)$ it actually suffices to consider only the q-difference operator D_1, since the spectrum of D_1 is simple for generic q. For the symmetric Jacobi polynomials associated with root systems, which are the classical ($q=1$) analogues of the symmetric Macdonald–Koornwinder polynomials, all the commuting differential operators are needed (see §8.4).

Note that the symmetric GL_n Macdonald polynomials are homogeneous Laurent polynomials. Hence $t^{-\lambda}P_\lambda^+(t)$, with $t^\lambda = t_1^{\lambda_1}\cdots t_n^{\lambda_n}$, depends only on $t_1/t_2,\ldots,t_{n-1}/t_n$. For $n=2$ the symmetric GL_2 Macdonald polynomial $t^{-\lambda}P_\lambda^+(t)$ is the continuous q-ultraspherical polynomials of degree $\lambda_1-\lambda_2$ in t_2/t_1. The above results then reduce to the orthogonality relations and the second order q-difference equation satisfied by the continuous q-ultraspherical polynomials, see §9.3.7 for further details.

9.1.2 Symmetric Koornwinder polynomials

The symmetric Koornwinder polynomials [61] form a six parameter family of linear bases of the space $\mathbb{C}[t^{\pm1}]^{W_0}$ of W_0-invariant Laurent polynomials in n variables $t=(t_1,\ldots,t_n)$, with W_0 the hyperoctahedral group $S_n \ltimes \{\pm1\}^n$ acting by permutations and inversions of the variables.

In this case $\mathbb{C}[t^{\pm1}]^{W_0}$ has a linear basis consisting of the W_0-symmetric monomials

$$m_\lambda(t) := \sum_{\mu\in W_0\lambda} t_1^{\mu_1}t_2^{\mu_2}\cdots t_n^{\mu_n}, \qquad \lambda\in\Lambda^+,$$

with $\Lambda^+ := \{\lambda\in\mathbb{Z}^n \mid \lambda_1\geq\cdots\geq\lambda_n\geq 0\}$ the partitions of length $\leq n$. Here the hyperoctahedral group W_0 acts on $\mathbb{Z}^n$ by permutations and sign changes. We consider Λ^+ as a partially ordered set with respect to the *dominance order*, which in this case is given by

$$\mu \leq \lambda \text{ if } \mu_1 + \cdots + \mu_i \leq \lambda_1 + \cdots + \lambda_i \text{ for } i = 1, \ldots, n.$$

The space $\mathbb{C}[t^{\pm 1}]^{W_0}$ is a dense subspace of the Hilbert space $L^2(T_u, v_+(t)d_ut)^{W_0}$ of W_0-invariant L^2-functions on the compact torus $T_u := \{t \in \mathbb{C}^n \mid |t_i| = 1\}$, with the six-parameter family of weight functions $v_+(t) = v_+(t; a, b, c, d, k, q)$ $(0 < a, b, c, d, q, k < 1)$ given by

$$v_+(t) := \prod_{i=1}^{n} \frac{(t_i^{\pm 2}; q)_\infty}{(at_i^{\pm 1}; q)_\infty (bt_i^{\pm 1}; q)_\infty (ct_i^{\pm 1}; q)_\infty (dt_i^{\pm 1}; q)_\infty} \prod_{1 \leq r < s \leq n} \frac{(t_r t_s^{\pm 1}; q)_\infty (t_r^{-1} t_s^{\pm 1}; q)_\infty}{(kt_r t_s^{\pm 1}; q)_\infty (kt_r^{-1} t_s^{\pm 1}; q)_\infty},$$

where $(uz^{\pm 1}; q)_\infty := (uz; q)_\infty (uz^{-1}; q)_\infty$. Denote $\langle \cdot, \cdot \rangle$ for the associated inner product.

Definition 9.1.2 With Λ^+, $\leq$, m_λ, W_0 defined as above, the monic *symmetric Koornwinder polynomial* $P_\lambda^+(t) = P_\lambda^+(t; a, b, c, d, k, q)$ of degree $\lambda \in \Lambda^+$ is the unique W_0-invariant Laurent polynomial in the variables $t_1, \ldots, t_n$ satisfying

1. $P_\lambda^+(t) = m_\lambda(t) + \sum_{\mu \in \Lambda^+ : \mu < \lambda} d_{\lambda,\mu} m_\mu(t)$ for certain $d_{\lambda,\mu} \in \mathbb{C}$,

2. $\langle P_\lambda^+, m_\mu \rangle = 0$ if $\mu \in \Lambda^+$ and $\mu < \lambda$.

Full orthogonality is again a nontrivial fact. In this case one can establish it by showing that the symmetric Koornwinder polynomials are the eigenfunctions of Koornwinder's [61] multivariable extension of the Askey–Wilson [1] second order q-difference operator, given by

$$(Df)(t) := \sum_{i=1}^{n} \sum_{\xi \in \{\pm 1\}} A_i^\xi(t)(f(q^{\xi \epsilon_i} t) - f(t)), \quad \text{where}$$

$$A_i^\xi(t) := \frac{(1 - at_i^\xi)(1 - bt_i^\xi)(1 - ct_i^\xi)(1 - dt_i^\xi)}{(1 - t_i^{2\xi})(1 - qt_i^{2\xi})} \prod_{j \neq i} \frac{(1 - kt_i^\xi t_j)(1 - kt_i^\xi t_j^{-1})}{(1 - t_i^\xi t_j)(1 - t_i^\xi t_j^{-1})}.$$

In fact, D is part of a commutative family $D = D_1, \ldots, D_n$ of algebraically independent linear q-difference operators acting on $\mathbb{C}[t^{\pm 1}]^{W_0}$, see [26] and §9.3.8.

For $n = 1$ the k-dependence drops out and the symmetric Koornwinder polynomials $P_\lambda^+(t)$ reduce to the monic Askey–Wilson [1] polynomials. See §9.3.8 for further details.

9.1.3 Detailed description of the contents

Precise references to the literature are given in the main text.

In §9.2 we give the definition of the affine braid group, affine Weyl group and affine Hecke algebra. We determine an explicit realization of the affine Hecke algebra, which will serve as starting point of the Cherednik–Macdonald theory on Macdonald–Koornwinder polynomials in the next section. In addition we introduce the initial data. We introduce the space of multiplicity functions associated to the fixed initial data D. We extend the duality on initial data to an isomorphism of the associated spaces of multiplicity functions. We give the basic representation of the extended affine Hecke algebra associated to D, using the explicit realization of the affine Hecke algebra.

In §9.3 we define and study the nonsymmetric and symmetric monic Macdonald–Koornwinder polynomials associated to the initial data D. We first focus on the nonsymmetric polynomials. We characterize them as common eigenfunctions of a family of commuting q-difference reflection operators. These operators are obtained as the images under the basic representation of the elements of the Bernstein–Zelevinsky abelian subalgebra of the extended affine Hecke algebra. We determine the biorthogonality relations of the nonsymmetric monic polynomials and use the finite Hecke symmetrizer to obtain the symmetric monic polynomials. We give the orthogonality relations of the symmetric monic polynomials and show that they are common eigenfunctions of the commuting Macdonald q-difference operators (also known as Ruijsenaars operators in the GL case). We finish this section by describing three cases in detail: the GL case, the $\mathrm{C}^\vee\mathrm{C}$ case, and a new nonreduced rank two case not covered in Cherednik's [18] and Macdonald's [73] treatments.

In §9.4 we introduce the double affine braid group and the double affine Hecke algebra associated to D and (D,κ) respectively, where κ is a choice of a multiplicity function on R. We lift the duality on initial data to a duality anti-isomorphism on the level of the associated double affine braid groups. We show how it descends to the level of double affine Hecke algebras and how it leads to an explicit evaluation formula for the monic Macdonald–Koornwinder polynomials. We proceed by defining the associated normalized nonsymmetric and symmetric Macdonald–Koornwinder polynomials and deriving their duality and quadratic norms.

In §9.5 we give the norm and evaluation formulas in terms of q-shifted factorials for the GL case, the $\mathrm{C}^\vee\mathrm{C}$ case, and for the new nonreduced rank two case.

In the Appendix we give a short introduction to (the classification of) affine root systems, following closely [66] but with some adjustments. We close the Appendix with a list of all the affine Dynkin diagrams.

Remark 9.1.3 Four years after the appearance of this chapter as preprint on the arXiv, Ion and Sahi [48] developed an approach to the theory of double affine braid groups and double affine Hecke algebras based on double affine Coxeter type data. It leads to a uniform treatment of this theory which is closely related to the treatment in §9.4. The finite root system we use as part of the initial data plays the role of the dual finite root system in [48]. What is called untwisted (respectively twisted) in the present chapter, is called twisted (respectively untwisted) in [48].

9.1.4 Further topics

We list here various important developments involving Macdonald–Koornwinder polynomials that are not discussed in this chapter. For a more extensive discussion of the ramifications of these polynomials we refer to the introductory chapter of Cherednik's [18] book.

1. Shift operators for the Macdonald–Koornwinder polynomials (see, e.g., [12, 94]). It leads to explicit evaluation formulas for the constant terms, which are q-analogs of Selberg integrals (Chapter 5).
2. Connections to combinatorics (Chapter 10).

3. Connections to algebraic geometry, see, e.g., [39, 40, 92].
4. Connections to representation theory, see, e.g., [18, 17, 19, 50, 46, 47].
5. Applications to harmonic analysis on quantum groups, see, e.g., [75, 77, 78, 64, 30, 32, 33, 79].
6. Connections to quantum integrable systems, such as Ruijsenaars' [89] quantum many body systems and integrable one-dimensional spin chains, see, e.g., [10, 9, 55, 93, 7, 8, 84, 52, 53, 25, 97, 100].
7. Integrable probabilistic systems (Macdonald processes), see, e.g., [3, 4].
8. Applications to torus knot homology (DAHA-Jones polynomials), see, e.g., [21, 22, 23, 29] and (Chapter 10).
9. Limit cases of symmetric Macdonald–Koornwinder polynomials. Limits $q \to 1$ to classical multivariable orthogonal polynomials such as the multivariable Jacobi polynomials (Chapter 8); p-adic limits $q \to 0$ to Hall–Littlewood type polynomials, see, e.g., [70, 72, 42]; Whittaker limits, see, e.g., [20]; and multivariable analogues of limit transitions within the q-Askey scheme, see, e.g., [99, 95, 2].
10. Macdonald–Mehta type integrals and basic hypergeometric functions associated with root systems, i.e., the q-analogues of the hypergeometric functions associated with root systems discussed in Chapter 8, see, e.g., [15, 96, 20, 98].
11. Interpolation Macdonald–Koornwinder polynomials, see, e.g., [56, 57, 90, 81, 82, 63]. Their elliptic versions are discussed in Chapter 6.
12. Special parameter values (e.g. roots of unity), see, e.g., [18, 14, 28, 54, 50, 51, 34, 7, 8].
13. Affine and elliptic generalizations, see, e.g., [31, 11, 89] and [59, 85, 86, 24] respectively. See Chapter 6 for a discussion of the elliptic generalization of the symmetric Koornwinder polynomial.

9.2 The basic representation of the extended affine Hecke algebra

We first introduce the affine Hecke algebra and the appropriate initial data for the Cherednik–Macdonald theory on Macdonald–Koornwinder polynomials. Then we introduce the basic representation of the extended affine Hecke algebra, which is fundamental in the development of the theory.

9.2.1 Affine Hecke algebras

A convenient reference for this subsection is [44]. For unexplained notations and terminology regarding affine Weyl groups we refer to the Appendix.

For a generalized Cartan matrix $A = (a_{ij})_{1\le i,j\le r}$ let $M = (m_{ij})_{1\le i,j\le r}$ be the matrix with entries $m_{ii} = 1$ and, for $i \neq j$, $m_{ij} = 2, 3, 4, 6, \infty$ according to whether $a_{ij}a_{ji} = 0, 1, 2, 3, \geq 4$, respectively.

Definition 9.2.1 Let $A = (a_{ij})_{1\le i,j\le r}$ be a generalized Cartan matrix.

1. The *braid group* $\mathcal{B}(A)$ is the group generated by T_i ($1 \leq i \leq r$) with defining relations $T_iT_jT_i\cdots = T_jT_iT_j\cdots$ (m_{ij} factors on each side) if $1 \leq i \neq j \leq r$ (which should be interpreted as no relation if $m_{ij} = \infty$).
2. The *Coxeter group* $\mathcal{W}(A)$ associated to A is the quotient of $\mathcal{B}(A)$ by the normal subgroup generated by T_i^2 ($1 \leq i \leq r$).

It is convenient to denote by s_i the element in $\mathcal{W}(A)$ corresponding to T_i for $1 \leq i \leq r$. They are the *Coxeter generators* of the Coxeter group $\mathcal{W}(A)$.

Let $A = (a_{ij})_{1\leq i,j\leq r}$ be a generalized Cartan matrix. Suppose that k_i ($1 \leq i \leq r$) are nonzero complex numbers such that $k_i = k_j$ if s_i is conjugate to s_j in $\mathcal{W}(A)$. We write k for the collection $\{k_i\}_i$. Let $\mathbb{C}[\mathcal{B}(A)]$ be the complex group algebra of the braid group $\mathcal{B}(A)$.

Definition 9.2.2 The *Hecke algebra* $H(A, k)$ is the complex unital associative algebra given by $\mathbb{C}[\mathcal{B}(A)]/I_k$, where I_k is the two-sided ideal of $\mathbb{C}[\mathcal{B}(A)]$ generated by $(T_i - k_i)(T_i + k_i^{-1})$ for $1 \leq i \leq r$.

If $k_i = 1$ for all $1 \leq i \leq r$ then the associated affine Hecke algebra is the complex group algebra of $\mathcal{W}(A)$.

If $w = s_{i_1}s_{i_2}\cdots s_{i_r}$ is a reduced expression in $\mathcal{W}(A)$, i.e. a shortest expression of w as product of Coxeter generators, then $T_w := T_{i_1}T_{i_2}\cdots T_{i_r} \in H(A, k)$ is well defined (already in the braid group $\mathcal{B}(A)$), and the T_w ($w \in \mathcal{W}(A)$) form a complex linear basis of $H(A, k)$.

Suppose $R_0 \subset V$ is a finite crystallographic root system with ordered basis $\Delta_0 = (\alpha_1, \ldots, \alpha_n)$ and write A_0 for the associated Cartan matrix. Then $W_0 \simeq \mathcal{W}(A_0)$ by mapping the simple reflections $s_{\alpha_i} \in W_0$ to the Coxeter generators s_i of $\mathcal{W}(A_0)$ for $1 \leq i \leq n$. The associated finite Hecke algebra $H(A_0, k)$ depends only on k and W_0 (as Coxeter group). We will sometimes denote it by $H(W_0, k)$.

Similarly, if R is an irreducible affine root system with ordered basis $\Delta = (a_0, \ldots, a_n)$ and if A is the associated affine Cartan matrix, then $\mathcal{W}(A) \simeq W(R)$ by $s_i \mapsto s_{a_i}$ ($0 \leq i \leq n$). Again we write $H(W(R), k)$ for the associated affine Hecke algebra $H(A, k)$.

9.2.2 Realizations of the affine Hecke algebra

We use the notations on affine root systems as introduced in the Appendix. The following construction is motivated by Cherednik's polynomial representation [18, Thm. 3.2.1] of the affine Hecke algebra and its extension to the nonreduced case by Noumi [74].

Let $R \subset \widehat{E}$ be an irreducible (possibly nonreduced) affine root system on the affine Euclidean space E of dimension n, with affine Weyl group W. Let $\Delta = (a_0, a_1, \ldots, a_n)$ be an ordered basis of R. Write $A = A(R, \Delta)$ for the associated affine Cartan matrix.

Consider the lattice $\mathbb{Z}R$ in $\widehat{E}$. It is a full W-stable lattice with the simple affine roots as $\mathbb{Z}$-basis. Denote by F the quotient field $\mathrm{Quot}(\mathbb{C}[\mathbb{Z}R])$ of the complex group algebra $\mathbb{C}[\mathbb{Z}R]$ of $\mathbb{Z}R$. It is convenient to write e^λ ($\lambda \in \mathbb{Z}R$) for the natural complex linear basis of $\mathbb{C}[\mathbb{Z}R]$. The multiplicative structure of F is determined by $e^0 = 1$, $e^{\lambda+\mu} = e^\lambda e^\mu$ ($\lambda, \mu \in \mathbb{Z}R$).

The affine Weyl group W canonically acts by field automorphisms on F. On the basis elements e^λ the W-action reads $w(e^\lambda) = e^{w\lambda}$ ($w \in W$, $\lambda \in \mathbb{Z}R$). Since W acts by algebra automorphisms on F, we can form the semidirect product algebra $W \ltimes F$.

Let $k\colon a \mapsto k_a\colon R \to \mathbb{C}^* := \mathbb{C} \setminus \{0\}$, be a W-equivariant map, i.e., $k_{wa} = k_a$ for all $w \in W$ and $a \in R$. If $a \in R$ but $2a \notin R$ then we set $k_{2a} := k_a$. Note that $k^{\mathrm{ind}} := k|_{R^{\mathrm{ind}}}$ is a W-equivariant map on R^{ind}, which is determined by its values $k_i := k_{a_i}$ ($0 \le i \le n$) on the simple affine roots. It satisfies $k_i = k_j$ if s_i is conjugate to s_j in W. Hence we can form the associated affine Hecke algebra $H(W(R), k^{\mathrm{ind}})$.

Theorem 9.2.3 *With the above notations and conventions, there exists a unique algebra monomorphism* $\beta = \beta_{R,\Delta,k}\colon H(W(R), k^{\mathrm{ind}}) \hookrightarrow W \ltimes F$ *satisfying*

$$\beta(T_i) = k_i s_i + \frac{k_i - k_i^{-1} + (k_{2a_i} - k_{2a_i}^{-1})e^{a_i}}{1 - e^{2a_i}}(1 - s_i), \qquad 0 \le i \le n. \tag{9.2.1}$$

The proof uses the Bernstein–Zelevinsky presentation of the affine Hecke algebra, which we present in a slightly more general context in §9.3.1.

Remark 9.2.4 If $2a_i \notin R$ then (9.2.1) simplifies to $\beta(T_i) = k_i s_i + \dfrac{k_i - k_i^{-1}}{1 - e^{a_i}}(1 - s_i)$.

The notion of similarity of pairs (R, Δ) (see the Appendix) can be extended to triples (R, Δ, k) in the obvious way. The algebra homomorphisms β that are associated to different representatives of the similarity class of (R, Δ, k) are then equivalent in a natural sense. Starting from the next subsection we therefore will focus on the explicit representatives of the similarity classes as described in §A.2.

Recall from the Appendix that the classification of irreducible affine root systems leads to a subdivision of irreducible affine root systems in three types, namely untwisted, twisted and mixed type. It is easy to show that the algebra map $\beta_{R,\Delta,k}$ in case R is of mixed type (possibly nonreduced) can alternatively be written as $\beta_{(R',\Delta',k')}$ with appropriately chosen triple (R', Δ', k') and with R' of untwisted or of twisted type. In the Cherednik–Macdonald theory, the mixed type can therefore safely be ignored.

9.2.3 Initial data

It is tempting to believe that the initial data for the Macdonald–Koornwinder polynomials should be similarity classes of irreducible affine root systems R together with a choice of a deformation parameter q and a multiplicity function (playing the role of the free parameters in the theory). In such a parametrization the untwisted and twisted cases should relate to the similarity classes of the irreducible reduced affine root systems of untwisted and twisted type, respectively: the GL case to the irreducible reduced affine root system of type A with a "reductive" extension of the affine Weyl group, and the Koornwinder case with the nonreduced irreducible affine root system of type $\mathrm{C}^\vee\mathrm{C}$ (we refer here to the classification of affine root systems from [66], see also the Appendix). It turns out, though, that a more subtle labelling is needed in order to come to a uniform theory capturing all cases and capturing all fundamental properties of the Macdonald–Koornwinder polynomials.

We will take as initial data quintuples $D = (R_0, \Delta_0, \bullet, \Lambda, \Lambda^d)$ with R_0 a finite reduced irreducible root system, Δ_0 an ordered basis of R_0, $\bullet \in \{u, t\}$ (u stands for *untwisted* and t stands for *twisted*), and Λ, Λ^d two lattices satisfying appropriate compatibility conditions with respect to the (co)root lattice of R_0 (see (9.2.2)). We build from D an irreducible affine root system R and an extended affine Weyl group W. The extended affine Weyl group W is simply the semidirect product group $W_0 \ltimes \Lambda^d$ with W_0 the Weyl group of R_0. The affine root system R will be constructed as follows. We associate to R_0 and $\bullet$ the reduced irreducible affine root system $R^\bullet$ of type $\bullet$ with gradient root system R_0. Then R is an irreducible affine root system obtained from $R^\bullet$ by adding $2a$ if $a \in R^\bullet$ has the property that the pairings of the associated coroot $a^\vee$ to elements of Λ take value in $2\mathbb{Z}$ (see (9.2.5)). The Macdonald–Koornwinder polynomials associated to D are the ones naturally related to R in the labelling proposed in the previous paragraph.

Duality will be related to a simple involution on initial data, $D \mapsto D^d = (R_0^d, \Delta_0^d, \bullet, \Lambda^d, \Lambda)$ with R_0^d the coroot system $R_0^\vee$ if $\bullet = u$ and the root system R_0 if $\bullet = t$. This duality is subtle on the level of affine root systems (it can for instance happen that the affine root system R^d associated to D^d is reduced while R is nonreduced). The reason that the present choice of initial data is convenient is the fact that the duality map $D \mapsto D^d$ on initial data naturally lifts to the duality anti-isomorphism of the associated double affine braid group (see [41] and §9.4). This duality anti-isomorphism turns out to be the key tool to prove duality, evaluation formulas and norm formulas for the Macdonald–Koornwinder polynomials.

In this subsection we carefully introduce the initial data and we explain how it relates to affine root systems. As always, we refer for basic notations and facts on affine root systems to the Appendix.

Definition 9.2.5 The set $\mathcal{D}$ of *initial data* consists of quintuples $D = (R_0, \Delta_0, \bullet, \Lambda, \Lambda^d)$ with

1. R_0 is a finite set of nonzero vectors in an Euclidean space Z forming a finite, irreducible, reduced crystallographic root system within the real span V of R_0;
2. $\Delta_0 = (\alpha_1, \ldots, \alpha_n)$ is an ordered basis of R_0;
3. $\bullet = u$ or $\bullet = t$;
4. Λ and Λ^d are full lattices in Z, satisfying

$$\mathbb{Z}R_0 \subseteq \Lambda, \quad (\Lambda, \mathbb{Z}R_0^\vee) \subseteq \mathbb{Z}, \qquad \mathbb{Z}R_0^d \subseteq \Lambda^d, \quad (\Lambda^d, \mathbb{Z}R_0^{d\vee}) \subseteq \mathbb{Z}, \tag{9.2.2}$$

where $R_0^d = \{\alpha^d := \mu_\alpha^\bullet \alpha^\vee\}_{\alpha \in R_0}$ with $\quad \mu_\alpha^u := 1 \quad (\alpha \in R_0) \quad$ and $\quad \mu_\alpha^t := |\alpha|^2/2 \quad (\alpha \in R_0)$.

Note that $R_0^d = R_0^\vee$ if $\bullet = u$ and $= R_0$ if $\bullet = t$.

We view the vector space $\widehat{V}$ of real-valued affine linear functions on V as the subspace of $\widehat{Z}$ consisting of affine linear functions on Z which are constant on the orthocomplement $V^\perp$ of V in Z. We write c for the constant function one on V as well as on Z. In a similar fashion we view the orthogonal group $\mathrm{O}(V)$ as subgroup of $\mathrm{O}(Z)$ and $\mathrm{O}_c(\widehat{V})$ as subgroup of $\mathrm{O}_c(\widehat{Z})$, where $\mathrm{O}_c(\widehat{V})$ is the subgroup of linear automorphisms of $\widehat{V}$ preserving c and preserving the natural semi-positive definite form on $\widehat{V}$ (see the Appendix for further details).

Fix $D = (R_0, \Delta_0, \bullet, \Lambda, \Lambda^d) \in \mathcal{D}$. We associate to D a triple (R, Δ, W) of an affine root system $R = R(D)$, an ordered basis $\Delta = \Delta(D)$ of R and an extended affine Weyl group $W = W(D)$ as follows. We first define a reduced irreducible affine root system $R^\bullet \subset \widehat{V}$ to D as

$$R^\bullet := \{m\mu_\alpha^\bullet c + \alpha\}_{m\in\mathbb{Z},\,\alpha\in R_0} \tag{9.2.3}$$

for $\bullet \in \{u, t\}$. In other words, $R^u := \mathcal{S}(R_0)$ and $R^t := \mathcal{S}(R_0^\vee)^\vee$ in the notations of the Appendix (see §A.2). Let $\varphi \in R_0$ (respectively $\theta \in R_0$) be the highest root (respectively highest short root) of R_0 with respect to the ordered basis Δ_0 of R_0. Then the ordered basis $\Delta = \Delta(D)$ of $R^\bullet$ is set to be $\Delta := (a_0, a_1, \ldots, a_n)$ with $a_i := \alpha_i$ for $1 \leq i \leq n$ and

$$a_0 := \begin{cases} c - \varphi & \text{if} \quad \bullet = u, \\ \frac{1}{2}|\theta|^2 c - \theta & \text{if} \quad \bullet = t. \end{cases} \tag{9.2.4}$$

Remark 9.2.6 Suppose that (R', Δ') is a pair consisting of a reduced irreducible affine root system R' and an ordered basis Δ' of R'. If R' is similar to $R^\bullet$ then there exists a similarity transformation realizing $R' \simeq R^\bullet$ and mapping Δ' to $\Delta^\bullet$ as unordered sets.

The affine root system R is the following extension of $R^\bullet$. Define the subset $S = S(D)$ by

$$S := \{i \in \{0, \ldots, n\} \mid (\Lambda, a_i^\vee) = 2\mathbb{Z}\}. \tag{9.2.5}$$

Let $W^\bullet$ be the affine Weyl group of $R^\bullet$. Then we set

$$R = R(D) := R^\bullet \cup \Big(\bigcup_{i\in S} W^\bullet(2a_i)\Big), \tag{9.2.6}$$

which is an irreducible affine root system since $\mathbb{Z}R_0 \subseteq \Lambda$. Note that Δ is also an ordered basis of R.

Remark 9.2.7 Note that R is an irreducible affine root system of untwisted or twisted type, but never of mixed type (see the Appendix for the terminology). But the irreducible affine root systems of mixed type are affine root subsystems of the affine root system of type $\mathrm{C}^\vee\mathrm{C}$, which is the nonreduced extension of the affine root system R^t with R_0 of type B (see §9.3.8 for a detailed description of the affine root system of type $\mathrm{C}^\vee\mathrm{C}$). Accordingly, special cases of the Koornwinder polynomials are naturally attached to affine root systems of mixed type, see §9.2.2 and Remark 9.3.29.

Remark 9.2.8 The nonreduced extension of the affine root system R^u with R_0 of type B_2 is not an affine root subsystem of the affine root system of type $\mathrm{C}^\vee\mathrm{C}_2$. It can actually be better viewed as the rank two case of the family R^u with R_0 of type C_n since, in the corresponding affine Dynkin diagram (see §A.4), the vertex labelled by the affine simple root a_0 is double bonded with the finite Dynkin diagram of R_0. The nonreduced extension of R^u with R_0 of type C_2 was missing in Macdonald's [66] classification list. It was added in [73, (1.3.17)], but the associated theory of Macdonald–Koornwinder polynomials was not developed. In the present setup it is a special case of the general theory. We will describe this particular case in detail in §9.3.9.

Finally we define the extended affine Weyl group $W = W(D)$. Write $s_i := s_{a_i}$ $(0 \leq i \leq n)$ for the simple reflections of $W^\bullet$. Note that $s_i = s_{\alpha_i}$ $(1 \leq i \leq n)$ are the simple reflections of the finite Weyl group W_0 of R_0. Furthermore, $s_0 = \tau(\varphi^\vee)s_\varphi$ if $\bullet = u$ and $s_0 = \tau(\theta)s_\theta$ if $\bullet = t$, where $\tau(v)$ stands for the translation by v (see the Appendix). Consequently, $W^\bullet \simeq W_0 \ltimes \tau(\mathbb{Z}R_0^d)$. We omit τ from the notations if no confusion can arise. The *extended affine Weyl group* $W = W(D)$ is now defined as $W := W_0 \ltimes \Lambda^d$. It contains the affine Weyl group $W^\bullet$ of $R^\bullet$ as normal subgroup, and $W/W^\bullet \simeq \Lambda^d/\mathbb{Z}R_0^d$.

The affine root system $R^\bullet \subset \widehat{Z}$ is W-stable since

$$\tau(\xi)\,(m\,\mu^\bullet_\beta\,c + \beta) = \big(m - (\xi, \beta^{d\vee})\big)\,\mu^\bullet_\beta\,c + \beta, \qquad m \in \mathbb{Z},\, \beta \in R_0, \tag{9.2.7}$$

and $(\xi, \beta^{d\vee}) \in \mathbb{Z}$ for $\xi \in \Lambda^d$ and $\beta \in R_0$. Moreover, the affine root system R is W-invariant.

We now proceed by giving key examples of initial data. Recall that, for a finite root system $R_0 \subset V$,

$$P(R_0) := \{\lambda \in V \mid (\lambda, \alpha^\vee) \in \mathbb{Z} \quad \forall \alpha \in R_0\}$$

is the weight lattice of R_0. If $\Delta_0 = (\alpha_1, \ldots, \alpha_n)$ is an ordered basis of R_0 then we write $\varpi_i \in P(R_0)$ $(1 \leq i \leq n)$ for the corresponding fundamental weights, which are characterized by $(\varpi_i, \alpha_j^\vee) = \delta_{i,j}$.

Example 9.2.9

(i) Take an arbitrary finite reduced irreducible root system R_0 in $V = Z$ with ordered basis Δ_0. Choose $\bullet \in \{u, t\}$ and let Λ, Λ^d be lattices in V satisfying

$$\mathbb{Z}R_0 \subseteq \Lambda \subseteq P(R_0), \qquad \mathbb{Z}R_0^d \subseteq \Lambda^d \subseteq P(R_0^d).$$

Then $(R_0, \Delta_0, \bullet, \Lambda, \Lambda^d) \in \mathcal{D}$. Note that, if $\Lambda = P(R_0)$, then $S = \emptyset$ and $R = R^\bullet$ is reduced. (Cherednik's [18] theory corresponds to the special case $(\Lambda, \Lambda^d) = (P(R_0), P(R_0^d))$.)

(ii) Take $Z = \mathbb{R}^{n+1}$ with standard orthonormal basis $\{\epsilon_i\}_{i=1}^{n+1}$ and $R_0 = \{\epsilon_i - \epsilon_j\}_{1\leq i\neq j\leq n+1}$ for the realization of the finite root system of type A_n in Z. Then $V = (\epsilon_1 + \cdots + \epsilon_{n+1})^\perp$. As ordered basis take $\Delta_0 = (\alpha_1, \ldots, \alpha_n) = (\epsilon_1 - \epsilon_2, \ldots, \epsilon_n - \epsilon_{n+1})$. Then $(R_0, \Delta_0, u, \mathbb{Z}^{n+1}, \mathbb{Z}^{n+1}) \in \mathcal{D}$. Note that $\theta = \varphi = \epsilon_1 - \epsilon_{n+1}$, hence the simple affine root a_0 of R is $a_0 = c - \epsilon_1 + \epsilon_{n+1}$. This example is naturally related to the GL_{n+1} type Macdonald polynomials, see §9.3.7.

Given a quintuple $D = (R_0, \Delta_0, \bullet, \Lambda, \Lambda^d)$ we have the dual root system R_0^d with dual ordered basis $\Delta_0^d := (\alpha_1^d, \ldots, \alpha_n^d)$. This extends to an involution $D \mapsto D^d$ on $\mathcal{D}$ with

$$D^d := (R_0^d, \Delta_0^d, \bullet, \Lambda^d, \Lambda) \tag{9.2.8}$$

for $D = (R_0, \Delta, \bullet, \Lambda, \Lambda^d) \in \mathcal{D}$. We call D^d the *initial data dual to D*.

We write $\mu = \mu(D)$ and $\mu^d = \mu(D^d)$ for the function $\mu^\bullet$ on R_0 and R_0^d, respectively. Let $\varpi_i^d \in P(R_0^d)$ $(1 \leq i \leq n)$ be the fundamental weights with respect to Δ_0^d.

For a given $D = (R_0, \Delta_0, \bullet, \Lambda, \Lambda^d) \in \mathcal{D}$ we thus have a dual quintuple $(R_0^d, \Delta_0^d, \bullet, \Lambda^d, \Lambda) \in \mathcal{D}$, and hence an associated triple (R^d, Δ^d, W^d). Concretely, the highest root φ^d and the highest short root θ^d of R_0^d with respect to Δ_0^d are given by

$$\varphi^d = \begin{cases} \theta^\vee & \text{if } \bullet = u, \\ \varphi & \text{if } \bullet = t, \end{cases} \qquad \text{and} \qquad \theta^d = \begin{cases} \varphi^\vee & \text{if } \bullet = u, \\ \theta & \text{if } \bullet = t. \end{cases}$$

Hence $\Delta^d = (a_0^d, a_1^d, \ldots, a_n^d)$ with $a_i^d = \alpha_i^d$ $(1 \le i \le n)$ and $a_0^d = \mu_\theta(c - \theta^\vee)$. The dual affine root system $R^d = R(D^d)$ is $R^d = R^{d\bullet} \cup \bigcup_{i \in S^d} W^{d\bullet}(2a_i^d)$ with $S^d = \{i \in \{0, \ldots, n\} \mid (\Lambda^d, a_i^{d\vee}) = 2\mathbb{Z}\}$, with $R^{d\bullet} = \{m\mu_{\alpha^d} + \alpha^d\}_{m \in \mathbb{Z}, \alpha \in R_0}$ and with $W^{d\bullet} \simeq W_0 \ltimes \tau(\mathbb{Z}R_0)$ the affine Weyl group of $R^{d\bullet}$. The dual extended affine Weyl group is $W^d = W_0 \ltimes \Lambda$. The simple reflections $s_i^d := s_{a_i^d} \in W^{d\bullet}$ $(0 \le i \le n)$ are $s_i^d = s_i$ for $1 \le i \le n$ and $s_0^d = \tau(\theta)s_\theta$.

Example 9.2.10 The correpondence $R \leftrightarrow R^d$ can turn nonreduced affine root systems into reduced ones. We give here an example of untwisted type. An example for twisted type will be given in Example 9.3.28.

Take $n \ge 3$ and $R_0 \subseteq V = Z := \mathbb{R}^n$ of type B_n, realized as $R_0 = \{\pm\epsilon_i\} \cup \{\pm\epsilon_i \pm \epsilon_j\}_{i<j}$ (all sign combinations possible), with $\{\epsilon_i\}$ the standard orthonormal basis of V. Take as ordered basis of R_0: $\Delta_0 = (\alpha_1, \ldots, \alpha_{n-1}, \alpha_n) = (\epsilon_1 - \epsilon_2, \ldots, \epsilon_{n-1} - \epsilon_n, \epsilon_n)$. The highest root is $\varphi = \epsilon_1 + \epsilon_2 \in R_0$. We then have $\mathbb{Z}R_0^\vee \subset P(R_0^\vee) = \mathbb{Z}^n = \mathbb{Z}R_0 \subset P(R_0)$ with both sublattices $\mathbb{Z}R_0^\vee \subset \mathbb{Z}^n$ and $\mathbb{Z}^n \subset P(R_0)$ of index two. For $\Lambda = \mathbb{Z}^n = \Lambda^d$ we get the initial data $D = (R_0, \Delta_0, u, \mathbb{Z}^n, \mathbb{Z}^n) \in \mathcal{D}$. Then $S = S(D) = \{n\}$ and $R = R(D)$ is given by

$$R = \{\pm\epsilon_i + mc\}_{1 \le i \le n, m \in \mathbb{Z}} \cup \{\pm\epsilon_i \pm \epsilon_j + mc\}_{1 \le i < j \le n, m \in \mathbb{Z}} \cup \{\pm 2\epsilon_i + 2mc\}_{1 \le i \le n, m \in \mathbb{Z}}.$$

It is nonreduced and of untwisted type B_n. We have written it here as the disjoint union of the three $W = W_0 \ltimes \mathbb{Z}^n$-orbits of R. Note that a_0 lies in the orbit $\{\pm\epsilon_i \pm \epsilon_j + mc\}_{1 \le i < j \le n, m \in \mathbb{Z}}$.

Dually, $R^d = R(D^d)$ is the reduced affine root system of untwisted type C_n. Concretely,

$$R^d = \{\pm\epsilon_i \pm \epsilon_j + mc\}_{1 \le i < j \le n, m \in \mathbb{Z}} \cup \{\pm 2\epsilon_i + 2mc\}_{1 \le i \le n, m \in \mathbb{Z}} \cup \{\pm 2\epsilon_i + (2m+1)c\}_{1 \le i \le n, m \in \mathbb{Z}},$$

written here as the disjoint union of the three $W^d = W_0 \ltimes \mathbb{Z}^n$-orbits of R^d.

This example shows that basic features of the affine root system can alter under dualization. It turns out though that the number of orbits with respect to the action of the extended affine Weyl group is unaltered. To establish this fact it is convenient to use the concept of a multiplicity function on R.

Definition 9.2.11 Set $\mathcal{M} = \mathcal{M}(D)$ for the complex algebraic group of W-invariant functions $\kappa \colon R \to \mathbb{C}^*$, and $\nu = \nu(D)$ for the complex dimension of $\mathcal{M}$.

Note that $\nu = \nu(D)$ equals the number of W-orbits of R. The value of $\kappa \in \mathcal{M}$ at an affine root $a \in R$ is denoted by κ_a. We call $\kappa \in \mathcal{M}$ a *multiplicity function*. We write $\kappa^\bullet := \kappa|_{R^\bullet}$ for its restriction to $R^\bullet$. For a multiplicity function $\kappa \in \mathcal{M}$ we set $\kappa_{2a} := \kappa_a$ if $a \in R$ is unmultipliable (i.e., $2a \notin R$).

First we need a more precise description of the sets $S = S(D)$ and $S^d = S(D^d)$. It is obtained using the classification of affine root systems (see §A.2 and [66]).

Lemma 9.2.12 *Let $D = (R_0, \Delta_0, \bullet, \Lambda, \Lambda^d) \in \mathcal{D}$. Set $S_0 = S \cap \{1, \ldots, n\}$.*

a *If* $\bullet = u$ *then* $\#S \leq 2$. *If* $\#S = 2$ *then* R_0 *is of type* A_1. *If* $\#S = 1$ *then* R_0 *is of type* B_n *(*$n \geq 2$*) and* $S = S_0 = \{j\}$ *with* $\alpha_j \in \Delta_0$ *the unique simple short root.*

b *If* $\bullet = t$ *then* $\#S = 0$ *or* $\#S = 2$. *If* $\#S = 2$ *then* R_0 *is either of type* A_1 *or of type* B_n *(*$n \geq 2$*) and* $S = \{0, j\}$ *with* $\alpha_j \in \Delta_0$ *the unique short simple root.*

Note that in both the untwisted and the twisted case, $\alpha_j^d \in W_0(D(a_0^d))$ *if* $S_0 = \{j\}$.

The following lemma should be compared with [41, §5.7].

Lemma 9.2.13 *Let* $D \in \mathcal{D}$ *and* $\kappa \in \mathcal{M}(D)$. *Let* $\alpha_j \in \Delta_0$ *(respectively* $\alpha_{j^d}^d \in \Delta_0^d$*) be a simple short root. The assignments*

$$\kappa_{a_0^d}^d := \kappa_{2\alpha_j}, \quad \kappa_{\alpha_i^d}^d := \kappa_{\alpha_i} \ \ (i \in \{1, \ldots, n\}), \quad \kappa_{2a_0^d}^d := \kappa_{2a_0} \ \ (0 \in S^d), \quad \kappa_{2\alpha_{j^d}^d}^d := \kappa_{a_0} \ \ (j^d \in S^d)$$

uniquely extend to a multiplicity function $\kappa^d \in \mathcal{M}^d := \mathcal{M}(D^d)$. *For fixed* $D \in \mathcal{D}$ *the map* $\kappa \mapsto \kappa^d$ *defines an isomorphism* $\phi_D \colon \mathcal{M}(D) \xrightarrow{\sim} \mathcal{M}(D^d)$ *of complex tori, with inverse* ϕ_{D^d}.

Remark 9.2.14 Let $\kappa \in \mathcal{M}(D)$. Recall the convention that $\kappa_{2a} = \kappa_a$ for $a \in R$ such that $2a \notin R$. Then, for all $\alpha \in R_0$, $\kappa_{2\alpha} = \kappa^d_{\mu_{\alpha^d} c + \alpha^d}$.

Corollary 9.2.15 *Let* $D \in \mathcal{D}$. *The number* ν *of* W*-orbits of* R *is equal to the number* ν^d *of* W^d*-orbits of* R^d.

Remark 9.2.16 Returning to Example 9.2.10, note that the correspondence from Lemma 9.2.13 links the orbit $\{\pm\epsilon_i + mc\}$ of R to the orbit $\{2\epsilon_i + 2mc\}$ of R^d, the orbit $\{\pm\epsilon_i \pm \epsilon_j + mc\}$ of R to the orbit $\{\pm\epsilon_i \pm \epsilon_j + mc\}$ of R^d and the orbit $\{\pm 2\epsilon_i + 2mc\}$ of R to the orbit $\{\pm 2\epsilon_i + (2m+1)c\}$ of R^d.

9.2.4 The basic representation

We fix throughout this subsection a quintuple $D = (R_0, \Delta_0, \bullet, \Lambda, \Lambda^d) \in \mathcal{D}$ of initial data. Recall that it gives rise to a triple (R, Δ, W) of an irreducible affine root system R containing $R^\bullet$, an ordered basis Δ of R as well as of $R^\bullet$, and an extended affine Weyl group $W = W_0 \ltimes \Lambda^d$. In addition we fix a multiplicity function $\kappa \in \mathcal{M}(D)$ and we write $\kappa^\bullet := \kappa|_{R^\bullet}$. It is a W-equivariant map $R^\bullet \to \mathbb{C}^*$. Write $\kappa_i := \kappa^\bullet_{a_i}$ for $0 \leq i \leq n$. Note that $\kappa_i = \kappa_j$ if s_i is conjugate to s_j in $W = \Omega \ltimes W^\bullet$.

We write $R^\pm$ and $R^{\bullet\pm}$ for the positive respectively negative affine roots of R and $R^\bullet$ with respect to Δ. Since the affine root system $R^\bullet$ is W-stable, we can define the *length* function by

$$l(w) = l_D(w) := \#(R^{\bullet+} \cap w^{-1} R^{\bullet-}), \qquad w \in W.$$

If $w \in W^\bullet = W(R^\bullet)$ then $l(w)$ equals the number of simple reflections s_i $(0 \leq i \leq n)$ in a reduced expression of w. We have $W = \Omega \ltimes W^\bullet$ with $\Omega = \Omega(D) := \{w \in W \mid l(w) = 0\}$, a subgroup of W. Then $\Omega \simeq \Lambda^d / \mathbb{Z} R_0^d$. The abelian group Ω permutes the simple affine roots a_i $(0 \leq i \leq n)$, which thus gives rise to an action of Ω on the index set $\{0, \ldots, n\}$. Consequently the action of Ω on $W^\bullet$ by conjugation permutes the set $\{s_i\}_{i=0}^n$ of simple reflections, $w s_i w^{-1} =$

$s_{w(i)}$ for $w \in \Omega$ and $0 \le i \le n$ (cf., e.g., [73, §2.5]). A detailed description of the group Ω in terms of a complete set of representatives of $\Lambda^d/\mathbb{Z}R_0^d$ will be given in §9.3.4.

Extended versions of the affine braid group and of the affine Hecke algebra are defined as follows. Let $A = A(D) := A(R^\bullet, \Delta)$ be the affine Cartan matrix associated to $(R^\bullet, \Delta)$. Recall that the affine braid group $\mathcal{B}^\bullet := \mathcal{B}(A)$ is isomorphic to the abstract group generated by T_w ($w \in W^\bullet$) with defining relations $T_vT_w = T_{vw}$ for all $v, w \in W^\bullet$ satisfying $l(vw) = l(v) + l(w)$.

Definition 9.2.17

(i) The *extended affine braid group* $\mathcal{B} = \mathcal{B}(D)$ is the group generated by T_w ($w \in W$) with defining relations $T_vT_w = T_{vw}$ for all $v, w \in W$ satisfying $l(vw) = l(v) + l(w)$.

(ii) The *extended affine Hecke algebra* $H(\kappa^\bullet) = H(D, \kappa^\bullet)$ is the quotient of $\mathbb{C}[\mathcal{B}]$ by the two-sided ideal generated by $(T_i - \kappa_i)(T_i + \kappa_i^{-1})$ $(0 \le i \le n)$.

Similarly to the semidirect product decomposition $W \simeq \Omega \ltimes W^\bullet$ we have $\mathcal{B} \simeq \Omega \ltimes \mathcal{B}^\bullet$ and $H(\kappa^\bullet) \simeq \Omega \ltimes H(W^\bullet, \kappa^\bullet)$, where the action of Ω on $\mathcal{B}$ by group automorphisms (respectively on $H(W^\bullet, \kappa^\bullet)$ by algebra automorphisms) is determined by $w \cdot T_i = T_{w(i)}$ for $w \in \Omega$ and $0 \le i \le n$. For $\omega \in \Omega$ we will denote the element T_ω in the extended affine Hecke algebra $H(\kappa^\bullet)$ simply by ω.

The algebra homomorphism $\beta_{(R,\Delta,\kappa)} \colon H(W^\bullet, \kappa^\bullet) \hookrightarrow W^\bullet \ltimes F \subseteq W \ltimes F$ from §9.2.2 extends to an injective algebra map $\beta_{D,\kappa} : H(\kappa^\bullet) \hookrightarrow W \ltimes F$ by $\beta_{D,\kappa}(T_\omega) = \omega$ ($\omega \in \Omega$). We will now show that it gives rise to an action of $H(\kappa^\bullet)$ as q-difference reflection operators on a complex torus T_Λ. It is called the *basic representation* of $H(\kappa^\bullet)$. It is fundamental for the development of the Cherednik–Macdonald theory.

The complex torus $T_\Lambda := \mathrm{Hom}_{\mathbb{Z}}(\Lambda, \mathbb{C}^*)$ (of rank $\dim_{\mathbb{R}}(Z)$) is the algebraic group of complex characters of the lattice Λ. The algebra $\mathbb{C}[T_\Lambda]$ of regular functions on T_Λ is isomorphic to the group algebra $\mathbb{C}[\Lambda]$, where the standard basis element e^λ ($\lambda \in \Lambda$) of $\mathbb{C}[\Lambda]$ is viewed as the regular function $t \mapsto t(\lambda)$ on T_Λ. We write t^λ for the value of e^λ at $t \in T_\Lambda$. Since Λ is W_0-stable, W_0 acts on T_Λ, giving in turn rise to an action of W_0 on $\mathbb{C}[T_\Lambda]$ by algebra automorphisms. Then $w(e^\lambda) = e^{w\lambda}$ ($w \in W_0$ and $\lambda \in \Lambda$). We now first extend it to an action of the extended affine Weyl group W on $\mathbb{C}[T_\Lambda]$ depending on a fixed parameter $q \in \mathbb{R}_{>0} \setminus \{1\}$.

For $\alpha \in R_0$ set $q_\alpha := q^{\mu_\alpha}$ and define $q^\xi \in T_\Lambda$ ($\xi \in \Lambda^d$) to be the character $\lambda \mapsto q^{\langle\lambda,\xi\rangle}$ of Λ. The action of W_0 on T_Λ extends to a left W-action $(w, t) \mapsto w_q t$ on T_Λ by

$$\tau(\xi)_q t := q^\xi t, \qquad \xi \in \Lambda^d,\ t \in T_\Lambda.$$

Then $(w_q p)(t) := p(w_q^{-1} t)$ ($w \in W$, $p \in \mathbb{C}[T_\Lambda]$) is a W-action by algebra automorphisms on $\mathbb{C}[T_\Lambda]$. In particular,

$$\tau(\xi)_q(e^\lambda) = q^{-\langle\lambda,\xi\rangle} e^\lambda, \qquad \xi \in \Lambda^d,\ \lambda \in \Lambda.$$

It extends to a W-action by field automorphisms on the quotient field $\mathbb{C}(T_\Lambda)$ of $\mathbb{C}[T_\Lambda]$. It is useful to introduce the notation $t_q^{rc+\lambda} := q^r t^\lambda$ ($t \in T_\Lambda$, $r \in \mathbb{R}$, $\lambda \in \Lambda$). Then $(w_q^{-1} t)_q^{rc+\lambda} = t_q^{w(rc+\lambda)}$ for $w \in W$, $t \in T_\Lambda$, $r \in \mathbb{R}$ and $\lambda \in \Lambda$.

We write $W \ltimes_q \mathbb{C}(T_\Lambda)$ for the resulting semidirect product algebra. It canonically acts on

$\mathbb{C}(T_\Lambda)$ by q-difference reflection operators. We thus have a sequence of algebra maps

$$H(\kappa^\bullet) \to W \ltimes F \to W \ltimes_q \mathbb{C}(T_\Lambda) \to \mathrm{End}_{\mathbb{C}}(\mathbb{C}(T_\Lambda)),$$

where the first map is $\beta_{D,\kappa}$ and the second map sends $e^{\mu_\alpha mc+\alpha}$ to $q_\alpha^m e^\alpha$ for $m\mu_\alpha c + \alpha \in R^\bullet$. It gives the following result, which is closely related to [41, 5.13] in the present generality.

Theorem 9.2.18 *Let $D = (R_0, \Delta_0, \bullet, \Lambda, \Lambda^d) \in \mathcal{D}$ and $\kappa \in \mathcal{M}(D)$. For $a \in R$ we set $\kappa_{2a} := \kappa_a$ if $2a \notin R$. There exists a unique algebra monomorphism*

$$\pi_{\kappa,q} = \pi_{D;\kappa,q} \colon H(D, \kappa^\bullet) \hookrightarrow \mathrm{End}_{\mathbb{C}}(\mathbb{C}[T_\Lambda])$$

satisfying, for $p \in \mathbb{C}[T_\Lambda]$ and $t \in T_\Lambda$,

$$(\pi_{\kappa,q}(T_i)p)(t) = \kappa_{a_i}(s_{i,q}p)(t) + \frac{\kappa_{a_i} - \kappa_{a_i}^{-1} + (\kappa_{2a_i} - \kappa_{2a_i}^{-1})t_q^{a_i}}{1 - t_q^{2a_i}}\,(p(t) - (s_{i,q}p)(t)), \quad 0 \leq i \leq n,$$

$$(\pi_{\kappa,q}(\omega)p)(t) = (\omega_q p)(t), \quad \omega \in \Omega.$$

If $2a_i \notin R$ then, by the convention $\kappa_{2a_i} = \kappa_{a_i}$, the first formula reduces to

$$(\pi_{\kappa,q}(T_i)p))(t) = \kappa_{a_i}(s_{i,q}p)(t) + \frac{\kappa_{a_i} - \kappa_{a_i}^{-1}}{1 - t_q^{a_i}}\,(p(t) - (s_{i,q}p)(t)).$$

The theorem is due to Cherednik (see [18] and references therein) in the GL_{n+1} case (see Example 9.2.9(**ii**)) and when $D = (R_0, \Delta_0, \bullet, P(R_0), P(R_0^d))$ with R_0 an arbitrary reduced irreducible root system. The theorem is due to Noumi [74] for $D = (R_0, \Delta_0, t, \mathbb{Z}R_0, \mathbb{Z}R_0)$ with R_0 of type A_1 or of type B_n ($n \geq 2$). This case is special due to its large degree of freedom ($\nu(D) = 4$ if $n = 1$ and $\nu(D) = 5$ if $n \geq 2$). We will describe this case in detail in §9.3.8.

Remark 9.2.19

(i) From Theorem 9.2.3 one first obtains $\pi_{\kappa,q}$ as an algebra map from $H(\kappa^\bullet)$ to $\mathrm{End}_{\mathbb{C}}(\mathbb{C}(T_\Lambda))$. The image is contained in the subalgebra of endomorphisms preserving $\mathbb{C}[T_\Lambda]$ since, for $\lambda \in \Lambda$, we have $(\lambda, a_i^\vee) \in \mathbb{Z}$ if $2a_i \notin R$ and $(\lambda, a_i^\vee) \in 2\mathbb{Z}$ if $2a_i \in R$.

(ii) Let s be the number of W-orbits of $R \setminus R^\bullet$. Extending a W-equivariant map $\kappa^\bullet \colon R^\bullet \to \mathbb{C}^*$ to a multiplicity function $\kappa \in \mathcal{M}$ on R amounts to choosing s nonzero complex parameters. Hence, the maps $\pi_{\kappa,q}$ define a family of algebra monomorphisms of the extended affine Hecke algebra $H(\kappa^\bullet)$ into $\mathrm{End}_{\mathbb{C}}(\mathbb{C}[T_\Lambda])$, parametrized by $s+1$ parameters $\kappa|_{R\setminus R^\bullet}$ and q.

9.3 Monic Macdonald–Koornwinder polynomials

In this section we introduce the monic nonsymmetric and symmetric Macdonald–Koornwinder polynomials. The terminology *Macdonald polynomials* is employed in the literature for the cases that R is reduced (i.e., the cases $D = (R_0, \Delta_0, \bullet, P(R_0), P(R_0^d))$ and the GL_{n+1} case). The *Koornwinder polynomials* correspond to the initial data $D = (R_0, \Delta_0, t, \mathbb{Z}R_0, \mathbb{Z}R_0)$ with R_0 of type A_1 or of type B_n ($n \geq 2$), in which case $R = R(D)$ is nonreduced and of type $\mathrm{C}^\vee\mathrm{C}_n$.

To have uniform terminology we will speak of *Macdonald–Koornwinder polynomials* when discussing the theory for arbitrary initial data.

The monic nonsymmetric Macdonald–Koornwinder polynomials will be introduced as the common eigenfunctions in $\mathbb{C}[T_\Lambda]$ of a family of commuting q-difference reflection operators. The operators are obtained as images under the basic representation $\pi_{\kappa,q}$ of elements from a large commutative subalgebra of the extended affine Hecke algebra $H(\kappa^\bullet)$. A Hecke algebra symmetrizer turns the monic nonsymmetric Macdonald–Koornwinder polynomials into monic symmetric Macdonald–Koornwinder polynomials, which are W_0-invariant regular functions on T_Λ solving a suitable spectral problem of a commuting family of q-difference operators, called Macdonald operators. In addition we determine in this section the (bi)orthogonality relations of the polynomials.

Throughout this section we fix

1. a quintuple $D = (R_0, \Delta_0, \bullet, \Lambda, \Lambda^d)$ of initial data,
2. a deformation parameter $q \in \mathbb{R}_{>0} \setminus \{1\}$,
3. a multiplicity function $\kappa \in \mathcal{M}(D)$,

and we freely use the associated notations from §9.2.

9.3.1 Bernstein–Zelevinsky presentation

For a given expression $w = \omega s_{i_1} s_{i_2} \cdots s_{i_r} \in W$ ($\omega \in \Omega$, $0 \le i_j \le n$) which is *reduced*, i.e., $r = l(w)$, put $T_w := \omega T_{i_1} T_{i_2} \cdots T_{i_r} \in H(\kappa^\bullet)$. This is well defined, and $\{T_w\}_{w \in W}$ is a complex linear basis of $H(\kappa^\bullet)$.

The cones $\Lambda^{d\pm} := \{\xi \in \Lambda^d \mid (\xi, \alpha^{d\vee}) \ge 0 \ \forall \alpha \in R_0^\pm\}$ form fundamental domains for the W_0-action on Λ^d. Any $\xi \in \Lambda^d$ can be written as $\xi = \mu - \nu$ with $\mu, \nu \in \Lambda^{d+}$ (and similarly for Λ^{d-}). Furthermore, if $\xi, \xi' \in \Lambda^{d+}$ then $l(\tau(\xi + \xi')) = l(\tau(\xi)) + l(\tau(\xi'))$. It follows that there exists a unique group homomorphism $\xi \mapsto Y^\xi \colon \Lambda^d \to \mathcal{B}$ such that $Y^\xi = T_{\tau(\xi)}$ ($\xi \in \Lambda^{d+}$). On the level of the extended affine Hecke algebra it gives rise to an algebra homomorphism

$$p \mapsto p(Y) \colon \mathbb{C}[T_{\Lambda^d}] \to H(\kappa^\bullet), \tag{9.3.1}$$

where $p(Y) = \sum_\xi c_\xi Y^\xi$ if $p(t) = \sum_\xi c_\xi t^\xi$. The image of the map (9.3.1) is denoted by $\mathbb{C}_Y[T_{\Lambda^d}]$.

As in §9.2.4, the lattice Λ^d is W_0-stable, giving rise to a W_0-action on T_{Λ^d}, and hence a W_0-action on $\mathbb{C}[T_{\Lambda^d}]$ by algebra automorphisms.

Denote by H_0 the subalgebra of $H(\kappa^\bullet)$ generated by T_i ($1 \le i \le n$). We have a natural surjective algebra map $H(W_0, \kappa|_{R_0}) \to H_0$, sending the algebraic generator T_i of $H(W_0, \kappa|_{R_0})$ to $T_i \in H_0$.

The analog of the semidirect product decomposition $W = W_0 \ltimes \Lambda^d$ for the extended affine Hecke algebra $H(\kappa^\bullet)$ is the following *Bernstein–Zelevinsky presentation* of $H(\kappa^\bullet)$ (see [65])

Theorem 9.3.1

1. *The algebra maps $\mathbb{C}[T_{\Lambda^d}] \to \mathbb{C}_Y[T_{\Lambda^d}]$ and $H(W_0, \kappa|_{R_0}) \to H_0$ are isomorphisms.*
2. *Multiplication defines a linear isomorphism $H_0 \otimes \mathbb{C}_Y[T_{\Lambda^d}] \simeq H(\kappa^\bullet)$.*

3. *For $i \in \{1,\ldots,n\}$ such that $(\Lambda^d, \alpha_i^{d\vee}) = \mathbb{Z}$ we have in $H(\kappa^\bullet)$ that*

$$p(Y)T_i - T_i(s_ip)(Y) = \left(\frac{\kappa_i - \kappa_i^{-1}}{1 - Y^{-\alpha_i^d}}\right)(p(Y) - (s_ip)(Y)), \quad p \in \mathbb{C}[T_{\Lambda^d}]. \tag{9.3.2}$$

4. *For $i \in \{1,\ldots,n\}$ such that $(\Lambda^d, \alpha_i^{d\vee}) = 2\mathbb{Z}$ we have in $H(\kappa^\bullet)$ that*

$$p(Y)T_i - T_i(s_ip)(Y) = \left(\frac{\kappa_i - \kappa_i^{-1} + (\kappa_0 - \kappa_0^{-1})Y^{-\alpha_i^d}}{1 - Y^{-2\alpha_i^d}}\right)(p(Y) - (s_ip)(Y)), \quad p \in \mathbb{C}[T_{\Lambda^d}]. \tag{9.3.3}$$

These properties characterize $H(\kappa^\bullet)$ as a unital complex associative algebra.

With the notion of the dual multiplicity parameter κ^d (see Lemma 9.2.13), the *cross relations* (9.3.2) and (9.3.3) in the affine Hecke algebra $H(\kappa^\bullet)$ can be uniformly written as

$$p(Y)T_i - T_i(s_ip)(Y) = \left(\frac{\kappa^d_{\alpha_i^d} - (\kappa^d_{\alpha_i^d})^{-1} + (\kappa^d_{2\alpha_i^d} - (\kappa^d_{2\alpha_i^d})^{-1})Y^{-\alpha_i^d}}{1 - Y^{-2\alpha_i^d}}\right)(p(Y) - (s_ip)(Y)) \tag{9.3.4}$$

for $p \in \mathbb{C}[T_{\Lambda^d}]$ and $1 \leq i \leq n$. It follows from the theorem that the center $Z(H(\kappa^\bullet))$ of the extended affine Hecke algebra $H(\kappa^\bullet)$ equals $\mathbb{C}_Y[T_{\Lambda^d}]^{W_0}$.

9.3.2 Monic nonsymmetric Macdonald–Koornwinder polynomials

The results in this subsection are from [71, 14, 91, 41]. For detailed proofs see, e.g., [73, §2.8, §4.6, §5.2]. We put the following conditions on q and $\kappa \in \mathcal{M}(D)$:

$$0 < q < 1 \;\text{ and }\; 0 < \kappa_a < 1 \; (a \in R) \quad \text{ or } \quad q > 1 \;\text{ and }\; \kappa_a > 1 \; (a \in R). \tag{9.3.5}$$

Set $\eta(x) := 1$ for $x > 0$ and $\eta(x) := -1$ for $x \leq 0$. Define a W_0-equivariant map $\upsilon\colon R_0^d \to \mathbb{R}_{>0}$ (depending on $\kappa^\bullet$) by $\upsilon_{\alpha^d} := \kappa_\alpha^{1/2}\kappa_{\mu_\alpha c+\alpha}^{1/2}$ $(\alpha \in R_0)$.

Definition 9.3.2 For $\lambda \in \Lambda$ define $\gamma_{\lambda,q} = \gamma_{\lambda,q}(D;\kappa^\bullet) \in T_{\Lambda^d}$ by

$$\gamma_{\lambda,q} := q^\lambda \prod_{\alpha\in R_0^+} \upsilon_{\alpha^d}^{\eta((\lambda,\alpha^\vee))\alpha^{d\vee}}.$$

In other words, $\gamma_{\lambda,q}^\xi = q^{(\lambda,\xi)} \prod_{\alpha\in R_0^+} \upsilon_{\alpha^d}^{\eta((\lambda,\alpha^\vee))(\xi,\alpha^{d\vee})}$ for all $\xi \in \Lambda^d$.

As a special case we have $\gamma_{\lambda,q} = q^\lambda \prod_{\alpha\in R_0^+} \upsilon_{\alpha^d}^{-\alpha^{d\vee}}$ $(\lambda \in \Lambda^-)$, where we use the notation $\Lambda^\pm := \{\lambda \in \Lambda \mid (\lambda,\alpha^\vee) \geq 0 \;\forall \alpha \in R_0^\pm\}$.

Write $l^d = l_{D^d}$ for the length function on the dual extended affine Weyl group W^d and $\Omega^d = \Omega(D^d)$ for the subgroup of elements of W^d of length zero with respect to l^d. We have a q-dependent W^d-action on T_{Λ^d} extending the W_0-action by $\tau(\lambda)_q\gamma = q^\lambda\gamma$ for all $\lambda \in \Lambda$ and $\gamma \in T_{\Lambda^d}$. Then $\gamma_{\lambda,q} = \tau(\lambda)_q\gamma_{0,q}$ in T_{Λ^d} if $\lambda \in \Lambda^-$. This generalizes as follows.

Lemma 9.3.3 *We have in T_{Λ^d} that $\gamma_{\lambda,q} = u^d(\lambda)_q\gamma_{0,q}$ $(\lambda \in \Lambda)$, where $u^d(\lambda) \in W^d$ is the element of minimal length with respect to l^d in the coset $\tau(\lambda)W_0$.*

The condition (9.3.5) on the parameters, together with Lemma 9.3.3, implies:

Lemma 9.3.4 *The map $\lambda \mapsto \gamma_{\lambda,q} \colon \Lambda \to T_{\Lambda^d}$ is injective.*

For later purposes it is convenient to record the following compatibility between the q-dependent W^d-action on $\gamma_{\lambda,q} \in T_{\Lambda^d}$ and the W^d-action $(w\tau(\lambda), \lambda') \mapsto w(\lambda + \lambda')$ on Λ ($w \in W_0$, $\lambda, \lambda' \in \Lambda$).

Proposition 9.3.5 *Let $\lambda \in \Lambda$. Then*

a. *If $\omega \in \Omega^d$ then $\omega_q \gamma_{\lambda,q} = \gamma_{\omega\lambda,q}$.*

b. *If $0 \le i \le n$ and $s_i^d \lambda \neq \lambda$ then $s_{i,q}^d \gamma_{\lambda,q} = \gamma_{s_i^d \lambda,q}$.*

c. *If $0 \le i \le n$ and $s_i^d \lambda = \lambda$ then $s_{i,q}^d \gamma_{\lambda,q} = \gamma_{\lambda,q} \upsilon_{D(a_i^d)}^{2D(a_i^d)^\vee}$.*

Warning $D(a_i^d) = (Da_i)^d$ holds true for $1 \le i \le n$, and for $i = 0$ if $\bullet = t$ (then both sides equal $-\theta$). It is not correct when $i = 0$, $\bullet = u$ and R_0 has two root lengths, since then $(Da_0)^d = -\varphi^\vee$ and $D(a_0^d) = -\theta^\vee$.

For $\lambda, \mu \in \Lambda^+$ we write $\lambda \le \mu$ if $\mu - \lambda$ can be written as a sum of positive roots $\alpha \in R_0^+$. We also write $\le$ for the *Bruhat order* of W_0 with respect to the Coxeter generators s_i ($1 \le i \le n$), see [44, §5.9]. For $\lambda \in \Lambda$ let $\lambda_\pm$ be the unique element in $\Lambda^\pm \cap W_0\lambda$ and write $v(\lambda) \in W_0$ for the element of shortest length such that $v(\lambda)\lambda = \lambda_-$. Then $\tau(\lambda) = u^d(\lambda)v(\lambda)$ in W^d for $\lambda \in \Lambda$.

Definition 9.3.6 Let $\lambda, \mu \in \Lambda$. We write $\lambda \preceq \mu$ if $\lambda_+ < \mu_+$ or if $\lambda_+ = \mu_+$ and $v(\lambda) \ge v(\mu)$.

Note that $\preceq$ is a partial order on Λ. Furthermore, if $\lambda \in \Lambda^-$ then $\mu \preceq \lambda$ for all $\mu \in W_0\lambda$. For each $\lambda \in \Lambda$ the set of elements $\mu \in \Lambda$ satisfying $\mu \preceq \lambda$ thus is contained in the finite set

$$\{\mu \in \Lambda \mid \mu \preceq \lambda_-\} = \bigcup_{\mu_+ \in \Lambda^+ \colon \mu_+ \le \lambda_+} W_0\mu_+,$$

which is the smallest saturated subset $\mathrm{Sat}(\lambda_+)$ of Λ containing λ_+ (a subset $X \subseteq \Lambda$ is *saturated* if for each $\alpha \in R_0^+$ and $\lambda \in \Lambda$ we have $\lambda - r\alpha \in X$ for all integers r between zero and $(\lambda, \alpha^\vee)$, including both zero and $(\lambda, \alpha^\vee)$).

We write $p = d_\lambda e^\lambda + \text{l.o.t.}$ for an element $p = \sum_{\mu \in \Lambda} d_\mu e^\mu \in \mathbb{C}[T_\Lambda]$ satisfying $d_\mu = 0$ if $\mu \npreceq \lambda$. If in addition $d_\lambda \neq 0$ then we say that p is *of degree* λ.

Proposition 9.3.7 *In $\mathbb{C}[T_\Lambda]$ we have $\pi_{\kappa,q}(r(Y))e^\lambda = r(\gamma_{\lambda,q}^{-1})e^\lambda + \text{l.o.t.}$ ($r \in \mathbb{C}[T_{\Lambda^d}]$, $\lambda \in \Lambda$).*

Corollary 9.3.8 *For each $\lambda \in \Lambda$ there exists a unique $P_\lambda = P_\lambda(D; \kappa, q) \in \mathbb{C}[T_\Lambda]$ satisfying $\pi_{\kappa,q}(r(Y))P_\lambda = r(\gamma_{\lambda,q}^{-1})P_\lambda$ ($r \in \mathbb{C}[T_{\Lambda^d}]$) and $P_\lambda = e^\lambda + \text{l.o.t.}$*

Definition 9.3.9 $P_\lambda = P_\lambda(D; \kappa, q) \in \mathbb{C}[T_\Lambda]$ is called the monic *nonsymmetric Macdonald–Koornwinder polynomial* of degree $\lambda \in \Lambda$.

For $D = (R_0, \Delta_0, \bullet, P(R_0), P(R_0^d))$ the definition of the nonsymmetric Macdonald–Koornwinder polynomial is due to Macdonald [71] in the untwisted case ($\bullet = u$) and due to Cherednik [14] in the general case. For $D = (R_0, \Delta_0, t, \mathbb{Z}R_0, \mathbb{Z}R_0)$ with R_0 of type A_1 or of type B_n

($n \geq 2$) the nonsymmetric Macdonald–Koornwinder polynomials are Sahi's [91] nonsymmetric Koornwinder polynomials. In the present generality (with more flexible choices of lattices Λ and Λ^d) the definition is close to Haiman's definition [41, §6]. The same references apply for the biorthogonality relations of the nonsymmetric Macdonald–Koornwinder polynomials discussed in the next subsection.

The GL_{n+1} nonsymmetric Macdonald polynomials (corresponding to Example 9.2.9**(ii)**) are often studied separately, see, e.g., [56, 38].

9.3.3 Biorthogonality

We assume in this subsection that $\kappa \in \mathcal{M}$ and q satisfy

$$0 < q < 1, \qquad 0 < \kappa_a < 1 \ \ \forall a \in R, \qquad 0 < \kappa_a \kappa_{2a}^{\pm 1} \leq 1 \ \ \forall a \in R^{\bullet}, \tag{9.3.6}$$

and for $\lambda \in \Lambda$ we write $P_\lambda := P_\lambda(D, \kappa, q)$, $P_\lambda^\circ := P_\lambda(D, \kappa^{-1}, q^{-1})$, where $\kappa^{-1} \in \mathcal{M}(D)$ is the multiplicity function $a \mapsto \kappa_a^{-1}$. Define

$$c_a(t) = c_a^{\kappa,q}(t; D) := \frac{(1 - \kappa_a \kappa_{2a} t_q^a)(1 + \kappa_a \kappa_{2a}^{-1} t_q^a)}{1 - t_q^{2a}} \in \mathbb{C}(T_\Lambda), \qquad a \in R^{\bullet}. \tag{9.3.7}$$

Then $c_a(w_q^{-1} t) = c_{wa}(t)$ $(w \in W, a \in R^{\bullet})$. In addition, $\pi_{\kappa,q}(T_i) = \kappa_i + \kappa_i^{-1} c_{a_i}(s_{i,q} - 1)$ $(0 \leq i \leq n)$.

Since $0 < q < 1$, the infinite product $v := \prod_{a \in R^{\bullet +}} c_a^{-1}$ defines a meromorphic function

$$\begin{aligned} v(t) = & \prod_{\alpha \in R_0^+} \frac{1 - t^{2\alpha}}{(1 - \kappa_\alpha \kappa_{2\alpha} t^\alpha)(1 + \kappa_\alpha \kappa_{2\alpha}^{-1} t^\alpha)} \\ & \times \prod_{\beta \in R_0} \frac{(q_\beta^2 t^{2\beta}; q_\beta^2)_\infty}{(q_\beta^2 \kappa_\beta \kappa_{2\beta} t^\beta, -q_\beta^2 \kappa_\beta \kappa_{2\beta}^{-1} t^\beta, q_\beta \kappa_{\mu_\beta c+\beta} \kappa_{2\mu_\beta c+2\beta} t^\beta, -q_\beta \kappa_{\mu_\beta c+\beta} \kappa_{2\mu_\beta c+2\beta}^{-1} t^\beta; q_\beta^2)_\infty}. \end{aligned} \tag{9.3.8}$$

on T_Λ. Here we used *q-shifted factorials*

$$(x_1, \ldots, x_m; q)_r := (x_1; q)_r \ldots (x_m; q)_r, \qquad (x; q)_r := \prod_{j=0}^{r-1} (1 - q^j x), \qquad r \in \mathbb{Z}_{\geq 0} \cup \{\infty\}. \tag{9.3.9}$$

Remark 9.3.10 If $\beta \in R_0$ satisfies $(\Lambda^d, \beta^{d\vee}) = \mathbb{Z}$ then $\kappa_{\mu_\beta c+\beta} = \kappa_\beta$ and $\kappa_{2\mu_\beta c+2\beta} = \kappa_{2\beta}$, and the β-factor in the second line of (9.3.8) simplifies to $\dfrac{(q_\beta^2 t^{2\beta}; q_\beta^2)_\infty}{(q_\beta \kappa_\beta \kappa_{2\beta} t^\beta, -q_\beta \kappa_\beta \kappa_{2\beta}^{-1} t^\beta; q_\beta)_\infty}$. If in addition $\kappa_{2\beta} = \kappa_\beta$ (for instance, if $2\beta \notin R$) then the β-factor simplifies further to $\dfrac{(q_\beta t^\beta; q_\beta)_\infty}{(q_\beta \kappa_\beta^2 t^\beta; q_\beta)_\infty}$.

By the conditions (9.3.6) on the parameters, v is a continuous function on the compact torus $T_\Lambda^u = \mathrm{Hom}(\Lambda, S^1) \subset T_\Lambda$, where $S^1 = \{z \in \mathbb{C} \mid |z| = 1\}$. Write $d_u t$ for the normalized Haar measure on T_Λ^u.

Definition 9.3.11 Define a sesquilinear form $\langle\cdot,\cdot\rangle\colon \mathbb{C}[T_\Lambda]\times\mathbb{C}[T_\Lambda]\to\mathbb{C}$ by

$$\langle p,r\rangle := \int_{T^u_\Lambda} p(t)\,\overline{r(t)}\,v(t)\,d_ut.$$

Proposition 9.3.12 *Let $p,r\in\mathbb{C}[T_\Lambda]$ and $w\in W$. Then $\langle \pi_{\kappa,q}(T_w)p,r\rangle = \langle p,\pi_{\kappa^{-1},q^{-1}}(T_w^{-1})r\rangle$.*

The biorthogonality of the nonsymmetric Macdonald–Koornwinder polynomials readily follows from Proposition 9.3.12.

Theorem 9.3.13 *If $\lambda,\mu\in\Lambda$ and $\lambda\neq\mu$ then $\langle P_\lambda,P^\circ_\mu\rangle=0$.*

9.3.4 Macdonald operators

Special cases of what now are known as the Macdonald q-difference operators were explicitly written down by Macdonald [72] when he introduced the symmetric Macdonald polynomials. Earlier Ruijsenaars [89] had introduced these commuting q-difference operators for R_0 of type A as the quantum Hamiltonian of a relativistic version of the quantum trigonometric Calogero–Moser system (see §9.3.7). Koornwinder [61] introduced a multivariable extension of the second-order Askey–Wilson q-difference operator to define the symmetric Koornwinder polynomials. This case corresponds to $D=(R_0,\Delta_0,t,\mathbb{Z}R_0,\mathbb{Z}R_0)$ with R_0 of type A_1 or of type B_n with $n\geq 2$ (see §9.3.8).

The construction of the whole family of Macdonald q-difference operators using affine Hecke algebras is due to Cherednik [12] in case $D=(R_0,\Delta_0,\bullet,P(R_0),P(R_0^d))$ and due to Noumi [74] in case $D=(R_0,\Delta_0,t,\mathbb{Z}R_0,\mathbb{Z}R_0)$ with R_0 of type A_1 or of type B_n ($n\geq 2$). We explain this construction here, see [73, §4.4] for a treatment close to the present one.

In this subsection we assume that $\kappa\in\mathcal{M}$ and q satisfy (9.3.5). Consider the linear map

$$\mathrm{Res}_q\colon \sum_{w\in W_0} D_w w\mapsto \sum_{w\in W_0} D_w\colon \quad W\ltimes_q \mathbb{C}(T_\Lambda)\to \tau(\Lambda^d)\ltimes_q\mathbb{C}(T_\Lambda)$$

where $D_w\in\tau(\Lambda^d)\ltimes_q\mathbb{C}(T_\Lambda)$ ($w\in W_0$). Note that

$$L(p)=(\mathrm{Res}_q(L))(p),\qquad p\in\mathbb{C}(T_\Lambda)^{W_0}. \tag{9.3.10}$$

Lemma 9.3.14 *Let $\beta_{\kappa,q}(H_0)'$ be the commutant of the subalgebra $\beta_{\kappa,q}(H_0)$ in $W\ltimes_q\mathbb{C}(T_\Lambda)$. Then Res_q restricts to an algebra homomorphism $\mathrm{Res}_q\colon\beta_{\kappa,q}(H_0)'\to(\tau(\Lambda^d)\ltimes_q\mathbb{C}(T_\Lambda))^{W_0}$, where $(\tau(\Lambda^d)\ltimes_q\mathbb{C}(T_\Lambda))^{W_0}$ is the subalgebra of $W\ltimes_q\mathbb{C}(T_\Lambda)$ consisting of W_0-invariant q-difference operators.*

As $Z(H(\kappa^\bullet))=\mathbb{C}_Y[T_{\Lambda^d}]^{W_0}$, the lemma implies that the W_0-invariant q-difference operators

$$D_p:=\mathrm{Res}_q(\beta_{\kappa,q}(p(Y)))\in(\tau(\Lambda^d)\ltimes_q\mathbb{C}(T_\Lambda))^{W_0},\qquad p\in\mathbb{C}[T_{\Lambda^d}]^{W_0},$$

pairwise commute. The operator D_p is called the *Macdonald q-difference operator* associated to $p\in\mathbb{C}[T_{\Lambda^d}]^{W_0}$.

Define the orbit sums $m^d_\xi(t) := \sum_{\eta\in W_0\xi} t^\eta \in \mathbb{C}[T_{\Lambda^d}]^{W_0}$ ($\xi \in \Lambda^d$). Then $\{m^d_\xi\}_{\xi\in\Lambda^{d-}}$ is a linear basis of $\mathbb{C}[T_{\Lambda^d}]^{W_0}$. We write D_ξ for $D_{m^d_\xi}$. Set, for $w \in W$,

$$c_w := \prod_{a\in R^{\bullet+}\cap w^{-1}R^{\bullet-}} c_a \in \mathbb{C}(T_\Lambda) \qquad \text{and} \qquad \kappa_w := \prod_{a\in R^{\bullet+}\cap w^{-1}R^{\bullet-}} \kappa_a.$$

Write $W_{0,\xi}$ for the stabilizer subgroup of ξ in W_0.

Remark 9.3.15 If $\xi \in \Lambda^{d-}$ and $w \in W_{0,\xi}$ then $w(c_{\tau(-\xi)}) = c_{\tau(-\xi)}$ in $\mathbb{C}(T_\Lambda)$.

Proposition 9.3.16 *Let $\xi \in \Lambda^{d-}$. Then*

$$D_\xi = \kappa^{-1}_{\tau(-\xi)} \sum_{w\in W_0/W_{0,\xi}} w(c_{\tau(-\xi)})\tau(w\xi)_q + \sum_{\eta\in \mathrm{Sat}(\xi_+)\setminus W_0\xi} g_\eta \tau(\eta)_q$$

for certain $g_\eta \in \mathbb{C}(T_\Lambda)$ satisfying $g_{w\eta} = w(g_\eta)$ for all $w \in W_0$ and $\eta \in \Lambda^d$.

The set of *dominant minuscule weights* in Λ^d is defined by

$$\Lambda^{d+}_{min} := \{\xi \in \Lambda^d \mid (\xi, \alpha^{d\vee}) \in \{0,1\} \ \forall \alpha \in R^+_0\}.$$

Set $\Lambda^d_0 := \Lambda^d \cap V^\perp$. We now first give an explicit description of the dominant minuscule weights.

Recall that $Da_0 = -\varphi$ if $\bullet = u$ and $Da_0 = -\theta$ if $\bullet = t$. Hence $-(Da_0)^{d\vee}$ is the highest root of $R^{d\vee}_0$. Consider the expansion $-(Da_0)^{d\vee} = \sum_{i=1}^n m_i\alpha_i^{d\vee}$ of $-(Da_0)^{d\vee} \in R^{d\vee}_0$ with respect to the ordered basis $\Delta^{d\vee}_0$ of $R^{d\vee}_0$. Then $m_i \in \mathbb{Z}_{\geq 1}$ for all i. Set

$$J^+_{\Lambda^d} := \{i \in \{1,\ldots,n\} \mid m_i = 1 \ \& \ (\varpi^d_i + V^\perp) \cap \Lambda^d \neq \emptyset\},$$

where $V^\perp$ is the orthocomplement of V in Z.

Proposition 9.3.17

(i) *Λ^{d+}_{min} is a complete set of representatives of $\Lambda^d/\mathbb{Z}R^d_0$.*

(ii) *For $j \in J^+_{\Lambda^d}$ choose an element $\tilde{\varpi}^d_j \in (\varpi^d_j + V^\perp) \cap \Lambda^d$. Then*

$$\Lambda^{d+}_{min} = \Lambda^d_0 \cup \bigcup_{j\in J^+_{\Lambda^d}} (\tilde{\varpi}^d_j + \Lambda^d_0) \quad \text{(disjoint union)}.$$

(iii) *For $\eta \in \Lambda^d$ let $u(\eta) \in W$ be the unique element of minimal length (with respect to l) in the coset $\tau(\eta)W_0$. Then $\Omega = \{u(\xi) \mid \xi \in \Lambda^{d+}_{min}\}$.*

Since $-(Da_0)^{d\vee} \in R^{d\vee+}_0$ is the highest root, $-(Da_0)^d \in R^{d+}_0$ is *quasi-minuscule*, meaning that $(-(Da_0)^d, \alpha^{d\vee}) \in \{0,1\}$ for all $\alpha^d \in R^{d+}_0 \setminus \{-(Da_0)^d\}$.

Corollary 9.3.18 *Let $w_0 \in W_0$ be the longest Weyl group element.*

(i) *For $j \in J^+_{\Lambda^d}$ we have* $D_{w_0\tilde{\varpi}^d_j} = \kappa^{-1}_{\tau(-w_0\tilde{\varpi}^d_j)} \sum_{w\in W_0/W_{0,w_0\tilde{\varpi}^d_j}} w(c_{\tau(-w_0\tilde{\varpi}^d_j)})\,\tau(ww_0\tilde{\varpi}^d_j)_q.$

(ii) $D_{(Da_0)^d} = \kappa^{-1}_{\tau(-(Da_0)^d)} \sum_{w\in W_0/W_{0,(Da_0)^d}} w(c_{\tau(-(Da_0)^d)})\,\big(\tau(w(Da_0)^d)_q - 1\big) + m^d_{(Da_0)^d}(\gamma^{-1}_{0,q}).$

9.3.5 Monic symmetric Macdonald–Koornwinder polynomials

In §9.3.2 we introduced the nonsymmetric Macdonald–Koornwinder polynomials, but historically the symmetric Macdonald–Koornwinder polynomials were defined first. The monic symmetric Macdonald polynomials associated to initial data given by quintuples of the form $D = (R_0, \Delta_0, \bullet, P(R_0), P(R_0^d))$ were defined by Macdonald in [72] using the fact that the explicit Macdonald q-difference operator D_ξ for $\xi \in \{w_0\tilde{\varpi}_j^d\}_{j\in J^+_{\Lambda^d}} \cup \{(Da_0)^d\}$ (see Corollary 9.3.18) is a linear operator on $\mathbb{C}[T_\Lambda]^{W_0}$ which is triangular with respect to the suitable partially ordered basis of orbit sums and which has (generically) simple spectrum.

This approach was extended by Koornwinder [61] to the case corresponding to the initial data $D = (R_0, \Delta, t, \mathbb{Z}R_0, \mathbb{Z}R_0)$ with R_0 of type A_1 or of type B_n ($n \geq 2$), in which case $D_{-\theta}$ is Koornwinder's multivariable extension of the Askey–Wilson second-order q-difference operator (see §9.3.8). The corresponding symmetric Macdonald–Koornwinder polynomials are the Askey–Wilson polynomials [1] if R_0 is of rank one and the symmetric Koornwinder [61] polynomials if R_0 is of higher rank.

In this subsection we introduce the monic symmetric Macdonald–Koornwinder polynomials by symmetrizing the nonsymmetric ones, cf. [12, §4]. We assume throughout this subsection that $\kappa \in \mathcal{M}$ and q satisfy (9.3.5). Recall the notation $\kappa_w := \prod_{a\in R^{\bullet+}\cap w^{-1}R^{\bullet-}} \kappa_a$. It only depends on $\kappa^\bullet = \kappa|_{R^\bullet}$. It satisfies $\kappa_v\kappa_w = \kappa_{vw}$ if $l(vw) = l(v) + l(w)$. Hence there exists a unique linear character $\chi_+ \colon H(\kappa^\bullet) \to \mathbb{C}$ satisfying $\chi_+(T_w) = \kappa_w$ for all $w \in W$, the trivial linear character of $H(\kappa^\bullet)$. Define

$$C_+ := \frac{1}{\sum_{w\in W_0} \kappa_w^2} \sum_{w\in W_0} \kappa_w T_w \in H_0(\kappa^\bullet|_{R_0}) \subset H(\kappa^\bullet). \tag{9.3.11}$$

The normalization is such that $\chi_+(C_+) = 1$. Then $T_iC_+ = \kappa_iC_+ = C_+T_i$ for $1 \leq i \leq n$, and $C_+^2 = C_+$. The following lemma follows from the explicit expression of $\pi_{\kappa,q}(T_i)$ ($1 \leq i \leq n$).

Lemma 9.3.19 *The linear endomorphism $\pi_{\kappa,q}(C_+)$ of $\mathbb{C}[T_\Lambda]$ is an idempotent with image $\mathbb{C}[T_\Lambda]^{W_0}$.*

Consider the linear basis $\{m_\lambda\}_{\lambda\in\Lambda^+}$ of $\mathbb{C}[T_\Lambda]^{W_0}$ given by orbit sums, $m_\lambda(t) = \sum_{\mu\in W_0\lambda} t^\mu$. Recall that $P_\lambda = P_\lambda(D;\kappa,q)$ denotes the monic nonsymmetric Macdonald–Koornwinder polynomial of degree $\lambda \in \Lambda$.

Lemma 9.3.20 *If $\lambda \in \Lambda^+$ then $\pi_{\kappa,q}(C_+)P_\lambda = \sum_{\mu\in\Lambda^+:\mu\leq\lambda} c_{\lambda,\mu}m_\mu$ for certain $c_{\lambda,\mu} \in \mathbb{C}$ with $c_{\lambda,\lambda} \neq 0$.*

Definition 9.3.21 The monic *symmetric Macdonald–Koornwinder polynomial* of degree $\lambda \in \Lambda^+$ is defined by $P_\lambda^+ = P_\lambda^+(D;\kappa,q) := c_{\lambda,\lambda}^{-1}\pi_{\kappa,q}(C_+)P_\lambda \in \mathbb{C}[T_\Lambda]^{W_0}$.

Theorem 9.3.22 *P_λ^+, defined above, is the unique element in $\mathbb{C}[T_\Lambda]^{W_0}$ satisfying*

1. *$P_\lambda^+ = \sum_{\mu\in\Lambda^+:\mu\leq\lambda} d_{\lambda,\mu}m_\mu$ with $d_{\lambda,\mu} \in \mathbb{C}$ and $d_{\lambda,\lambda} = 1$,*
2. *$D_pP_\lambda^+ = p(q^{-\lambda}\prod_{\alpha\in R_0^+} \upsilon_{\alpha^d}^{-\alpha^{d\vee}})P_\lambda^+$ for all $p \in \mathbb{C}[T_{\Lambda^d}]^{W_0}$.*

(Note that for $p \in \mathbb{C}[T_{\Lambda^d}]^{W_0}$ and $\lambda \in \Lambda^+$ we have $p(q^{-\lambda}\prod_{\alpha\in R_0^+} \upsilon_{\alpha^d}^{-\alpha^{d\vee}}) = p(\gamma_{w_0\lambda,q}^{-1})$).

Remark 9.3.23 One may replace in Theorem 9.3.22 condition **2** by the weaker condition

$$D_{(Da_0)^d}P_\lambda^+ = m^d_{(Da_0)^d}\Big(q^{-\lambda}\prod_{\alpha\in R_0^+}\upsilon_{\alpha^d}^{-\alpha^{d\vee}}\Big)P_\lambda^+.$$

Note that the left-hand side of this equation is completely explicit by Corollary 9.3.18.

Explicit expressions of the monic symmetric Macdonald–Koornwinder polynomials when R_0 is of rank one are given in §9.3.7 and §9.3.8.

9.3.6 Orthogonality

In this subsection we assume that $\kappa \in \mathcal{M}$ and q satisfy the conditions (9.3.6). We thus have the monic symmetric Macdonald–Koornwinder polynomials $\{P_\lambda^+\}_{\lambda\in\Lambda^+}$ with respect to the parameters (κ, q), as well as the monic symmetric Macdonald–Koornwinder polynomials with respect to the parameters (κ^{-1}, q^{-1}), in which case we denote them by $\{P_\lambda^{\circ+}\}_{\lambda\in\Lambda^+}$.

Define the W_0-invariant meromorphic function

$$v_+ := \prod_{a\in R^\bullet;\, a(0)\geq 0} c_\alpha^{-1}$$

on T_Λ. It is related to the weight function v by

$$v = \mathcal{C}v_+, \qquad \mathcal{C} = \mathcal{C}(\,\cdot\,; D; \kappa, q) := \prod_{\alpha\in R_0^-} c_\alpha. \tag{9.3.12}$$

One recovers v_+ from v up to a multiplicative constant by symmetrization, in view of the following property of the rational function $\mathcal{C} \in \mathbb{C}(T_\Lambda)$ (cf. [67, Theorem (2.8) & (2.8 *nr*)]).

Lemma 9.3.24 *We have*

$$\sum_{w\in W_0} w\mathcal{C} = \mathcal{C}(\gamma^d_{0,q}) \tag{9.3.13}$$

as identity in $\mathbb{C}(T_\Lambda)$, *where* $\gamma^d_{\xi,q} := \gamma_{\xi,q}(D^d; \kappa^{d\bullet}) \in T_\Lambda$ $(\xi \in \Lambda^d)$. *More generally, if* $\xi \in \Lambda^{d-}$ *and if* (κ, q) *is generic then, as identity in* $\mathbb{C}(T_\Lambda)$,

$$\sum_{w\in W_0} w\mathcal{C} = \mathcal{C}(\gamma^d_{\xi,q})\sum_{\eta\in W_0\xi}\prod_{\alpha\in R_0^+\cap v(\eta)R_0^-}\frac{c_\alpha(\gamma^d_{\xi,q})}{c_{-\alpha}(\gamma^d_{\xi,q})},$$

where (recall) $v(\eta) \in W_0$ *is the element of shortest length such that* $v(\eta)\eta = \eta_-$.

The meromorphic function v_+ on T_Λ reads in terms of q-shifted factorials as follows.

$$v_+(t) = \prod_{\beta\in R_0}\frac{(t^{2\beta}; q_\beta^2)_\infty}{(\kappa_\beta\kappa_{2\beta}t^\beta, -\kappa_\beta\kappa_{2\beta}^{-1}t^\beta, q_\beta\kappa_{\mu_\beta c+\beta}\kappa_{2\mu_\beta c+2\beta}t^\beta, -q_\beta\kappa_{\mu_\beta c+\beta}\kappa^{-1}_{2\mu_\beta c+2\beta}t^\beta; q_\beta^2)_\infty}. \tag{9.3.14}$$

It is a nonnegative real-valued continuous function on T^u_Λ (it is nonnegative since it can be written on T^u_Λ as $v_+(t) = |\delta(t)|^2$ with $\delta(t)$ the expression (9.3.14) with product taken only over the set R_0^+ of positive roots).

Corollary 9.3.25 *$\langle\cdot,\cdot\rangle$ restricts to a positive definite, sesquilinear form on $\mathbb{C}[T_\Lambda]^{W_0}$. In fact,*

$$\langle p, r\rangle = \frac{\mathcal{C}(\gamma^d_{0,q})}{\#W_0}\int_{T^u_\Lambda} p(t)\,\overline{r(t)}\,v_+(t)\,d_ut, \qquad p, r\in\mathbb{C}[T_\Lambda]^{W_0}.$$

Symmetrization of the results on the monic nonsymmetric Macdonald–Koornwinder polynomials by using the idempotent $C_+\in H_0$ gives the following properties of the monic symmetric Macdonald–Koornwinder polynomials.

Theorem 9.3.26 *Let $\lambda\in\Lambda^+$.*

(a) *The symmetric Macdonald–Koornwinder polynomial $P^+_\lambda\in\mathbb{C}[T_\Lambda]^{W_0}$ satisfies the following characterizing properties:*

  i. $P^+_\lambda = \sum_{\mu\in\Lambda^+:\mu\le\lambda} d_{\lambda,\mu}m_\mu$ *with* $d_{\lambda,\lambda}=1$,

  ii. $\langle P^+_\lambda, m_\mu\rangle = 0$ *if* $\mu\in\Lambda^+$ *and* $\mu<\lambda$.

(b) $P^{\circ+}_\lambda = P^+_\lambda$.

(c) $\langle P^+_\lambda, P^+_\mu\rangle = 0$ *if* $\mu\in\Lambda^+$ *and* $\mu\neq\lambda$.

9.3.7 GL_n *Macdonald polynomials*

Take $n\ge 2$ and $V := (\epsilon_1+\cdots+\epsilon_n)^\perp\subset\mathbb{R}^n =: Z$ with $\{\epsilon_i\}_{i=1}^n$ the standard orthonormal basis of $\mathbb{R}^n$. Let $R_0 := \{\epsilon_i-\epsilon_j\}_{1\le i\neq j\le n}$ with ordered basis $\Delta_0 := (\alpha_1,\ldots,\alpha_{n-1}) := (\epsilon_1-\epsilon_2,\ldots,\epsilon_{n-1}-\epsilon_n)$. As lattices take $\Lambda=\mathbb{Z}^n=\Lambda^d$. Then $D := (R_0,\Delta_0,u,\mathbb{Z}^n,\mathbb{Z}^n)\in\mathcal{D}$.

The corresponding irreducible reduced affine root system is $R^u = \{mc+\alpha\}_{m\in\mathbb{Z},\,\alpha\in R_0}$, and the corresponding additional simple affine root is $a_0 = c-\epsilon_1+\epsilon_n$. There is no nonreduced extension of R^u involved since $(\Lambda, a_i^\vee)=\mathbb{Z}$ for $i\in\{0,\ldots,n\}$. Hence $R=R^u$.

The fundamental weights $\varpi_j=\varpi^u_j$ ($1\le j\le n-1$) are given by

$$\varpi_j = -\frac{j}{n}(\epsilon_1+\cdots+\epsilon_n)+\epsilon_1+\epsilon_2+\cdots+\epsilon_j.$$

Define for $1\le j\le n$ the elements $\tilde{\varpi}_j := \epsilon_1+\cdots+\epsilon_j$. The orthocomplement $V^\perp$ of V in Z is $\mathbb{R}\tilde{\varpi}_n$ and $\Lambda^d_0 := \Lambda^d\cap V^\perp = \mathbb{Z}\tilde{\varpi}_n$. Then $\tilde{\varpi}_j\in(\varpi_j+V^\perp)\cap\Lambda^d$ ($j\in\{1,\ldots,n-1\}$). Since $-(Da_0)^{d\vee} = \epsilon_1-\epsilon_n = \sum_{i=1}^{n-1}\alpha_i$ we conclude that $J^+_{\Lambda^d} = \{1,\ldots,n-1\}$. The minuscule dominant weights in Λ^d are thus given by $\Lambda^{d+}_{min} = \Lambda^d_0\cup\bigcup_{j=1}^{n-1}(\tilde{\varpi}_j+\Lambda^d_0)$. We have $u(\epsilon_1)=\tau(\epsilon_1)s_1s_2\cdots s_{n-1}$ in the extended affine Weyl group $W\simeq S_n\ltimes\mathbb{Z}^n$ (where S_n is the symmetric group in n letters). Note also that $u(\epsilon_1)(a_j)=a_{j+1}$ for $0\le j<n-1$ and $u(\epsilon_1)(a_{n-1})=a_0$. In addition we have $u(\epsilon_1)^j = u(\tilde{\varpi}_j)$ for $1\le j\le n-1$ and $u(\epsilon_1)^n = u(\tilde{\varpi}_n)=\tau(\tilde{\varpi}_n)$ in W. Hence $\mathbb{Z}\simeq\Omega$ by $m\mapsto u(\epsilon_1)^m$.

Observe that $R=R^u$ has one $W(R^u)$-orbit, hence also one W-orbit. The affine Hecke algebra $H(W(R);\kappa)$ and the extended affine Hecke algebra $H(\kappa)=H(D;\kappa)$ thus depend on a single nonzero complex number κ. $\mathbb{Z}$ acts on $H(W(R);\kappa)$ by algebra automorphisms, where $1\in\mathbb{Z}$ acts on T_i by mapping it to T_{i+1} (reading the subscript modulo n). We write $\mathbb{Z}\ltimes H(W(R);\kappa)$ for the associated semidirect product algebra.

Proposition 9.3.27

(i) *$\mathbb{Z} \ltimes H(W(R);\kappa) \simeq H(\kappa)$ by mapping the generator $1 \in \mathbb{Z}$ to $u(\tilde{\varpi}_1) = u(\epsilon_1)$.*

(ii) *For $1 \leq i \leq n$ we have $Y^{\epsilon_i} = T_{i-1}^{-1} \cdots T_2^{-1} T_1^{-1} u(\epsilon_1) T_{n-1} \cdots T_{i+1} T_i$ in $H(\kappa)$.*

We now turn to the explicit description of the Macdonald–Ruijsenaars q-difference operators and the orthogonality measure for the GL_n Macdonald polynomials.

The longest Weyl group element $w_0 \in W_0$ maps ϵ_i to ϵ_{n+1-i} for $1 \leq i \leq n$, hence

$$R^+ \cap \tau(w_0\tilde{\varpi}_j)R^- = \{\epsilon_r - \epsilon_s \mid 1 \leq r \leq n-j \;\&\; n+1-j \leq s \leq n\}.$$

Write $t_i = t^{\epsilon_i}$ for $t \in T_\Lambda$ and $1 \leq i \leq n$. Then, for $1 \leq j \leq n-1$ we obtain

$$c_{\tau(-w_0\tilde{\varpi}_j)}(t) = \prod_{\substack{1 \leq r \leq n-j \\ n+1-j \leq s \leq n}} \frac{1 - \kappa^2 t_r t_s^{-1}}{1 - t_r t_s^{-1}},$$

and consequently

$$D_{w_0\tilde{\varpi}_j} = \sum_{\substack{I \subset \{1,\ldots,n\}: \\ \#I = n-j}} \left(\prod_{r \in I,\, s \notin I} \frac{1 - \kappa^2 t_r t_s^{-1}}{\kappa(1 - t_r t_s^{-1})} \right) \tau\Big(\sum_{s \notin I} \epsilon_s\Big)_q = \sum_{\substack{I \subset \{1,\ldots,n\}: \\ \#I = j}} \left(\prod_{r \in I,\, s \notin I} \frac{\kappa^{-1} t_r - \kappa t_s}{t_r - t_s} \right) \tau\Big(\sum_{r \in I} \epsilon_r\Big)_q$$

for $1 \leq j \leq n-1$ by Corollary 9.3.18**(i)**. These commutative q-difference operators were introduced by Ruijsenaars [89] as the quantum Hamiltonians of a relativistic version of the trigonometric quantum Calogero–Moser system. These operators also go by the name (trigonometric) *Ruijsenaars* or *Macdonald–Ruijsenaars q-difference operators*.

The corresponding monic *symmetric Macdonald polynomials* $\{P_\lambda\}_{\lambda \in \Lambda^+}$ are parametrized by $\Lambda^+ = \{\lambda \in \mathbb{Z}^n \mid \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n\}$. The orthogonality weight function then becomes

$$v_+(t) = \prod_{1 \leq i \neq j \leq n} \frac{(t_i/t_j;q)_\infty}{(\kappa^2 t_i/t_j;q)_\infty}.$$

The GL_2 Macdonald polynomials are essentially the *continuous q-ultraspherical polynomials* (see [18, Ch. 2] or [73, §6.3]),

$$P_\lambda^+(t) = t_1^{\lambda_1} t_2^{\lambda_2}\, {}_2\phi_1\left(\begin{matrix} \kappa^2, q^{\lambda_2-\lambda_1} \\ q^{1+\lambda_2-\lambda_1}/\kappa^2 \end{matrix}; q, \frac{qt_2}{\kappa^2 t_1} \right), \qquad \lambda = (\lambda_1,\lambda_2) \in \mathbb{Z}^2,\ \lambda_1 \geq \lambda_2,$$

where we use standard notations for the basic hypergeometric ${}_r\phi_s$ series (see, e.g., [36]).

9.3.8 Koornwinder polynomials

Take $Z = V = \mathbb{R}^n$ with standard orthonormal basis $\{\epsilon_i\}_{i=1}^n$. We realize the root system $R_0 \subset V$ of type B_n as $R_0 = \{\pm\epsilon_i\}_{i=1}^n \cup \{\pm\epsilon_i \pm \epsilon_j\}_{1 \leq i < j \leq n}$ (all sign combinations allowed). The W_0-orbits are $\mathcal{O}_l = \{\pm\epsilon_i \pm \epsilon_j\}_{1 \leq i < j \leq n}$ and $\mathcal{O}_s = \{\pm\epsilon_i\}_{i=1}^n$. As an ordered basis of R_0 take

$$\Delta_0 := (\alpha_1, \ldots, \alpha_{n-1}, \alpha_n) := (\epsilon_1 - \epsilon_2, \ldots, \epsilon_{n-1} - \epsilon_n, \epsilon_n).$$

Note that $\mathbb{Z}R_0 = \mathbb{Z}^n$. For $n = 1$ this should be interpreted as $R_0 = \{\pm\epsilon_1\}$, the root system of type A_1, with basis element given by ϵ_1. We consider in this subsection the initial data $D = (R_0, \Delta_0, t, \mathbb{Z}^n, \mathbb{Z}^n) \in \mathcal{D}$.

The associated reduced affine root system is $R^t = \{\frac{1}{2}mc + \alpha\}_{n\in\mathbb{Z}, \alpha\in\mathcal{O}_s} \cup \{mc + \beta\}_{m\in\mathbb{Z}, \beta\in\mathcal{O}_l}$ (for $n = 1$ it should be read as $R^t = \{\frac{1}{2}mc \pm \epsilon_1\}_{m\in\mathbb{Z}}$, the affine root system R^t with R_0 of type A_1). The associated ordered basis of R^t is $\Delta = (a_0, a_1, \ldots, a_n) = (\frac{1}{2}c - \theta, \alpha_1, \ldots, \alpha_n)$ with $\theta = \epsilon_1 = \sum_{j=1}^n \alpha_j$ the highest short root of R_0 with respect to Δ_0. The affine root system R^t has three W-orbits, $R^t = \widehat{\mathcal{O}}_1 \cup \widehat{\mathcal{O}}_2 \cup \widehat{\mathcal{O}}_3$, where $\widehat{\mathcal{O}}_1 = Wa_0 = (\frac{1}{2} + \mathbb{Z})c + \mathcal{O}_s$, $\widehat{\mathcal{O}}_2 = Wa_i = \mathbb{Z}c + \mathcal{O}_l$ $(1 \leq i < n)$ and $\widehat{\mathcal{O}}_3 = Wa_n = \mathbb{Z}c + \mathcal{O}_s$. If $n = 1$ then $\Delta = (a_0, a_1) = (\frac{1}{2}c - \epsilon_1, \epsilon_1)$ and R^t has two W-orbits $\widehat{\mathcal{O}}_1$ and $\widehat{\mathcal{O}}_3$.

Note that $(\Lambda, a_i^\vee) = 2\mathbb{Z}$ for $i = 0$ and $i = n$, hence $S = \{0, n\} = S^d$ and

$$R = R^t \cup W(2a_0) \cup W(2a_n) = \widehat{\mathcal{O}}_1 \cup \widehat{\mathcal{O}}_2 \cup \widehat{\mathcal{O}}_3 \cup \widehat{\mathcal{O}}_4 \cup \widehat{\mathcal{O}}_5$$

with additional W-orbits $\widehat{\mathcal{O}}_4 = 2\widehat{\mathcal{O}}_1$ and $\widehat{\mathcal{O}}_5 = 2\widehat{\mathcal{O}}_3$. If $n = 1$ then R has four W-orbits $\widehat{\mathcal{O}}_i$ $(i = 1, 2, 4, 5)$. Since $D^d = D$ we have $R^d = R$, $\Delta^d = \Delta$ and $W^d = W$.

Suppose that $\kappa \in \mathcal{M}(D)$ and $q \in \mathbb{C}^*$ satisfy (9.3.5). Fix $1 \leq i < n$. Then $\kappa \in \mathcal{M}(D)$ is determined by five (four in case $n = 1$) independent numbers $\kappa_{a_0}, \kappa_{a_i}, \kappa_{a_n}, \kappa_{2a_0}$ and κ_{2a_n}. The corresponding Askey–Wilson parameters [1] are defined by

$$(a, b, c, d, k) = (\kappa_{a_n}\kappa_{2a_n}, -\kappa_{a_n}\kappa_{2a_n}^{-1}, q^{\frac{1}{2}}\kappa_{a_0}\kappa_{2a_0}, -q^{\frac{1}{2}}\kappa_{a_0}\kappa_{2a_0}^{-1}, \kappa_{a_i}^2) \tag{9.3.15}$$

(the parameter k is dropping out in case $n = 1$). The dual multiplicity function κ^d on $R^d = R$ is then determined by $\kappa_{a_0}^d := \kappa_{2a_n}$, $\kappa_{a_i}^d := \kappa_{a_i}$, $\kappa_{a_n}^d := \kappa_{a_n}$, $\kappa_{2a_0}^d := \kappa_{2a_0}$ and $\kappa_{2a_n}^d := \kappa_{a_0}$. The corresponding Askey–Wilson parameters are

$$(\tilde{a}, \tilde{b}, \tilde{c}, \tilde{d}, \tilde{k}) = (\kappa_{a_n}\kappa_{a_0}, -\kappa_{a_n}\kappa_{a_0}^{-1}, q^{\frac{1}{2}}\kappa_{2a_n}\kappa_{2a_0}, -q^{\frac{1}{2}}\kappa_{2a_n}\kappa_{2a_0}^{-1}, \kappa_{a_i}^2).$$

In terms of the Askey–Wilson parameters this can be expressed as $\tilde{k} = k$ and

$$(\tilde{a}, \tilde{b}, \tilde{c}, \tilde{d}) = \left(\sqrt{q^{-1}abcd}, \frac{ab}{\sqrt{q^{-1}abcd}}, \frac{ac}{\sqrt{q^{-1}abcd}}, \frac{ad}{\sqrt{q^{-1}abcd}}\right).$$

Note that

$$-(Da_0)^{d\vee} = \theta^\vee = 2\alpha_1^\vee + 2\alpha_2^\vee + \cdots + 2\alpha_{n-1}^\vee + \alpha_n^\vee.$$

Furthermore, denoting the fundamental weights of R_0 with respect to the ordered basis Δ_0 by $\{\varpi_i\}_{i=1}^n$, we have $\varpi_n = \frac{1}{2}(\epsilon_1 + \cdots + \epsilon_n) \notin \Lambda^d = \mathbb{Z}^n$. Consequently $J_{\Lambda^d}^+ = \emptyset$. Hence the only explicit Macdonald q-difference operator obtainable from Corollary 9.3.18 is $D_{-\theta} = D_{-\epsilon_1}$.

Write

$$\mathcal{D} = \sum_{w\in W_0/W_{0,\epsilon_1}} w(c_{\tau(\epsilon_1)})(\tau(-w\epsilon_1)_q - 1), \tag{9.3.16}$$

so that $D_{-\epsilon_1} = \kappa_{\tau(\epsilon_1)}^{-1}\mathcal{D} + m_{-\epsilon_1}^d(\gamma_{0,q}^{-1})$. Write $t_i = t^{\epsilon_i}$ for $t \in T_{\mathbb{Z}^n}$ and $1 \leq i \leq n$. Since

$$R^{t+} \cap \tau(-\epsilon_1)R^{t-} = \{\epsilon_1, \tfrac{1}{2}c + \epsilon_1\} \cup \{\epsilon_1 \pm \epsilon_j\}_{j=2}^n$$

($= \{\epsilon_1, \frac{1}{2}c + \epsilon_1\}$ if $n = 1$) it follows that $\kappa_{\tau(\epsilon_1)} = \sqrt{q^{-1}abcd}\,k^{n-1}$ and

$$\begin{aligned} c_{\tau(\epsilon_1)}(t) &= c_{\epsilon_1}(t)c_{\frac{1}{2}c+\epsilon_1}(t)\prod_{j=2}^{n} c_{\epsilon_1-\epsilon_j}(t)c_{\epsilon_1+\epsilon_j}(t) \\ &= \frac{(1-at_1)(1-bt_1)(1-ct_1)(1-dt_1)}{(1-t_1^2)(1-qt_1^2)}\prod_{j=2}^{n}\frac{(1-kt_1t_j^{-1})(1-kt_1t_j)}{(1-t_1t_j^{-1})(1-t_1t_j)}. \end{aligned} \tag{9.3.17}$$

For $n = 1$ the product over j is not present in (9.3.17). In particular, the operator $\mathcal{D}$ then only depends on q and on the four parameters a, b, c, d.

Hence $\mathcal{D}$ is Koornwinder's [61] second order q-difference operator

$$\begin{aligned} (\mathcal{D}f)(t) &= \sum_{i=1}^{n}\Big(A_i(t)\big((\tau(-\epsilon_i)_q f)(t) - f(t)\big) + A_i(t^{-1})\big((\tau(\epsilon_i)_q f)(t) - f(t)\big)\Big), \\ A_i(t) &= \frac{(1-at_i)(1-bt_i)(1-ct_i)(1-dt_i)}{(1-t_i^2)(1-qt_i^2)}\prod_{j\neq i}\frac{(1-kt_it_j)(1-kt_it_j^{-1})}{(1-t_it_j)(1-t_it_j^{-1})}, \end{aligned}$$

with the obvious adjustment for $n = 1$, in which case $\mathcal{D}$ is the Askey–Wilson second-order q-difference operator [1]. This derivation is due to Noumi [74].

The monic symmetric Macdonald–Koornwinder polynomials associated to (D, κ) are the monic *symmetric Koornwinder polynomials* [61]. They can be characterised as follows. Note that $\mathbb{C}[T_{\mathbb{Z}^n}] = \mathbb{C}[z_1^{\pm 1}, \ldots, z_n^{\pm 1}]$ with $z_i = e^{\epsilon_i}$. The finite Weyl group W_0 is the *hyperoctahedral group* $W_0 \simeq S_n \ltimes \{\pm 1\}^n$ acting on $\mathbb{C}[T_{\mathbb{Z}^n}]$ by permutations and inversions of the variables z_i. The symmetric Koornwinder polynomials are then parametrised by

$$\Lambda^+ := \{\lambda \in \mathbb{Z}^n \mid \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq 0\},$$

which we consider as a partially ordered set with respect to the *dominance order*: $\lambda \geq \mu$ if $\sum_{j=1}^{i}\lambda_j \geq \sum_{j=1}^{i}\mu_j$ for all i. The symmetric monomials are $m_\lambda = \sum_{\mu \in W_0\lambda} z^\mu \in \mathbb{C}[T_{\mathbb{Z}^n}]^{W_0}$ for $\lambda \in \Lambda^+$, with W_0 acting by permutations and sign changes on $\mathbb{Z}^n$. The monic symmetric Koornwinder polynomial $P_\lambda^+(\cdot) = P_\lambda^+(\cdot\,; a, b, c, d; q, k)$ of degree $\lambda \in \Lambda^+$ is now uniquely characterised by the eigenvalue equation

$$\mathcal{D}P_\lambda^+ = \left(\sum_{i=1}^{n}\Big(q^{-1}abcdk^{2n-i-1}(q^{\lambda_i} - 1) + k^{i-1}(q^{-\lambda_i} - 1)\Big)\right)P_\lambda^+$$

and the property that $P_\lambda^+ = m_\lambda + \sum\limits_{\mu \in \Lambda^+;\, \mu < \lambda} c_{\lambda,\mu}m_\mu$ for certain $c_{\lambda,\mu} \in \mathbb{C}$.

The weight function $v_+(t)$ becomes

$$v_+(t) = \prod_{i=1}^{n}\frac{(t_i^2, t_i^{-2}; q)_\infty}{(at_i, at_i^{-1}, bt_i, bt_i^{-1}, ct_i, ct_i^{-1}, dt_i, dt_i^{-1}; q)_\infty}\prod_{1\leq r<s\leq n}\frac{(t_rt_s, t_r^{-1}t_s^{-1}, t_rt_s^{-1}, t_r^{-1}t_s; q)_\infty}{(kt_rt_s, kt_r^{-1}t_s^{-1}, kt_rt_s^{-1}, kt_r^{-1}t_s; q)_\infty}. \tag{9.3.18}$$

For $n = 1$ the Koornwinder polynomials are the Askey–Wilson polynomials [1]. Concretely, with standard notations for basic hypergeometric series (cf. [36]), we have

$$P_m^+(t) = \frac{(ab, ac, ad; q)_m}{a^m(q^{m-1}abcd; q)_m}\, {}_4\phi_3\left(\begin{matrix} q^{-m}, q^{m-1}abcd, at, at^{-1} \\ ab, ac, ad \end{matrix}; q, q\right), \quad m \in \mathbb{Z}_{\geq 0} = \Lambda^+.$$

The Cherednik–Macdonald theory associated to the Askey–Wilson polynomials was worked out in detail in [76].

Example 9.3.28 Consider the initial data $D = (R_0, \Delta_0, t, \mathbb{Z}^n, P(R_0)) \in \mathcal{D}$ with the root system $R_0 \subset \mathbb{R}^n$ of type B_n ($n \geq 2$) and the ordered basis Δ_0 defined in terms of the standard orthonormal basis $\{\epsilon_i\}_{i=1}^n$ of $\mathbb{R}^n$ as above. Compared to the Koornwinder setup just discussed we have thus chosen a different lattice $\Lambda^d = P(R_0)$, which contains $\mathbb{Z}R_0 = \mathbb{Z}^n$ as index two sublattice. Still $S(D) = \{0, n\}$, hence R is the nonreduced affine root system with five $W_0 \ltimes \mathbb{Z}^n$-orbits $\widehat{\mathcal{O}}_i$ ($1 \leq i \leq 5$) as introduced above. But the number $\nu = \nu(D)$ of $W_0 \ltimes P(R_0)$-orbits is three: they are given by $\widehat{\mathcal{O}}_2$, $\widehat{\mathcal{O}}_1 \cup \widehat{\mathcal{O}}_3$ and $\widehat{\mathcal{O}}_4 \cup \widehat{\mathcal{O}}_5$.

On the other hand, $R^d = R^t$ since $S(D^d) = \emptyset$, in particular R^d is reduced. It has the three $W^d = W_0 \ltimes \mathbb{Z}^n$-orbits $\widehat{\mathcal{O}}_i$ ($i = 1, 2, 3$).

Remark 9.3.29 Consider the initial data $D = (R_0, \Delta_0, u, \mathbb{Z}^n, \mathbb{Z}^n)$ with (R_0, Δ_0) of type B_n ($n \geq 3$) as above. Compared to initial data $D^{\mathrm{C}^\vee\mathrm{C}_n} := (R_0, \Delta_0, t, \mathbb{Z}^n, \mathbb{Z}^n)$ related to Koornwinder polynomials, we thus have only changed the type from twisted to untwisted. Then R^u is the affine root subsystem $\widehat{\mathcal{O}}_2 \cup \widehat{\mathcal{O}}_3$ of the affine root system $R^{\mathrm{C}^\vee\mathrm{C}_n}$ of type $\mathrm{C}^\vee\mathrm{C}_n$ as defined above, and $R = R(D)$ is the nonreduced irreducible affine root subsystem $\widehat{\mathcal{O}}_2 \cup \widehat{\mathcal{O}}_3 \cup \widehat{\mathcal{O}}_5$ of $R(D^{\mathrm{C}^\vee\mathrm{C}_n})$. The dual affine root system $R^d = R(D^d)$ is the reduced irreducible affine root subsystem $R^d = \widehat{\mathcal{O}}_2 \cup \widehat{\mathcal{O}}_4 \cup \widehat{\mathcal{O}}_5$ (it is of untwisted type, with underlying finite root system $R_0^\vee$ of type C_n). The Macdonald–Koornwinder theory associated to the initial data D thus is the special case of the $\mathrm{C}^\vee\mathrm{C}_n$ theory when the multiplicity function $\kappa \in \mathcal{M}(D^{\mathrm{C}^\vee\mathrm{C}_n})$ takes the value one at the two orbits $\widehat{\mathcal{O}}_1 = W(a_0)$ and $\widehat{\mathcal{O}}_4 = W(2a_0)$ of $R^{\mathrm{C}^\vee\mathrm{C}_n} \setminus R$.

9.3.9 A new class of nonreduced rank two Macdonald–Koornwinder polynomials

Consider $R_0 \subset Z = V = \mathbb{R}^2$ the root system of type C_2 given by $R_0 = R_{0,s} \cup R_{0,l}$ with

$$R_{0,s} = \{\pm(\epsilon_1 + \epsilon_2), \pm(\epsilon_1 - \epsilon_2)\}, \qquad R_{0,l} = \{\pm 2\epsilon_1, \pm 2\epsilon_2\},$$

where $\{\epsilon_1, \epsilon_2\}$ is the standard orthonormal basis of $\mathbb{R}^2$. As ordered basis we take $\Delta_0 := (\alpha_1, \alpha_2)$ with $\alpha_1 := \epsilon_1 - \epsilon_2$ and $\alpha_2 := 2\epsilon_2$. Then $\varphi = 2\epsilon_1$ and $\theta = \epsilon_1 + \epsilon_2$. As quintuple of initial data we take

$$D = (R_0, \Delta_0, \bullet, \Lambda, \Lambda^d) := (R_0, \Delta_0, u, \mathbb{Z}R_0, \mathbb{Z}R_0^\vee).$$

Hence $\Lambda^d = \mathbb{Z}\epsilon_1 \oplus \mathbb{Z}\epsilon_2 \simeq \mathbb{Z}^2$ while $\Lambda = \{\lambda = (\lambda_1, \lambda_2) \in \mathbb{Z}^2 \mid \lambda_1 + \lambda_2 \text{ even}\}$. Hence the simple affine root a_0 of the associated reduced affine root system $R^u = \mathbb{Z}c + R_0$ is $c - 2\epsilon_1$. Then $S = \{1\}$. Hence $R = R^u \cup W(2\alpha_1)$ with $W = W^u = W_0 \ltimes \mathbb{Z}R_0^\vee$. Then R has four W-orbits,

$$W(a_0) = (2\mathbb{Z}+1)c + R_{0,l}, \quad W(\alpha_1) = \mathbb{Z}c + R_{0,s}, \quad W(\alpha_2) = 2\mathbb{Z}c + R_{0,l}, \quad W(2\alpha_1) = 2\mathbb{Z}c + 2R_{0,s}.$$

For the dual initial data $D^d = (R_0^\vee, \Delta_0^\vee, u, \mathbb{Z}R_0^\vee, \mathbb{Z}R_0)$ we have $R^{ud} = \mathbb{Z}c + R_0^\vee$ and furthermore $R^d = R^{ud} \cup W^d(2\alpha_2^\vee)$ with $W^d = W^{ud} = W_0 \ltimes \mathbb{Z}R_0$. The simple affine root a_0^d of R^{ud} is $a_0^d = c - \epsilon_1 - \epsilon_2$. The four W^d-orbits of R^d are

$$W^d(a_0^d) = (2\mathbb{Z}+1)c + R_{0,s}^\vee, \quad W^d(\alpha_1^\vee) = 2\mathbb{Z}c + R_{0,s}^\vee, \quad W^d(\alpha_2^\vee) = \mathbb{Z}c + R_{0,l}^\vee, \quad W^d(2\alpha_2^\vee) = 2\mathbb{Z}c + 2R_{0,l}^\vee.$$

Note that $R \simeq R^d$ but $(R, \Delta) \not\simeq (R^d, \Delta^d)$.

Let $\kappa \in \mathcal{M}(D)$. We write

$$\{a, b, c, d\} := \{\kappa_\theta\kappa_{2\theta}, -\kappa_\theta\kappa_{2\theta}^{-1}, \kappa_\varphi^2, q\kappa_0^2\}, \qquad \{\tilde{a}, \tilde{b}, \tilde{c}, \tilde{d}\} = \{\kappa_\varphi\kappa_0, -\kappa_\varphi\kappa_0^{-1}, \kappa_\theta^2, q\kappa_{2\theta}^2\} \tag{9.3.19}$$

(the dual parameters $(\tilde{a}, \tilde{b}, \tilde{c}, \tilde{d})$ are the parameters (a, b, c, d) with respect to the dual initial data D^d).

We view $\mathbb{C}[T_\Lambda]$ as the subalgebra $\bigoplus_{\lambda\in\Lambda} \mathbb{C}e^\lambda$ of $\mathbb{C}[T_{\mathbb{Z}^2}]$, which induces an embedding of $\mathbb{C}(T_\Lambda)$ as subfield of $\mathbb{C}(T_{\mathbb{Z}^2})$. More concretely, if $z_i := e^{\epsilon_i}$ for the standard coordinates of $\mathbb{C}[T_{\mathbb{Z}^2}]$, then $\mathbb{C}[T_\Lambda]$ is the subalgebra of $\mathbb{C}[T_{\mathbb{Z}^2}] = \mathbb{C}[z_1^{\pm 1}, z_2^{\pm 1}]$ generated by z_1z_2, z_2^2 and their inverses. We write $t_i = t^{\epsilon_i}$ for $i = 1, 2$ if $t \in T_{\mathbb{Z}^2} \subset T_\Lambda$. We sometimes abuse notation by writing $t_1^{\lambda_1}t_2^{\lambda_2}$ for t^λ $(t \in T_\Lambda, \lambda \in \Lambda)$.

The monic symmetric Macdonald–Koornwinder polynomials associated to (D, κ) can be characterized as eigenfunctions of the q-difference operator $D_{(Da_0)^d} = D_{-\epsilon_1}$ (see Theorem 9.3.22 and Remark 9.3.23). Write

$$\mathcal{D} := \alpha(D_{-\epsilon_1} - m^d_{-\epsilon_1}(\gamma_{0,q}^{-1})), \qquad \alpha := \sqrt{q^{-1}a^2b^2cd}. \tag{9.3.20}$$

Since $R^{u,+} \cap \tau(-\epsilon_1)R^{u,-} = \{\epsilon_1 + \epsilon_2, \epsilon_1 - \epsilon_2, 2\epsilon_1, c + 2\epsilon_1\}$ and $\gamma_{0,q} = \prod_{\alpha\in R_0^+} \upsilon_{\alpha^\vee}^{-\alpha} = (\alpha^{-1}, -ab\alpha^{-1})$, we have by Corollary 9.3.18 that

$$(\mathcal{D}f)(t) = \sum_{i=1}^{2} \Big(A_i(t)\big((\tau(-\epsilon_i)_q f)(t) - f(t)\big) + A_i(t^{-1})\big((\tau(\epsilon_i)_q f)(t) - f(t)\big)\Big), \qquad f \in \mathbb{C}[T_\Lambda]^{W_0},$$

with, for $t \in T_\Lambda$,

$$A_1(t) = \frac{(1-ct_1^2)(1-dt_1^2)}{(1-t_1^2)(1-qt_1^2)}\,\frac{(1-at_1t_2)(1-at_1t_2^{-1})(1-bt_1t_2)(1-bt_1t_2^{-1})}{(1-t_1^2t_2^2)(1-t_1^2t_2^{-2})},$$

$$A_2(t) = \frac{(1-ct_2^2)(1-dt_2^2)}{(1-t_2^2)(1-qt_2^2)}\,\frac{(1-at_2t_1)(1-at_2t_1^{-1})(1-bt_2t_1)(1-bt_2t_1^{-1})}{(1-t_2^2t_1^2)(1-t_2^2t_1^{-2})}.$$

The monic symmetric Macdonald–Koornwinder polynomial $P_\lambda^+(\cdot) = P_\lambda^+(\cdot\,; a, b, c, d; q) \in \mathbb{C}[T_\Lambda]^{W_0}$ associated to (D, κ) $(\lambda \in \Lambda^+)$ is the unique eigenfunction of the second order q-difference operator $\mathcal{D}$ with eigenvalue $\alpha^2(q^{\lambda_1}-1) - \alpha^2(ab)^{-1}(q^{\lambda_2}-1) + (q^{-\lambda_1}-1) - ab(q^{-\lambda_2}-1)$ satisfying $P_\lambda^+ = m_\lambda + \sum_{\mu\in\Lambda^+;\,\mu<\lambda} c_{\lambda,\mu}m_\mu$ for some $c_{\lambda,\mu} \in \mathbb{C}$, see Remark 9.3.23. Note here that

$$\Lambda^+ = \{\lambda = (\lambda_1, \lambda_2) \in \mathbb{Z}^2_{\geq 0} \mid \lambda_1 \geq \lambda_2 \ \&\ \lambda_1 + \lambda_2 \ \text{even}\}.$$

The P_λ^+ are orthogonal with respect to the pairing

$$\langle p, r\rangle_+ = \int_{T_\Lambda^u} p(t)\,\overline{r(t)}\,v_+(t)\,d_ut, \qquad p, r \in \mathbb{C}[T_\Lambda]^{W_0},$$

with weight function $v_+(t) = \delta(t)\delta(t^{-1})$ given by

$$\delta(t) := \frac{(t_1^2, t_2^2; q)_\infty}{(ct_1^2, ct_2^2, dt_1^2, dt_2^2; q^2)_\infty} \, \frac{(t_1t_2, t_1t_2^{-1}, -t_1t_2, -t_1t_2^{-1}; q)_\infty}{(at_1t_2, at_1t_2^{-1}, bt_1t_2, bt_1t_2^{-1}; q)_\infty}, \qquad t \in T_\Lambda.$$

For $t \in T_{\mathbb{Z}^2}$ it can be rewritten as

$$\delta(t) = \left(\prod_{i=1}^{2} \frac{(t_i^2; q)_\infty}{(\sqrt{c}\, t_i, -\sqrt{c}\, t_i, \sqrt{d}\, t_i, -\sqrt{d}\, t_i; q)_\infty} \right) \frac{(t_1t_2, t_1t_2^{-1}, -t_1t_2, -t_1t_2^{-1}; q)_\infty}{(at_1t_2, at_1t_2^{-1}, bt_1t_2, bt_1t_2^{-1}; q)_\infty}.$$

As far as we know, the $\{P_\lambda^+\}_{\lambda\in\Lambda^+}$ is a four parameter family of two-variable symmetric Macdonald–Koornwinder polynomials which has not appeared before in the literature.

Subfamilies of $\{P_\lambda^+\}_{\lambda\in\Lambda^+}$ can be related to symmetric Macdonald–Koornwinder polynomials. We give here one example by relating a three parameter subfamily of $\{P_\lambda\}_{\lambda\in\Lambda^+}$ to the rank two symmetric Koornwinder polynomial from §9.3.8. To distinguish the Koornwinder case from the present setup we will add a label K; in particular we write $P_\lambda^{\mathrm{K}+} = P_\lambda^{\mathrm{K}+}(\,\cdot\,; a, b, c, d; q, k) \in \mathbb{C}[T_{\mathbb{Z}^2}]^{W_0}$ for the $n = 2$ monic symmetric Koornwinder polynomial of degree $\lambda \in \Lambda^{\mathrm{K}+}$, where $\Lambda^{\mathrm{K}+} = \{(\lambda_1, \lambda_2) \in \mathbb{Z}_{\geq 0}^2 \mid \lambda_1 \geq \lambda_2\}$. By comparison of the characterisations of $P_\lambda \in \mathbb{C}[T_\Lambda]^{W_0}$ and $P_\lambda^{\mathrm{K}+} \in \mathbb{C}[T_{\mathbb{Z}^2}]^{W_0}$ for $\lambda \in \Lambda^+$ as eigenfunction of an explicit second-order q-difference operator $\mathcal{D}$ (9.3.20) and $\mathcal{D}^{\mathrm{K}}$ (9.3.16) respectively, we conclude that for $\lambda \in \Lambda^+$,

$$P_\lambda^+(\,\cdot\,; a, -1, \sqrt{c}, -\sqrt{c}, \sqrt{d}, -\sqrt{d}; q) = P_\lambda^{\mathrm{K}+}(\,\cdot\,; \sqrt{c}, -\sqrt{c}, \sqrt{d}, -\sqrt{d}; q, a)$$

as elements in $\mathbb{C}[T_\Lambda]^{W_0} \subset \mathbb{C}[z_1^{\pm 1}, z_2^{\pm 1}]^{W_0}$. Possibly there are other ways to relate special cases of P_λ^+ to rank two symmetric Koornwinder polynomials, for instance through quadratic transformations, cf. [62, §3.3], [87].

9.4 Double affine Hecke algebras and normalized Macdonald–Koornwinder polynomials

Cherednik's [9, 12] double affine braid group and double affine Hecke algebra are fundamental for proving properties of the (non)symmetric Macdonald–Koornwinder polynomials such as evaluation formula, duality and quadratic norms (see [13, 14]). We discuss these results in this section.

The first part closely follows [41]. We define a double affine braid group and a double affine Hecke algebra depending on q, on the initial data D and, in case of the double affine Hecke algebra, on a choice of a multiplicity function $\kappa \in \mathcal{M}(D)$. We extend the duality $D \mapsto D^d$ on initial data to an anti-isomorphism of the associated double affine braid groups, and to an algebra anti-isomorphism of the associated double affine Hecke algebras (on the dual side, the double affine Hecke algebra is taken with respect to the dual multiplicity function $\kappa^d \in \mathcal{M}(D^d)$ as defined in Lemma 9.2.13). These anti-isomorphisms are called *duality anti-isomorphisms* and find their origins in the work of Cherednik [13, 14] (see also [45, 91, 73, 41] for further results and generalizations).

As a consequence of the duality anti-isomorphism and of the theory of intertwiners we first derive the evaluation formula and duality of the Macdonald–Koornwinder polynomials following closely [14, 16, 91]. These results, together with quadratic norm formulas, were first conjectured by Macdonald (see, e.g., [72, 69]). For R_0 of type A the evaluation formula and duality were proven by Koornwinder [60] by different methods (see [70, Ch. VI] for a detailed account). Subsequently Cherednik [13] established the evaluation formula and duality when $(\Lambda, \Lambda^d) = (P(R_0), P(R_0^d))$. The $\mathrm{C}^\vee\mathrm{C}$ case was established using Koornwinder's [60] methods by van Diejen [27] for a suitable subset of multiplicity functions. Cherednik's double affine Hecke algebra methods were extended to the $\mathrm{C}^\vee\mathrm{C}$ case in the work of Noumi [74] and Sahi [91], leading to the evaluation formula and duality for all multiplicity functions. Our uniform approach is close to Haiman's [41]. Following Cherednik [14, 16], we will use the duality anti-isomorphism and intertwiners to establish quadratic norm formulas for the (non)symmetric Macdonald–Koornwinder polynomials.

We fix throughout this section initial data $D = (R_0, \Delta_0, \bullet, \Lambda, \Lambda^d) \in \mathcal{D}$, a deformation parameter $q \in \mathbb{R}_{>0} \setminus \{1\}$, and a multiplicity function $\kappa \in \mathcal{M}(D)$.

9.4.1 Double affine braid groups, Weyl groups and Hecke algebras

Consider the W-stable additive subgroup $\widehat{\Lambda} := \Lambda + \mathbb{R}c$ of $\widehat{Z}$. In the following definition it is convenient to write $X^{\widehat{\lambda}}$ ($\widehat{\lambda} \in \widehat{\Lambda}$) for the elements of $\widehat{\Lambda}$. In particular, $X^{\widehat{\lambda}}X^{\widehat{\mu}} = X^{\widehat{\lambda}+\widehat{\mu}}$ for $\widehat{\lambda}, \widehat{\mu} \in \widehat{\Lambda}$ and $X^0 = 1$.

Set $A = A(R, \Delta)$ and $A_0 = A(R_0, \Delta_0)$. The generators of the affine braid group $\mathcal{B}^\bullet = \mathcal{B}(A)$ are denoted by $T_0, T_1, \ldots, T_n$, the generators of $\mathcal{B}_0 = \mathcal{B}(A_0)$ by $T_1, \ldots, T_n$. Recall that elements $T_w \in \mathcal{B}$ ($w \in W^\bullet$) and $T_w \in \mathcal{B}_0$ ($w \in W_0$) can be defined using reduced expressions of w in the Coxeter groups $W^\bullet = W(A^\bullet)$ and $W_0 = W(A_0)$, respectively. Recall furthermore that $\mathcal{B} = \mathcal{B}(D) \simeq \Omega \ltimes \mathcal{B}^\bullet$ denotes the extended affine braid group (cf. Definition 9.2.17).

Definition 9.4.1 The *double affine braid group* $\mathbb{B} = \mathbb{B}(D)$ is the group generated by the groups $\mathcal{B}$ and $\widehat{\Lambda}$ together with the relations:

$$T_i X^{\widehat{\lambda}} = X^{\widehat{\lambda}} T_i \qquad \text{if } \widehat{\lambda} \in \widehat{\Lambda} \text{ and } 0 \le i \le n \text{ such that } (\widehat{\lambda}, a_i^\vee) = 0; \tag{9.4.1}$$

$$T_i X^{\widehat{\lambda}} T_i = X^{s_i\widehat{\lambda}} \qquad \text{if } \widehat{\lambda} \in \widehat{\Lambda} \text{ and } 0 \le i \le n \text{ such that } (\widehat{\lambda}, a_i^\vee) = 1; \tag{9.4.2}$$

$$\omega X^{\widehat{\lambda}} = X^{\omega\widehat{\lambda}} \omega \qquad \text{if } \widehat{\lambda} \in \widehat{\Lambda} \text{ and } \omega \in \Omega. \tag{9.4.3}$$

Note that $X^{\mathbb{R}c}$ is contained in the center of $\mathbb{B}$. It is not necessarily true that any element $g \in \mathbb{B}$ can be written as $g = bX^{\widehat{\lambda}}$ (or $g = X^{\widehat{\lambda}}b$) with $b \in \mathcal{B}$ and $\widehat{\lambda} \in \widehat{\Lambda}$. This possibly fails to be true in the cases where $S(D) \neq \emptyset$ (these are the cases for which $R \neq R^\bullet$, i.e. for which nonreduced extensions of $R^\bullet$ play a role in the theory, cf. §9.2.4). Straightening the elements of $\mathbb{B}$ is always possible as soon as quotients to double affine Weyl groups and double affine Hecke algebras are taken, as we shall see in a moment.

Recall from §9.3.1 the construction of commuting group elements $Y^\xi \in \mathcal{B}$ ($\xi \in \Lambda^d$). It leads to a presentation of $\mathcal{B}$ in terms of the group Λ^d (with its elements denoted by Y^ξ, $\xi \in \Lambda^d$) and the braid group $\mathcal{B}_0 = \mathcal{B}(A_0)$, see e.g. [73, §3.3]. On the level of double affine braid groups it implies the following alternative presentation of $\mathbb{B}$ (recall that Da_0 equals $-\varphi$ if $\bullet = u$ and $-\theta$ if $\bullet = t$).

Proposition 9.4.2 *The double affine braid group $\mathbb{B}$ is isomorphic to the group generated by the groups $\mathcal{B}_0$, Λ^d and $\widehat{\Lambda}$, satisfying for $1 \le i \le n$, $\lambda \in \Lambda$ and $\xi \in \Lambda^d$:*

a. $X^{\mathbb{R}c}$ *is contained in the center,*

b. 1. $Y^{-\xi}T_i = T_iY^{-\xi}$ *if* $(\xi, \alpha_i^{d\vee}) = 0$,

2. $T_iX^\lambda = X^\lambda T_i$ *if* $(\lambda, \alpha_i^\vee) = 0$,

c. 1. $T_iY^{-\xi}T_i = Y^{-s_i\xi}$ *if* $(\xi, \alpha_i^{d\vee}) = 1$,

2. $T_iX^\lambda T_i = X^{s_i\lambda}$ *if* $(\lambda, \alpha_i^\vee) = 1$,

d. *if* $(\lambda, a_0^\vee) = 0$ *then* $(Y^{-(Da_0)^d}T^{-1}_{s_{Da_0}})X^\lambda = X^\lambda(Y^{-(Da_0)^d}T^{-1}_{s_{Da_0}})$,

e. *if* $(\lambda, a_0^\vee) = 1$ *then* $(Y^{-(Da_0)^d}T^{-1}_{s_{Da_0}})X^\lambda(Y^{-(Da_0)^d}T^{-1}_{s_{Da_0}}) = p^{-1}_{Da_0}X^{s_{Da_0}\lambda}$, *where* $p_\alpha := X^{\mu_\alpha c}$ *for* $\alpha \in R_0$.

Recall that $u(\eta) \in W \simeq W_0 \ltimes \Lambda^d$ denotes the element of minimal length in $\tau(\eta)W_0$ for all $\eta \in \Lambda^d$. Then $u(\eta) = \tau(\eta)v(\eta)^{-1}$ with $v(\eta) \in W_0$ the element of minimal length such that $v(\eta)\eta = \eta_-$. Recall furthermore that $\Omega = \{u(\xi) \,|\, \xi \in \Lambda^{d+}_{min}\}$.

The identification of the two different sets of generators of $\mathbb{B}$ is as follows: $T_0 = Y^{-(Da_0)^d}T^{-1}_{s_{Da_0}}$ and $u(\xi) = Y^\xi T^{-1}_{v(\xi)}$ for $\xi \in \Lambda^{d+}_{min}$. Conversely, for $\xi = \eta_1 - \eta_2 \in \Lambda^d$ with $\eta_1, \eta_2 \in \Lambda^{d+}$, we have $Y^\xi = Y^{\eta_1}(Y^{\eta_2})^{-1}$ and $Y^{\eta_s} = T_{\tau(\eta_s)} = \omega T_{i_1}T_{i_2}\cdots T_{i_r}$ if $\tau(\eta_s) = \omega s_{i_1}s_{i_2}\cdots s_{i_r} \in W$ is a reduced expression ($\omega \in \Omega$ and $0 \le i_j \le n$).

Recall the set $S = S(D)$ given by (9.2.5). Write

$$V_i := X^{-a_i}T_i^{-1} \in \mathbb{B} \qquad \forall i \in S. \tag{9.4.4}$$

Recall furthermore that $R = R^\bullet \cup \bigcup_{i\in S} W^\bullet(2a_i)$.

Definition 9.4.3

(i) The *double affine Weyl group* $\mathbb{W} = \mathbb{W}(D)$ is the quotient of $\mathbb{B}$ by the normal subgroup generated by T_j^2 and V_i^2 for $j \in \{0, \ldots, n\}$ and $i \in S$.

(ii) The *double affine Hecke algebra* $\mathbb{H}(\kappa, q) = \mathbb{H}(D; \kappa, q)$ is $\mathbb{C}[\mathbb{B}]/\widehat{I_{\kappa,q}}$, where $\widehat{I_{\kappa,q}}$ is the two-sided ideal of $\mathbb{C}[\mathbb{B}]$ generated by $X^{rc} - q^r$, $(T_j - \kappa_j)(T_j + \kappa_j^{-1})$ and $(V_i - \kappa_{2a_i})(V_i + \kappa_{2a_i}^{-1})$ for $r \in \mathbb{R}$, $j \in \{0, \ldots, n\}$ and $i \in S$.

Recall that W acts on $\widehat{\Lambda}$ by group automorphisms.

Proposition 9.4.4 $\mathbb{W} \simeq W \ltimes \widehat{\Lambda}$.

For the double affine Hecke algebra, note that we have canonical algebra homomorphisms $H(\kappa^\bullet) \to \mathbb{H}(\kappa, q)$ and $\mathbb{C}[\widehat{\Lambda}] \to \mathbb{H}(\kappa, q)$. We write $\tilde{h}$, $\tilde{X}_q^{\widehat{\lambda}}$ and $\tilde{X}^\lambda$ for the images of $h \in \widehat{H}$, $X^{\widehat{\lambda}} \in \mathbb{C}[\widehat{\Lambda}]$ ($\widehat{\lambda} \in \widehat{\Lambda}$) and $X^\lambda \in \mathbb{C}[\Lambda]$ ($\lambda \in \Lambda$) in $\mathbb{H}(\kappa, q)$, respectively. Define a linear map

$$m\colon H(\kappa^\bullet) \otimes_{\mathbb{C}} \mathbb{C}[\Lambda] \to \mathbb{H}(\kappa, q), \quad h \otimes X^\lambda \mapsto \tilde{h}\,\tilde{X}^\lambda \qquad (h \in H(\kappa^\bullet),\ \lambda \in \Lambda).$$

If $\widehat{\lambda} = \lambda + rc \in \widehat{\Lambda}$ then we interpret $e_q^{\widehat{\lambda}}$ as the endomorphism of $\mathbb{C}[T_\Lambda]$ by

$$(e_q^{\widehat{\lambda}}p)(t) := t_q^{\widehat{\lambda}}p(t) = q^r t^\lambda p(t), \qquad r \in \mathbb{R},\ \lambda \in \Lambda.$$

Theorem 9.4.5

(i) *We have a unique algebra monomorphism* $\mathbb{H}(\kappa, q) \hookrightarrow \mathrm{End}_{\mathbb{C}}(\mathbb{C}[T_\Lambda])$ *defined by*

$$\tilde{h} \mapsto \pi_{\kappa,q}(h) \quad (h \in H(\kappa^\bullet)), \qquad \tilde{X}_q^{\widehat{\lambda}} \mapsto e_q^{\widehat{\lambda}} \quad (\widehat{\lambda} \in \widehat{\Lambda}).$$

(ii) *The linear map* m *defines a complex linear isomorphism* $H(\kappa^\bullet) \otimes_{\mathbb{C}} \mathbb{C}[\Lambda] \xrightarrow{\sim} \mathbb{H}(\kappa, q)$.

To simplify notations, we omit the tilde when writing the elements in $\mathbb{H}$. With this convention $X_q^{\widehat{\lambda}} = q^r X^\lambda$ in $\mathbb{H}(\kappa, q)$ if $\widehat{\lambda} = \lambda + rc$.

Together with the Bernstein–Zelevinsky presentation of the extended affine Hecke algebra $H(\kappa^\bullet)$ (see §9.3.1) we conclude that $\mathbb{C}[T_{\Lambda^d}] \otimes_{\mathbb{C}} H_0(\kappa|_{R_0}) \otimes_{\mathbb{C}} \mathbb{C}[T_\Lambda] \simeq \mathbb{H}(\kappa, q)$ as complex vector spaces by mapping $e^\xi \otimes h \otimes e^\lambda$ to $Y^\xi h X^\lambda$ ($\xi \in \Lambda^d$, $h \in H_0(\kappa|_{R_0})$ and $\lambda \in \Lambda$). This is the *Poincaré–Birkhoff–Witt property* of the double affine Hecke algebra $\mathbb{H}(\kappa, q)$.

The *dual version of the cross relations* (9.3.2) and (9.3.3), now also including a commutation relation for T_0, is given as follows. Write $p(X) \in \mathbb{H}(\kappa, q)$ for the element corresponding to $p \in \mathbb{C}[T_\Lambda]$. In other words, $p(X) = \sum_{\lambda\in\Lambda} c_\lambda X^\lambda$ if $p(t) = \sum_{\lambda\in\Lambda} c_\lambda t^\lambda$.

Corollary 9.4.6 *Let* $0 \leq i \leq n$ *and* $p \in \mathbb{C}[T_\Lambda] \subset \mathbb{H}(\kappa, q)$. *Then in* $\mathbb{H}(\kappa, q)$ *we have*

$$T_i p(X) - (s_{i,q}p)(X)T_i = \left(\frac{\kappa_i - \kappa_i^{-1} + (\kappa_{2a_i} - \kappa_{2a_i}^{-1})X_q^{a_i}}{1 - X_q^{2a_i}}\right)(p(X) - (s_{i,q}p)(X)) \tag{9.4.5}$$

9.4.2 Duality anti-isomorphism

Recall from (9.2.8) that we have associated to $D = (R_0, \Delta_0, \bullet, \Lambda, \Lambda^d)$ the dual initial data $D^d = (R_0^d, \Delta_0^d, \bullet, \Lambda^d, \Lambda)$. We define the *dual double affine braid group* by $\mathbb{B}^d := \mathbb{B}(D^d)$. We add a superscript d to elements of $\mathbb{B}^d$ if confusion may arise. So we write ${}^dY^\lambda$ ($\lambda \in \Lambda$), ${}^dX^\xi$ ($\xi \in \Lambda^d$) and T_i^d ($0 \leq i \leq n$) in $\mathbb{B}^d$. The group generators T_i^d ($1 \leq i \leq n$) of the homomorphic image of $\mathcal{B}_0$ in $\mathbb{B}^d$ will usually be written without superscripts.

Recall that $Da_0 = -\varphi$ or $-\theta$ if $\bullet = u$ or t, respectively. Then $s_0 = \tau(-Da_0)^d)\, s_{Da_0} \in W^\bullet$ and $T_0 = Y^{-(Da_0)^d} T_{s_{Da_0}}^{-1} \in \mathbb{B}$. Dually, $D(a_0^d) = -\theta^d$, $s_0^d = \tau(\theta)s_\theta \in W^d$ and $T_0^d = Y^\theta T_{s_\theta}^{-1} \in \mathbb{B}^d$. The following result, in the present generality, is from [41, §4] (in [41] the duality isomorphism is constructed, which is related to the duality anti-isomorphism below via an elementary anti-isomorphism). See also [13, 14, 45, 91, 73] for special cases.

Theorem 9.4.7 *There exists a unique anti-isomorphism* $\delta\colon \mathbb{B} \to \mathbb{B}^d$ *satisfying*

$$\begin{aligned} &\delta(X^{rc}) = X^{rc} \quad (r \in \mathbb{R}), && \delta(Y^\xi) = {}^dX^{-\xi} \quad (\xi \in \Lambda^d),\\ &\delta(T_i) = T_i \quad (i \in \{1, \ldots, n\}), && \delta(X^\lambda) = {}^dY^{-\lambda} \quad (\lambda \in \Lambda). \end{aligned}$$

Note that $\delta^d = \delta^{-1}$, where δ^d is the duality anti-isomorphism with respect to the dual initial data D^d.

Recall that $v(\lambda)$ ($\lambda \in \Lambda$) is the element in W_0 of smallest length such that $\lambda_- = v(\lambda)\lambda$. Then $\tau(\lambda) = u^d(\lambda)v(\lambda)$ in W^d, where $u^d(\lambda)$ is the shortest element (with respect to the length function l^d on W^d) of the coset $\tau(\lambda)W_0$ in W^d. Then $\Omega = \{u^d(\lambda) \,|\, \lambda \in \Lambda^+_{min}\}$, and in $\mathbb{B}^d$ we have ${}^dY^\lambda = u^d(\lambda)T_{v(\lambda)}$. Hence $\delta^d(u^d(\lambda)) = T^{-1}_{v(\lambda)^{-1}}X^{-\lambda}$ in $\mathbb{B}$. On the other hand, $\tau(\theta) = s^d_0 s_\theta$ and $l^d(\tau(\theta)) = l(s_\theta) + 1$. Hence $T^d_0 = {}^dY^\theta T^{-1}_{s_\theta}$ in $\mathbb{B}^d$ and $\delta^d(T^d_0) = T^{-1}_{s_\theta}X^{-\theta}$ in $\mathbb{B}$.

The next result shows that the duality anti-isomorphism from Theorem 9.4.7 descends to an anti-isomorphism between double affine Hecke algebras. In order to do so, we use the isomorphism $\mathcal{M} \xrightarrow{\sim} \mathcal{M}^d$ from the complex torus $\mathcal{M} = \mathcal{M}(D)$ onto $\mathcal{M}^d = \mathcal{M}(D^d)$ from Lemma 9.2.13.

Theorem 9.4.8 *The anti-isomorphism $\delta\colon \mathbb{B} \to \mathbb{B}^d$ descends to an anti-isomorphism $\delta\colon \mathbb{H}(\kappa, q) \to \mathbb{H}^d(\kappa^d, q) := \mathbb{H}(D^d; \kappa^d, q)$.*

For instance, the cross relations (9.4.5) in $\mathbb{H}(\kappa, q)$ for $i \in \{1, \ldots, n\}$ match with the the cross relations (9.3.4) in $\mathbb{H}^d(\kappa^d, q)$ through the anti-isomorphism δ. Note that $\delta^d = \delta^{-1}$ also on the level of the double affine Hecke algebra, where δ^d is the duality anti-isomorphism with respect to the dual data (D^d, κ^d).

9.4.3 Evaluation formulas

We follow closely [14] (which corresponds to the special case that $(\Lambda, \Lambda^d) = (P(R_0), P(R^d_0))$). See also [94, 73, 41]. We assume in this subsection that q and $\kappa \in \mathcal{M}(D)$ satisfy (9.3.5). We consider the nonsymmetric Macdonald–Koornwinder polynomials $P_\lambda \in \mathbb{C}[T_\Lambda]$ ($\lambda \in \Lambda$) associated to (D, κ, q), as well as the dual nonsymmetric Macdonald–Koornwinder polynomials $P^d_\xi \in \mathbb{C}[T_{\Lambda^d}]$ ($\xi \in \Lambda^d$) associated to (D^d, κ^d, q).

Write $\gamma^d_{\xi,q} = \gamma_{\xi,q}(D^d; \kappa^{d\bullet}) \in T_\Lambda$ ($\xi \in \Lambda^d$) for the spectral points with respect to the dual initial data (D^d, κ^d, q). Concretely, they are given by

$$\gamma^d_{\xi,q} = q^\xi \prod_{\alpha \in R^+_0} {}^d\upsilon_\alpha^{\eta((\xi,\alpha^{d\vee}))\alpha^\vee} \text{ with } {}^d\upsilon_\alpha := \left(\kappa^d_{\alpha^d}\right)^{\frac{1}{2}}\left(\kappa^d_{\mu_{\alpha^d}c+\alpha^d}\right)^{\frac{1}{2}} = \kappa_\alpha^{\frac{1}{2}}\kappa_{2\alpha}^{\frac{1}{2}} \text{ for } \alpha \in R^+_0.$$

Cherednik's basic representations

$$\pi_{D;\kappa,q}\colon H(\kappa^\bullet) \hookrightarrow \mathrm{End}_{\mathbb{C}}(\mathbb{C}[T_\Lambda]), \quad \pi_{D^d;\kappa^d,q}\colon H(\kappa^{d\bullet}) \hookrightarrow \mathrm{End}_{\mathbb{C}}(\mathbb{C}[T_{\Lambda^d}])$$

extend to algebra maps

$$\widehat{\pi}\colon \mathbb{H}(\kappa, q) \hookrightarrow \mathrm{End}_{\mathbb{C}}(\mathbb{C}[T_\Lambda]), \quad \widehat{\pi}^d\colon \mathbb{H}(\kappa^d, q) \hookrightarrow \mathrm{End}_{\mathbb{C}}(\mathbb{C}[T_{\Lambda^d}])$$

by $\widehat{\pi}\big(X^{\widehat{\lambda}}_q\big) = e^{\widehat{\lambda}}_q$ for $\widehat{\lambda} \in \widehat{\Lambda}$ and $\widehat{\pi}^d\big({}^dX^{\widehat{\xi}}_q\big) = e^{\widehat{\xi}}_q$ for $\widehat{\xi} \in \widehat{\Lambda^d}$. Note that

$$\gamma^d_{0,q} = \prod_{\alpha \in R^+_0} {}^d\upsilon_\alpha^{-\alpha^\vee} \in T_\Lambda, \qquad \gamma_{0,q} = \prod_{\alpha \in R^+_0} \upsilon_{\alpha^d}^{-\alpha^{d\vee}} \in T_{\Lambda^d}.$$

Definition 9.4.9 Define *evaluation maps* $\mathrm{Ev}\colon \mathbb{H}(\kappa, q) \to \mathbb{C}$ and $\mathrm{Ev}^d\colon \mathbb{H}(\kappa^d, q) \to \mathbb{C}$ by

$$\mathrm{Ev}(Z) := \big(\widehat{\pi}(Z)1\big)\big(\gamma^d_{0,q}\big) \quad (Z \in \mathbb{H}(\kappa, q)), \qquad \mathrm{Ev}^d(Z) := \big(\widehat{\pi}^d(Z)1\big)\big(\gamma_{0,q}\big) \quad (Z \in \mathbb{H}^d(\kappa^d, q)).$$

The following lemma is crucial for the duality of the (nonsymmetric) Macdonald–Koornwinder polynomials.

Lemma 9.4.10 *For all* $Z \in \mathbb{H}(\kappa, q)$ *we have* $\mathrm{Ev}^d(\delta(Z)) = \mathrm{Ev}(Z)$.

Write $\mathbb{H} = \mathbb{H}(\kappa, q)$ and $\mathbb{H}^d = \mathbb{H}^d(\kappa^d, q)$. Define bilinear forms

$$B\colon \mathbb{H} \times \mathbb{H}^d \to \mathbb{C}, \ \ (Z, \tilde{Z}) \mapsto \mathrm{Ev}(\delta^d(\tilde{Z})\, Z), \qquad B^d\colon \mathbb{H}^d \times \mathbb{H} \to \mathbb{C}, \ \ (\tilde{Z}, Z) \mapsto \mathrm{Ev}^d(\delta(Z)\, \tilde{Z}).$$

Corollary 9.4.11 $B(Z, \tilde{Z}) = B^d(\tilde{Z}, Z)$ *for* $Z \in \mathbb{H}$ *and* $\tilde{Z} \in \mathbb{H}^d$.

The following elementary lemma provides convenient tools to derive the evaluation formula for nonsymmetric Macdonald–Koornwinder polynomials.

Lemma 9.4.12 *Let* $p \in \mathbb{C}[T_\Lambda]$, $\tilde{p} \in \mathbb{C}[T_{\Lambda^d}]$, $Z, Z_1, Z_2 \in \mathbb{H}$ *and* $\tilde{Z}, \tilde{Z}_1, \tilde{Z}_2 \in \mathbb{H}^d$. *Then*

(i) $B(Z_1 Z_2, \tilde{Z}) = B(Z_2, \delta(Z_1)\tilde{Z})$.

(ii) $B(ZT_i, \tilde{Z}) = \kappa_i B(Z, \tilde{Z})$ *for* $0 \le i \le n$.

(iii) $B((\widehat{\pi}(Z)(p))(X), \tilde{Z}) = B(Zp(X), \tilde{Z})$.

Lemma 9.4.12 and Corollary 9.4.11 imply

Proposition 9.4.13

(i) *For* $\xi \in \Lambda^d$ *and* $p \in \mathbb{C}[T_\Lambda]$ *we have* $P^d_\xi(\gamma_{0,q})p(\gamma^d_{\xi,q}) = B(p, P^d_\xi)$.

(ii) *For* $\lambda \in \Lambda$ *and* $\tilde{p} \in \mathbb{C}[T_{\Lambda^d}]$ *we have* $P_\lambda(\gamma^d_{0,q})\tilde{p}(\gamma_{\lambda,q}) = B^d(\tilde{p}, P_\lambda)$.

Corollary 9.4.14 (duality) *For* $\lambda \in \Lambda$ *and* $\xi \in \Lambda^d$ *we have*

$$P^d_\xi(\gamma_{0,q})P_\lambda(\gamma^d_{\xi,q}) = P_\lambda(\gamma^d_{0,q})P^d_\xi(\gamma_{\lambda,q}).$$

The next aim is to explicitly evaluate $\mathrm{Ev}(P_\lambda) = P_\lambda(\gamma^d_{0,q})$. From the results on the pairing B above and from Proposition 9.3.5 one first derives an important intermediate result which shows how the action of generators of the double affine Hecke algebra on monic nonsymmetric Macdonald–Koornwinder polynomials P_λ can be explicitly expressed in terms of an action on the degree $\lambda \in \Lambda$ of P_λ.

Recall that W^d acts on Λ by $(w\tau(\lambda), \lambda') \mapsto w(\lambda + \lambda')$ ($w \in W_0$ and $\lambda, \lambda' \in \Lambda$).

Proposition 9.4.15

(i) *Let* $i \in \{1, \ldots, n\}$. *If* $s^d_i \lambda = \lambda$ *then* $\widehat{\pi}(T_i)P_\lambda = \kappa^d_i P_\lambda$.

(ii) *If* $s^d_0 \lambda = \lambda$ *then* $\widehat{\pi}(\delta^d(T^d_0))P_\lambda = \kappa^d_0 P_\lambda$.

(iii) *Let* $\lambda \in \Lambda^+_{min}$, *then* $\widehat{\pi}(\delta^d(u^d(\lambda)))1 = \kappa^d_{v(\lambda)} P_{-\lambda_-}$.

(iv) *Suppose* $1 \le i \le n$, $\lambda \in \Lambda$ *such that* $(\lambda, \alpha_i^\vee) > 0$. *Then*

$$\widehat{\pi}(T_i)P_\lambda = \frac{\kappa^d_i - (\kappa^d_i)^{-1} + \left(\kappa^d_{2\alpha^d_i} - (\kappa^d_{2\alpha^d_i})^{-1}\right)\gamma^{\alpha^d_i}_{\lambda,q}}{1 - \gamma^{2\alpha^d_i}_{\lambda,q}}\, P_\lambda + (\kappa^d_i)^{-1} P_{s^d_i \lambda}.$$

(v) *Suppose $\lambda \in \Lambda$ such that $a_0^d(\lambda) > 0$. Then*

$$\widehat{\pi}(\delta^{-1}(T_0^d))P_\lambda = \frac{\kappa_0^d - (\kappa_0^d)^{-1} + \left(\kappa_{2a_0^d}^d - (\kappa_{2a_0^d}^d)^{-1}\right) q_\theta\, \gamma_{\lambda,q}^{-\theta^d}}{1 - q_\theta^2 \gamma_{\lambda,q}^{-2\theta^d}} P_\lambda + \kappa_{v(s_0^d\lambda)}^d (\kappa_{v(\lambda)}^d)^{-1} P_{s_0^d\lambda}$$

(note that $a_0^d = \mu_\theta c - \theta^d = \mu_\theta(c - \theta^\vee)$ and $s_0^d\lambda = \lambda + (1 - (\lambda, \theta^\vee))\theta$).

The proposition gives the following recursion relations for $\mathrm{Ev}(P_\lambda) = P_\lambda(\gamma_{0,q}^d)$.

Corollary 9.4.16

(i) $\mathrm{Ev}(P_\lambda) = (\kappa_{v(\lambda)}^d)^{-1}$ *($\lambda \in \Lambda_{min}^+$).*

(ii) $$\mathrm{Ev}(P_{s_i^d\lambda}) = \frac{\left(1 - \kappa_i^d \kappa_{2\alpha_i^d}^d \gamma_{\lambda,q}^{\alpha_i^d}\right)\left(1 + \kappa_i^d (\kappa_{2\alpha_i^d}^d)^{-1} \gamma_{\lambda,q}^{\alpha_i^d}\right)}{1 - \gamma_{\lambda,q}^{2\alpha_i^d}} \mathrm{Ev}(P_\lambda) \quad (\lambda \in \Lambda,\ 1 \le i \le n,\ a_i^d(\lambda) > 0).$$

(iii) $$\mathrm{Ev}(P_{s_0^d\lambda}) = \frac{\kappa_{v(\lambda)}^d}{\kappa_0^d\, \kappa_{v(s_0^d\lambda)}^d} \frac{\left(1 - q_\theta \kappa_0^d \kappa_{2a_0^d}^d \gamma_{\lambda,q}^{-\theta^d}\right)\left(1 + q_\theta \kappa_0^d (\kappa_{2a_0^d}^d)^{-1} \gamma_{\lambda,q}^{-\theta^d}\right)}{1 - q_\theta^2 \gamma_{\lambda,q}^{-2\theta^d}} \mathrm{Ev}(P_\lambda) \quad (\lambda \in \Lambda,\ a_0^d(\lambda) > 0).$$

Recall the definition $c_a = c_a^{\kappa,q}(\,\cdot\,; D) \in \mathbb{C}(T_\Lambda)$ for $a \in R^\bullet$ from (9.3.7). The dual version is denoted by $c_a^d = c_a^{\kappa^d,q}(\,\cdot\,; D^d)$ $(a \in R^{d\bullet})$. Concretely, for $a \in R^{d\bullet}$, $c_a^d \in \mathbb{C}(T_{\Lambda^d})$ is given by

$$c_a^d(t) := \frac{(1 - \kappa_a^d \kappa_{2a}^d t_q^a)(1 + \kappa_a^d (\kappa_{2a}^d)^{-1} t_q^a)}{1 - t_q^{2a}}.$$

We also set $c_w^d := \prod_{a \in R^{d\bullet,+} \cap w^{-1}(R^{d\bullet,-})} c_a^d \in \mathbb{C}(T_{\Lambda^d})$ $(w \in W^d)$. An induction argument gives now the explicit evaluation formula for the nonsymmetric Macdonald–Koornwinder polynomials (see [14, 94, 73]).

Theorem 9.4.17 *For $\lambda \in \Lambda$ we have $\mathrm{Ev}(P_\lambda) = (\kappa_{\tau(\lambda)}^d)^{-1} c_{u^d(\lambda)}^d(\gamma_{0,q})$.*

9.4.4 Normalized nonsymmetric Macdonald–Koornwinder polynomials and duality

The treatment in this subsection is close to [14] and [94], which deal with the case that $(\Lambda, \Lambda^d) = (P(R_0), P(R_0^d))$ and the C$^\vee$C case, respectively. We assume that q and $\kappa \in \mathcal{M}(D)$ satisfy (9.3.5). Then $P_\lambda(\gamma_{0,q}^d) \neq 0$ and $P_\xi^d(\gamma_{0,q}) \neq 0$ for all $\lambda \in \Lambda$ and $\xi \in \Lambda^d$ in view of the evaluation formula (Theorem 9.4.17).

Recall that the Macdonald–Koornwinder polynomials $P_\lambda \in \mathbb{C}[T_\Lambda]$ $(\lambda \in \Lambda)$ and $P_\xi^d \in \mathbb{C}[T_{\Lambda^d}]$ $(\xi \in \Lambda^d)$ satisfy $\widehat{\pi}(p(Y))P_\lambda = p(\gamma_{\lambda,q}^{-1})P_\lambda$ and $\widehat{\pi}^d(r(^dY))P_\xi^d = r((\gamma_{\xi,q}^d)^{-1})P_\xi^d$ for all $p \in \mathbb{C}[T_{\Lambda^d}]$ and $r \in \mathbb{C}[T_\Lambda]$. This motivates the following notation for the normalized nonsymmetric Macdonald–Koornwinder polynomials.

Definition 9.4.18 (normalized nonsymmetric Macdonald–Koornwinder polynomials)

$$E(\gamma_{\lambda,q}^{-1};\,\cdot\,) := \frac{P_\lambda}{P_\lambda(\gamma_{0,q}^d)} \in \mathbb{C}[T_\Lambda] \ \ (\lambda \in \Lambda), \qquad E^d((\gamma_{\xi,q}^d)^{-1};\,\cdot\,) := \frac{P_\xi^d}{P_\xi^d(\gamma_{0,q})} \in \mathbb{C}[T_{\Lambda^d}] \ \ (\xi \in \Lambda^d).$$

We denote by $E^{\circ}(\gamma_{\lambda,q};\,\cdot\,)\in\mathbb{C}[T_\Lambda]$ the normalized Macdonald–Koornwinder polynomial with respect to the inverted parameters (κ^{-1},q^{-1}) (and similarly for $E^{d\circ}(\gamma^d_{\xi,q};\,\cdot\,)$).

For $\lambda\in\Lambda$ we have $E(\gamma^{-1}_{\lambda,q};\gamma^d_{0,q})=1$. From §9.4.3 we immediately get the following self-duality of the normalized nonsymmetric Macdonald–Koornwinder polynomials (cf. [14, 91, 73, 41]).

Corollary 9.4.19 *For all $p\in\mathbb{C}[T_\Lambda]$, $r\in\mathbb{C}[T_{\Lambda^d}]$, $\lambda\in\Lambda$ and $\xi\in\Lambda^d$ we have*

$$B(p,E^d((\gamma^d_{\xi,q})^{-1};\,\cdot\,))=p(\gamma^d_{\xi,q}),\qquad B(E(\gamma^{-1}_{\lambda,q};\,\cdot\,),r)=r(\gamma_{\lambda,q}).$$

In particular, $E(\gamma^{-1}_{\lambda,q};\gamma^d_{\xi,q})=E^d((\gamma^d_{\xi,q})^{-1};\gamma_{\lambda,q})$ for $\lambda\in\Lambda$ and $\xi\in\Lambda^d$.

9.4.5 Polynomial Fourier transform

For the remainder of the text we assume that the parameters q and $\kappa\in\mathcal{M}(D)$ satisfy the more restrictive parameter conditions (9.3.6).

In order to compute the norms of the normalized nonsymmetric Macdonald–Koornwinder polynomials it is convenient to formulate the explicit formulas from Proposition 9.4.15 in terms of properties of a Fourier transform whose kernel is given by the normalized nonsymmetric Macdonald–Koornwinder polynomial. For this we first need to consider the adjoint of the double affine Hecke algebra action with respect to the sesquilinear form

$$\langle p_1,p_2\rangle:=\int_{T^u_\Lambda}p_1(t)\,\overline{p_2(t)}\,\nu(t)\,d_ut,\qquad p_1,p_2\in\mathbb{C}[T_\Lambda],$$

and express it in terms of an explicit antilinear anti-isomorphism of the double affine Hecke algebra.

Lemma 9.4.20 *There exists a unique antilinear antialgebra isomorphism ${}^{\ddagger}\colon\mathbb{H}(\kappa,q)\to\mathbb{H}(\kappa^{-1},q^{-1})$ satisfying $T_w^{\ddagger}=T_w^{-1}$ ($w\in W$) and $(X^\lambda)^{\ddagger}=X^{-\lambda}$ ($\lambda\in\Lambda$). In addition,*

$$\langle\widehat{\pi}_{\kappa,q}(h)p_1,p_2\rangle=\langle p_1,\widehat{\pi}_{\kappa^{-1},q^{-1}}(h^{\ddagger})p_2\rangle,\qquad h\in\mathbb{H}.$$

Define

$$\mathcal{S}:=\{\gamma_{\lambda,q}\mid\lambda\in\Lambda\}\subset T_{\Lambda^d},\qquad\mathcal{S}^d:=\{\gamma^d_{\xi,q}\mid\xi\in\Lambda^d\}\subset T_\Lambda,$$

and write $F(\mathcal{S})$ (respectively $F(\mathcal{S}^d)$) for the space of finitely supported complex-valued functions on $\mathcal{S}$ (respectively $\mathcal{S}^d$). The following lemma follows easily from the results in §9.2.2 and from Proposition 9.3.5.

Lemma 9.4.21 *There exists a unique algebra homomorphism $\widehat{\rho}^d\colon\mathbb{H}^d\to\mathrm{End}_{\mathbb{C}}(F(\mathcal{S}))$ satisfying*

$$(\widehat{\rho}^{\,d}(T^d_i)g)(\gamma_{\lambda,q})=\begin{cases}\kappa^d_i g(\gamma_{\lambda,q})+(\kappa^d_i)^{-1}c^d_{a^d_i}(\gamma_{\lambda,q})(g(\gamma_{s^d_i\lambda,q})-g(\gamma_{\lambda,q})) & \text{if } s^d_i\lambda\neq\lambda,\ 0\le i\le n,\\ \kappa^d_i g(\gamma_{\lambda,q}) & \text{if } s^d_i\lambda=\lambda,\ 0\le i\le n,\end{cases}$$

$$(\widehat{\rho}^{\,d}(\omega)g)(\gamma_{\lambda,q})=g(\gamma_{\omega^{-1}\lambda,q})\qquad\text{if }\omega\in\Omega^d,$$

$$(\widehat{\rho}^{\,d}({}^dX^{\xi})g)(\gamma_{\lambda,q}) = \gamma_{\lambda,q}^{\xi}g(\gamma_{\lambda,q}) \qquad \text{if } \xi \in \Lambda^d.$$

Definition 9.4.22 We define the *polynomial Fourier transform* $\mathcal{F}\colon \mathbb{C}[T_\Lambda] \to F(\mathcal{S})$ by

$$(\mathcal{F}p)(\gamma) := \langle p, E^{\circ}(\gamma;\,\cdot\,)\rangle, \qquad p \in \mathbb{C}[T_\Lambda],\ \gamma \in \mathcal{S}.$$

For generators $X \in \mathbb{H}$ and $p \in \mathbb{C}[T_\Lambda]$ we can re-express $\mathcal{F}(\widehat{\pi}(X)p)$ as an explicit linear operator acting on $\mathcal{F}p \in F(\mathcal{S})$ by using Lemma 9.4.20, Proposition 9.4.15 and Theorem 9.4.17. It gives the following result (the first part of the theorem follows easily from the duality anti-isomorphism).

Theorem 9.4.23

(i) *The following formulas define an algebra isomorphism* $\Phi\colon \mathbb{H} \to \mathbb{H}^d$*:*

$$\Phi(T_i) = T_i^d \quad (1 \le i \le n), \qquad \Phi(T_{s_\theta}^{-1}X^{-\theta}) = T_0^d,$$
$$\Phi(T_{v(\lambda)^{-1}}^{-1}X^{-\lambda}) = u^d(\lambda)^{-1} \quad (\lambda \in \Lambda_{min}^+), \qquad \Phi(Y^{\xi}) = {}^dX^{-\xi} \quad (\xi \in \Lambda^d).$$

(ii) $\mathcal{F} \circ \widehat{\pi}(h) = \widehat{\rho}^{\,d}(\Phi(h)) \circ \mathcal{F}$ *for all* $h \in \mathbb{H}$.

9.4.6 Intertwiners and norm formulas

Define $I_i^d := [T_i^d, {}^dX_q^{a_i^d}] \in \mathbb{H}^d$ $(0 \le i \le n)$ and $I_\omega^d := \omega \in \mathbb{H}^d$ $(\omega \in \Omega^d)$. The following theorem extends results from [16, 91, 94].

Theorem 9.4.24 *For a reduced expression* $w = \omega s_{i_1}^d s_{i_2}^d \cdots s_{i_r}^d \in W^d$ $(\omega \in \Omega^d,\ 0 \le i_j \le n)$*, the expression*

$$I_w^d := I_\omega^d I_{i_1}^d I_{i_2}^d \cdots I_{i_r}^d \in \mathbb{H}^d$$

is well defined (independent of the choice of reduced expression). In addition,

$$\widehat{\pi}^{\,d}(I_w^d) = r_w^d \cdot w_q \in \mathrm{End}_{\mathbb{C}}(\mathbb{C}[T_{\Lambda^d}]),$$

where $r_w^d := \prod_{a \in {}^dR^{\bullet +} \cap w({}^dR^{\bullet -})} r_a^d \in \mathbb{C}[T_{\Lambda^d}]$ *with* $r_a^d(t) := (\kappa_a^d)^{-1}(1 - \kappa_a^d\kappa_{2a}^d t_q^a)(1 + \kappa_a^d(\kappa_{2a}^d)^{-1}t_q^a)$ *for* $t \in T_{\Lambda^d}$*. In addition, in* $\mathbb{H}^d$*,*

$$I_w^d I_{w^{-1}}^d = r_w^d({}^dX)(w_q r_{w^{-1}}^d)({}^dX) \quad \text{and} \quad I_w^d p({}^dX) = (w_q p)({}^dX) I_w^d \quad (w \in W^d,\ p \in \mathbb{C}[T_{\Lambda^d}]).$$

The $I_w^d \in \mathbb{H}^d$ $(w \in W^d)$ are called the *dual intertwiners*. The *intertwiners* are defined by $\mathcal{I}_w := \delta^d(I_w^d) \in \mathbb{H}$ $(w \in W^d)$. Then in $\mathbb{H}$,

$$\mathcal{I}_{w^{-1}}\mathcal{I}_w = r_w^d(Y^{-1})(w_q r_{w^{-1}}^d)(Y^{-1}) \quad \text{and} \quad p(Y^{-1})\mathcal{I}_w = \mathcal{I}_w(w_q p)(Y^{-1}) \quad (w \in W^d,\ p \in \mathbb{C}[T_{\Lambda^d}]).$$

Proposition 9.4.15 and Theorem 9.4.17 give the following result.

Proposition 9.4.25

(i) *If* $\lambda \in \Lambda$ *and* $0 \le i \le n$ *satisfy* $s_i^d\lambda \ne \lambda$ *then* $\widehat{\pi}(\mathcal{I}_i)E(\gamma_{\lambda,q}^{-1};\,\cdot\,) = \gamma_{\lambda,q}^{-a_i^d} r_{a_i^d}^d(\gamma_{\lambda,q})E(\gamma_{s_i^d\lambda,q}^{-1};\,\cdot\,)$.

(ii) *If* $\lambda \in \Lambda$ *and* $0 \le i \le n$ *satisfy* $s_i^d\lambda = \lambda$ *then* $\widehat{\pi}(\mathcal{I}_i)E(\gamma_{\lambda,q}^{-1};\,\cdot\,) = 0$.

(iii) *If* $\omega \in \Omega^d$ *and* $\lambda \in \Lambda$ *then* $\widehat{\pi}(\mathcal{I}_\omega)E(\gamma_{\lambda,q}^{-1};\,\cdot\,) = E(\gamma_{\omega^{-1}\lambda,q}^{-1};\,\cdot\,)$.

This proposition shows that intertwiners can be used to create the nonsymmetric Macdonald–Koornwinder polynomial from the constant polynomial $E(\gamma_{0,q}^{-1};\,\cdot\,)\equiv 1$ (cf., e.g., [73, §5.10]). We now use this observation to express the norms of the nonsymmetric Macdonald–Koornwinder in terms of the nonzero constant term

$$\langle 1,1\rangle=\int_{T_\Lambda^u}v(t)\,d_ut=\frac{\mathcal{C}(\gamma_{0,q}^d)}{\#W_0}\int_{T_\Lambda^u}v_+(t)\,d_ut.$$

The constant term $\langle 1,1\rangle$ is a q-analog and root system generalization of the Selberg integral. Its explicit evaluation was conjectured by Macdonald [72] in case $(\Lambda,\Lambda^d)=(P(R_0),P(R_0^d))$. By various methods it was evaluated in special cases (for references we refer to the detailed discussions in [72, 12]; see [35] for a survey on Selberg integrals and their generalizations and applications). A uniform proof in case $(\Lambda,\Lambda^d)=(P(R_0),P(R_0^d))$ using shift operators was given in [12, Theorem 0.1] (see [94] for the C$^\vee$C case). Write

$$N(\lambda):=\frac{\langle E(\gamma_{\lambda,q}^{-1};\,\cdot\,),E^\circ(\gamma_{\lambda,q};\,\cdot\,)\rangle}{\langle 1,1\rangle}\qquad(\lambda\in\Lambda),\tag{9.4.6}$$

$$c_w^d:=\prod_{a\in R^{d\bullet,+}\cap w^{-1}(R^{d\bullet,-})}c_a^d\in\mathbb{C}[T_{\Lambda^d}],\qquad c_{-w}^d:=\prod_{a\in R^{d\bullet,+}\cap w^{-1}(R^{d\bullet,-})}c_{-a}^d\in\mathbb{C}[T_{\Lambda^d}]\qquad(w\in W^d).$$

Warning: $c_{-w}^d(t)=c_w^d(t^{-1})$ is only valid if $w\in W_0$.

Theorem 9.4.26

(i) *For $\lambda\in\Lambda$, $N(\lambda)=\dfrac{c_{-u^d(\lambda)}^d(\gamma_{0,q})}{c_{u^d(\lambda)}^d(\gamma_{0,q})}$, which is nonzero for all $\lambda\in\Lambda$.*

(ii) *The transform $\mathcal{F}\colon\mathbb{C}[T_\Lambda]\to F(\mathcal{S})$ is a linear bijection with inverse $\mathcal{G}\colon F(\mathcal{S})\to\mathbb{C}[T_\Lambda]$ given by $(\mathcal{G}f)(t):=\langle 1,1\rangle^{-1}\sum_{\lambda\in\Lambda}N(\lambda)^{-1}f(\gamma_{\lambda,q})E(\gamma_{\lambda,q}^{-1};t)$ for $f\in F(\mathcal{S})$ and $t\in T_\Lambda$.*

Part **(i)** of the above theorem should be compared with [18, Proposition 3.4.1] and [73, (5.2.11)] (it originates from [14] in case $(\Lambda,\Lambda^d)=(P(R_0),P(R_0^d))$ and [91, 94] in the C$^\vee$C case). The first part of the theorem follows from Proposition 9.4.25 and the fact that the intertwiners behave nicely with respect to the anti-involution ‡ (see Lemma 9.4.20): $\mathcal{I}_i^\ddagger=\mathcal{I}_i$ $(0\le i\le n)$ in $\mathbb{H}(\kappa^{-1},q^{-1})$ and $\mathcal{I}_\omega^\ddagger=\mathcal{I}_{\omega^{-1}}$ $(\omega\in\Omega^d)$ in $\mathbb{H}(\kappa^{-1},q^{-1})$. The second part of the theorem is immediate from the first and from the biorthogonality of the nonsymmetric Macdonald–Koornwinder polynomials (see Theorem 9.3.13).

In combination with the evaluation formula (see Theorem 9.4.17) we get the following *norm formula*.

Corollary 9.4.27 *For all $\lambda\in\Lambda$,* $\dfrac{\langle P_\lambda,P_\lambda^\circ\rangle}{\langle 1,1\rangle}=(\kappa_{u^d(\lambda)}^d)^2c_{u^d(\lambda)}^d(\gamma_{0,q})c_{-u^d(\lambda)}^d(\gamma_{0,q})$.

9.4.7 Normalized symmetric Macdonald–Koornwinder polynomials

We still assume that q and $\kappa\in\mathcal{M}(D)$ satisfy (9.3.6). The results in this subsection are from [14] in case $(\Lambda,\Lambda^d)=(P(R_0),P(R_0^d))$ and from [91, 94] in the C$^\vee$C case.

For $\lambda \in \Lambda^-$ define $E^+(\gamma_{\lambda,q}^{-1};\,\cdot\,) \in \mathbb{C}[T_\Lambda]^{W_0}$ by $E^+(\gamma_{\lambda,q}^{-1};\,\cdot\,) := \widehat{\pi}_{\kappa,q}(C_+)E(\gamma_{\lambda,q}^{-1};\,\cdot\,)$, where (recall) $C_+ := \dfrac{1}{\sum_{w\in W_0}\kappa_w^2}\sum_{w\in W_0}\kappa_w T_w$. We call $E^+(\gamma_{\lambda,q}^{-1};\,\cdot\,)$ the *normalized symmetric Macdonald–Koornwinder polynomial* of degree $\lambda \in \Lambda^-$.

Lemma 9.4.28

(i) *$E^+(\gamma_{\lambda,q}^{-1};\,\cdot\,) = \widehat{\pi}(C_+)E(\gamma_{w\lambda,q}^{-1};\,\cdot\,)$ for all $w \in W_0$ and $\lambda \in \Lambda^-$.*

(ii) *$E^+(\gamma_{\lambda,q}^{-1};\gamma_{0,q}^d) = 1$ for all $\lambda \in \Lambda^-$.*

(iii) *$\{E^+(\gamma_{\lambda,q}^{-1};\,\cdot\,)\}_{\lambda\in\Lambda^-}$ is a basis of $\mathbb{C}[T_\Lambda]^{W_0}$ satisfying, for all $p \in \mathbb{C}[T_{\Lambda^d}]^{W_0}$,*

$$D_p(E^+(\gamma_{\lambda,q}^{-1};\,\cdot\,)) = p(\gamma_{\lambda,q}^{-1})\,E^+(\gamma_{\lambda,q}^{-1};\,\cdot\,) = p(q^{-\lambda}\gamma_{0,q}^{-1})\,E^+(\gamma_{\lambda,q}^{-1};\,\cdot\,).$$

(iv) *$E^+(\gamma_{\lambda,q}^{-1};\,\cdot\,) = P_{\lambda_+}^+(\,\cdot\,)/P_{\lambda_+}^+(\gamma_{0,q}^d)$ for all $\lambda \in \Lambda^-$.*

(v) *$E^+(\gamma_{\lambda,q}^{-1};\gamma_{\xi,q}^d) = E^{+\,d}((\gamma_{\xi,q}^d)^{-1};\gamma_{\lambda,q})$ for all $\lambda \in \Lambda^-$ and $\xi \in \Lambda^{d-}$.*

As before we write superindex $\circ$ to indicate that the parameters (κ, q) are inverted. The nonsymmetric and symmetric Macdonald–Koornwinder polynomials with inverted parameters (κ^{-1}, q^{-1}) can be explicitly expressed in terms of the ones with parameters (κ, q). The result is as follows (see [18, §3.3.2] for $(\Lambda, \Lambda^d) = (P(R_0), P(R_0^d))$ and [98, (2.5)] in the twisted case).

Proposition 9.4.29

(i) *For all $\lambda \in \Lambda$, $E^\circ(\gamma_{\lambda,q};t^{-1}) = \kappa_{w_0}^{-1}(\widehat{\pi}_{\kappa,q}(T_{w_0})E(\gamma_{-w_0\lambda,q}^{-1};\,\cdot\,))(t)$ in $\mathbb{C}[T_\Lambda]$, where $w_0 \in W_0$ is the longest Weyl group element and $t \in T_\Lambda$.*

(ii) *For all $\lambda \in \Lambda^-$ we have in $\mathbb{C}[T_\Lambda]^{W_0}$ that*

$$E^{+\,\circ}(\gamma_{\lambda,q};t^{-1}) = E^+(\gamma_{-w_0\lambda,q}^{-1};t), \qquad E^{+\,\circ}(\gamma_{\lambda,q};t) = E^+(\gamma_{\lambda,q}^{-1};t).$$

By means of intertwiners or by Proposition 9.4.15 it is now possible to expand the normalized symmetric Macdonald–Koornwinder polynomials in nonsymmetric ones. This in turn leads to an explicit expression of the quadratic norms of the symmetric Macdonald–Koornwinder polynomials in terms of those of the nonsymmetric ones. Recall the rational function $\mathcal{C}(\,\cdot\,) = \mathcal{C}(\,\cdot\,;D;\kappa,q) \in \mathbb{C}(T_\Lambda)$ from (9.3.12). Write $\mathcal{C}^d(\,\cdot\,) = \mathcal{C}(\,\cdot\,;D^d;\kappa^d,q) \in \mathbb{C}(T_{\Lambda^d})$ for its dual version.

Theorem 9.4.30

(i) *For $\lambda \in \Lambda^-$ we have* $P_{\lambda_+}^+(t) = \displaystyle\sum_{\mu\in W_0\lambda}\Bigg(\prod_{\alpha\in R_0^+\cap v(\mu)R_0^-} c_\alpha^d(\gamma_{\lambda,q})\Bigg)P_\mu(t).$

(ii) *For $\lambda \in \Lambda^-$ we have* $E^+(\gamma_{\lambda,q}^{-1};t) = \displaystyle\sum_{\mu\in W_0\lambda}\frac{\mathcal{C}^d(\gamma_{\mu,q})}{\mathcal{C}^d(\gamma_{0,q})}E(\gamma_{\mu,q}^{-1};t).$

(iii) *For $\lambda,\mu \in \Lambda^-$ we have* $\dfrac{\langle E^+(\gamma_{\lambda,q}^{-1};\,\cdot\,), E^+(\gamma_{\mu,q}^{-1};\,\cdot\,)\rangle_+}{\langle 1,1\rangle_+} = \delta_{\lambda,\mu}\dfrac{\mathcal{C}^d(\gamma_{\lambda,q})N(\lambda)}{\mathcal{C}^d(\gamma_{0,q})},$

where $\langle p,r\rangle_+ := \int_{T_\Lambda^u} p(t)\,\overline{r(t)}\,v_+(t)\,d_ut$.

As a consequence we get the following explicit evaluation formulas and quadratic norm formulas for the monic symmetric Macdonald–Koornwinder polynomials. Set

$$N^+(\lambda) := \frac{\langle E^+(\gamma_{\lambda,q}^{-1};\,\cdot\,), E^+(\gamma_{\lambda,q}^{-1};\,\cdot\,)\rangle_+}{\langle 1,1\rangle_+}, \qquad \lambda\in\Lambda^-.$$

Corollary 9.4.31

(i) *For $\lambda\in\Lambda^-$ we have* $P^+_{\lambda_+}(\gamma^d_{0,q}) = \dfrac{\mathcal{C}^d(\gamma_{0,q})c^d_{\tau(\lambda)}(\gamma_{0,q})}{\mathcal{C}^d(\gamma_{\lambda,q})\kappa^d_{\tau(\lambda)}}.$

(ii) *For $\lambda\in\Lambda^-$ we have* $N^+(\lambda) = \dfrac{\mathcal{C}^d(\gamma_{\lambda,q})c^d_{-\tau(\lambda)}(\gamma_{0,q})}{\mathcal{C}^d(\gamma_{0,q})c^d_{\tau(\lambda)}(\gamma_{0,q})}.$

Remark 9.4.32 For $\lambda\in\Lambda^-$ we have $\sum_{\mu\in W_0\lambda} N(\mu)^{-1} = N^+(\lambda)^{-1}$.

9.5 Explicit evaluation and norm formulas

We rewrite in this subsection the explicit evaluation formulas and the quadratic norm espressions for the symmetric Macdonald–Koornwinder polynomials in terms of q-shifted factorials (9.3.9). The explicit formulas for the GL_{n+1} symmetric Macdonald polynomials (see §9.3.7) can be immediately obtained as special cases of the explicit formulas below. We keep the conditions (9.3.6) on the parameters (κ, q).

9.5.1 The twisted cases

In case we have initial data $D=(R_0,\Delta_0,\bullet,\Lambda,\Lambda^d)$ with $\bullet = t$ the evaluation and norm formulas take the following explicit form.

Corollary 9.5.1

(i) *Suppose $\bullet = t$ and $S=\emptyset=S^d$. Then $R=R^\bullet=R^d$, $\kappa=\kappa^d$ and $\kappa_{m\mu_\alpha c+\alpha}=\kappa_\alpha$ for all $m\in\mathbb{Z}$ and $\alpha\in R_0$. Then for all $\lambda\in\Lambda^-$,*

$$P^+_{\lambda_+}(\gamma_{0,q}) = \gamma_{0,q}^{-\lambda}\prod_{\alpha\in R_0^+}\frac{(\kappa_\alpha^2\gamma_{0,q}^{-\alpha};q_\alpha)_{-(\lambda,\alpha^\vee)}}{(\gamma_{0,q}^{-\alpha};q_\alpha)_{-(\lambda,\alpha^\vee)}},$$

$$N^+(\lambda) = \gamma_{0,q}^{2\lambda}\prod_{\alpha\in R_0^+}\left(\frac{1-\gamma_{0,q}^{-\alpha}}{1-q_\alpha^{-(\lambda,\alpha^\vee)}\gamma_{0,q}^{-\alpha}}\right)\frac{(q_\alpha\kappa_\alpha^{-2}\gamma_{0,q}^{-\alpha};q_\alpha)_{-(\lambda,\alpha^\vee)}}{(\kappa_\alpha^2\gamma_{0,q}^{-\alpha};q_\alpha)_{-(\lambda,\alpha^\vee)}}.$$

(ii) *Suppose $\bullet = t$ and $(\Lambda,\Lambda^d)=(\mathbb{Z}R_0,\mathbb{Z}R_0)$ (the $\mathrm{C}^\vee\mathrm{C}$ case), realized concretely as in §9.3.8. Recall the relabelling of the multiplicity functions κ and κ^d given by $k=\kappa_\varphi^2=\tilde{k}$ and*

$$\{a,b,c,d\} = \{\kappa_\theta\kappa_{2\theta}, -\kappa_\theta\kappa_{2\theta}^{-1}, q_\theta\kappa_0\kappa_{2a_0}, -q_\theta\kappa_0\kappa_{2a_0}^{-1}\},$$

$$\{\tilde{a},\tilde{b},\tilde{c},\tilde{d}\} = \{\kappa_\theta\kappa_0, -\kappa_\theta\kappa_0^{-1}, q_\theta\kappa_{2\theta}\kappa_{2a_0}, -q_\theta\kappa_{2\theta}\kappa_{2a_0}^{-1}\}.$$

Then for $\lambda \in \Lambda^-$ we have

$$P^+_{\lambda_+}(\gamma^d_{0,q}) = (\gamma^d_{0,q})^{-\lambda} \prod_{\alpha \in R^+_{0,l}} \frac{(\kappa^2_\varphi \gamma^{-\alpha}_{0,q}; q_\varphi)_{-(\lambda,\alpha^\vee)}}{(\gamma^{-\alpha}_{0,q}; q_\varphi)_{-(\lambda,\alpha^\vee)}} \prod_{\alpha \in R^+_{0,s}} \frac{(\tilde{a}\gamma^{-\alpha}_{0,q}, \tilde{b}\gamma^{-\alpha}_{0,q}, \tilde{c}\gamma^{-\alpha}_{0,q}, \tilde{d}\gamma^{-\alpha}_{0,q}; q^2_\theta)_{-(\lambda,\alpha^\vee)/2}}{(\gamma^{-2\alpha}_{0,q}; q^2_\theta)_{-(\lambda,\alpha^\vee)}},$$

$$N^+(\lambda) = (\gamma^d_{0,q})^{2\lambda} \prod_{\alpha \in R^+_{0,l}} \left(\frac{1-\gamma^{-\alpha}_{0,q}}{1-q_\varphi^{-(\lambda,\alpha^\vee)}\gamma^{-\alpha}_{0,q}} \right) \frac{(q_\varphi k^{-2}_\varphi \gamma^{-\alpha}_{0,q}; q_\varphi)_{-(\lambda,\alpha^\vee)}}{(k^2_\varphi \gamma^{-\alpha}_{0,q}; q_\varphi)_{-(\lambda,\alpha^\vee)}}$$

$$\times \prod_{\beta \in R^+_{0,s}} \left(\frac{1-\gamma^{-2\beta}_{0,q}}{1-q_\theta^{-2(\lambda,\beta^\vee)}\gamma^{-2\beta}_{0,q}} \right) \frac{(q^2_\theta \tilde{a}^{-1}\gamma^{-\beta}_{0,q}, q^2_\theta \tilde{b}^{-1}\gamma^{-\beta}_{0,q}, q^2_\theta \tilde{c}^{-1}\gamma^{-\beta}_{0,q}, q^2_\theta \tilde{d}^{-1}\gamma^{-\beta}_{0,q}; q^2_\theta)_{-(\lambda,\beta^\vee)/2}}{(\tilde{a}\gamma^{-\beta}_{0,q}, \tilde{b}\gamma^{-\beta}_{0,q}, \tilde{c}\gamma^{-\beta}_{0,q}, \tilde{d}\gamma^{-\beta}_{0,q}; q^2_\theta)_{-(\lambda,\beta^\vee)/2}},$$

where $R_{0,s} \subset R_0$ (respectively $R_{0,l} \subset R_0$) are the short (respectively long) roots in R_0. In addition, $q_\varphi = q^2_\theta$.

The formulas in the intermediate case $\bullet = t$ and $S = \emptyset \neq S^d$ (or $S \neq \emptyset = S^d$) are special cases of the $\mathrm{C}^\vee\mathrm{C}$ case (by choosing an appropriate specialization of the multiplicity function).

9.5.2 The untwisted cases

If the initial data is of the form $D = (R_0, \Delta_0, u, \Lambda, \Lambda^d)$ then there are essentially two cases to be considered, namely the case $S = \emptyset = S^d$ and the special rank two case treated in §9.3.9. Indeed, the cases corresponding to a nonreduced extension of the untwisted affine root system with underlying finite root system R_0 of type B_n ($n \geq 3$) or BC_n ($n \geq 1$) are special cases of the twisted $\mathrm{C}^\vee\mathrm{C}_n$ case.

Corollary 9.5.2

(i) *Suppose $\bullet = u$ and $S = \emptyset = S^d$. Then $R = \mathbb{Z}c + R_0$, $R^d = \mathbb{Z}c + R_0^\vee$ and $\kappa_{mc+\alpha} = \kappa_\alpha = \kappa^d_{mc+\alpha^\vee}$ for all $m \in \mathbb{Z}$ and $\alpha \in R_0$. Then for all $\lambda \in \Lambda^-$,*

$$P^+_{\lambda_+}(\gamma^d_{0,q}) = \gamma^{-\lambda}_{0,q} \prod_{\alpha \in R^+_0} \frac{(\kappa^2_\alpha \gamma^{-\alpha^\vee}_{0,q}; q)_{-(\lambda,\alpha^\vee)}}{(\gamma^{-\alpha^\vee}_{0,q}; q)_{-(\lambda,\alpha^\vee)}}, \quad N^+(\lambda) = \gamma^{2\lambda}_{0,q} \prod_{\alpha \in R^+_0} \frac{1-\gamma^{-\alpha^\vee}_{0,q}}{1-q^{-(\lambda,\alpha^\vee)}\gamma^{-\alpha^\vee}_{0,q}} \frac{(q\kappa^{-2}_\alpha \gamma^{-\alpha^\vee}_{0,q}; q)_{-(\lambda,\alpha^\vee)}}{(\kappa^2_\alpha \gamma^{-\alpha^\vee}_{0,q}; q)_{-(\lambda,\alpha^\vee)}}.$$

(ii) *Suppose $D = (R_0, \Delta_0, u, \mathbb{Z}R_0, \mathbb{Z}R_0^\vee)$ with R_0 of type C_2, realized concretely as in §9.3.9. Recall the relabelling (9.3.19) of κ and κ^d. Then for $\lambda \in \Lambda^-$ we have*

$$P^+_{\lambda_+}(\gamma^d_{0,q}) = (\gamma^d_{0,q})^{-\lambda} \prod_{\alpha \in R^+_{0,s}} \frac{(\tilde{c}\gamma^{-\alpha^\vee}_{0,q}, \tilde{d}\gamma^{-\alpha^\vee}_{0,q}; q^2)_{-(\lambda,\alpha^\vee)/2}}{(\gamma^{-\alpha^\vee}_{0,q}; q)_{-(\lambda,\alpha^\vee)}} \prod_{\beta \in R^+_{0,l}} \frac{(\tilde{a}\gamma^{-\beta^\vee}_{0,q}, \tilde{b}\gamma^{-\beta^\vee}_{0,q}; q)_{-(\lambda,\beta^\vee)}}{(\gamma^{-2\beta^\vee}_{0,q}; q^2)_{-(\lambda,\beta^\vee)}},$$

$$N^+(\lambda) = (\gamma^d_{0,q})^{2\lambda} \prod_{\alpha \in R^+_{0,s}} \frac{1-\gamma^{-\alpha^\vee}_{0,q}}{1-q^{-(\lambda,\alpha^\vee)}\gamma^{-\alpha^\vee}_{0,q}} \frac{(q^2\tilde{c}^{-1}\gamma^{-\alpha^\vee}_{0,q}, q^2\tilde{d}^{-1}\gamma^{-\alpha^\vee}_{0,q}; q^2)_{-(\lambda,\alpha^\vee)/2}}{(\tilde{c}\gamma^{-\alpha^\vee}_{0,q}, \tilde{d}\gamma^{-\alpha^\vee}_{0,q}; q^2)_{-(\lambda,\alpha^\vee)/2}}$$

$$\times \prod_{\beta \in R^+_{0,l}} \frac{1-\gamma^{-2\beta^\vee}_{0,q}}{1-q^{-2(\lambda,\beta^\vee)}\gamma^{-2\beta^\vee}_{0,q}} \frac{(q\tilde{a}^{-1}\gamma^{-\beta^\vee}_{0,q}, q\tilde{b}^{-1}\gamma^{-\beta^\vee}_{0,q}; q)_{-(\lambda,\beta^\vee)}}{(\tilde{a}\gamma^{-\beta^\vee}_{0,q}, \tilde{b}\gamma^{-\beta^\vee}_{0,q}; q)_{-(\lambda,\beta^\vee)}},$$

where $R_{0,s} \subset R_0$ (respectively $R_{0,l} \subset R_0$) are the short (respectively long) roots in R_0.

A Appendix

A.1 Affine root systems

The main reference for this subsection is Macdonald [66].

Let E be a real affine space with the associated space of translations V of dimension $n \geq 1$. Fix a real scalar product $(\cdot,\cdot)$ on V and set $|v|^2 := (v,v)$ for all $v \in V$. It turns E into a metric space, called an *affine Euclidean space* [6, (1.3.1)].

A map $\Psi\colon E \to E$ is called an *affine linear endomorphism* of E if there exists a linear endomorphism $d\Psi$ of V such that $\Psi(e+v) = \Psi(e) + d\Psi(v)$ for all $e \in E$ and all $v \in V$. Set $\mathrm{O}(E)$ for the group of affine linear isometric automorphisms of E. Let $\tau_E\colon V \to \mathrm{O}(E)$ be the group monomorphism defined by $\tau_E(v)(e) := e + v$. Then we have a short exact sequence of groups $\tau_E(V) \hookrightarrow \mathrm{O}(E) \overset{d}{\twoheadrightarrow} \mathrm{O}(V)$. For a subgroup $W \subseteq \mathrm{O}(E)$ let $L_W \subseteq V$ be the additive subgroup such that $\tau_E(L_W) = \mathrm{Ker}(d|_W)$.

A function $a\colon E \to \mathbb{R}$ is said to be *affine linear* if there exists a linear functional $\alpha\colon V \to \mathbb{R}$ such that $a(e+v) = a(e) + \alpha(v)$ for $e \in E$ and $v \in V$. Set $\widehat{E}$ for the real $(n+1)$-dimensional vector space of affine linear functions $a\colon E \to \mathbb{R}$. Let $c \in \widehat{E}$ be the constant function one. The *gradient* of $a \in \widehat{E}$ is the unique vector $Da \in V$ such that $a(e+v) = a(e) + (Da, v)$ for all $e \in E$ and $v \in V$. The *gradient map* $D\colon \widehat{E} \to V$ is linear, surjective, with kernel consisting of the constant functions on E.

For $a, b \in \widehat{E}$ set $(a,b) := (Da, Db)$, which defines a semi-positive definite symmetric bilinear form on $\widehat{E}$. The radical consists of the constant functions on E. We write $|a|^2 := (a,a)$ for $a \in \widehat{E}$. Let $\mathrm{O}(\widehat{E})$ be the form-preserving linear automorphisms of $\widehat{E}$ and $\mathrm{O}_c(\widehat{E})$ its subgroup of automorphisms fixing the constant functions.

The contragredient action of $g \in \mathrm{O}(E)$ on $\widehat{E}$, given by $(ga)(e) := a(g^{-1}e)$ ($a \in \widehat{E}$, $e \in E$), realizes a group isomorphism $\mathrm{O}(E) \simeq \mathrm{O}_c(\widehat{E})$. Note that $\tau_E(v)a = a - (Da, v)c$ for $a \in \widehat{E}$ and $v \in V$.

A vector $a \in \widehat{E}$ is called *nonisotropic* if $Da \neq 0$. For such a let $s_a\colon E \to E$ be the orthogonal reflection in the affine hyperplane $a^{-1}(0)$ of E. It is explicitly given by

$$s_a(e) = e - a(e)Da^\vee, \qquad e \in E,$$

where $v^\vee := 2v/|v|^2 \in V$ is the covector of $v \in V \setminus \{0\}$. Viewed as element of $\mathrm{O}_c(\widehat{E})$ it reads

$$s_a(b) = b - (a^\vee, b)a, \qquad b \in \widehat{E},$$

where $a^\vee := 2a/|a|^2 \in \widehat{E}$ is the covector of a. For a subset R of nonisotropic vectors in $\widehat{E}$ let $W(R)$ be the subgroup of $\mathrm{O}(E) \simeq \mathrm{O}_c(\widehat{E})$ generated by the orthogonal reflections s_a ($a \in R$).

Definition A.1 A set R of nonisotropic vectors in $\widehat{E}$ is an *affine root system* on E if

1 R spans $\widehat{E}$,
2 $W(R)$ stabilizes R,
3 $(a^\vee, b) \in \mathbb{Z}$ for all $a, b \in R$,
4 $W(R)$ acts properly on E (i.e., if K_1 and K_2 are two compact subsets of E then $w(K_1) \cap K_2 \neq \emptyset$ for at most finitely many $w \in W(R)$),

5 $L_{W(R)}$ spans V.

The elements $a \in R$ are called *affine roots*. The group $W(R)$ is called the *affine Weyl group* of R. The real dimension n of V is the *rank* of R.

Definition A.2 Let R be an affine root system on E. A nonempty subset $R' \subseteq R$ is called an *affine root subsystem* if $W(R')$ stabilizes R' and if $L_{W(R')}$ generates the real span V' of the set $\{Da\}_{a\in R'}$ of gradients of R' in V.

Remark A.3 With the notations from the previous definitions, let E' be the set of $V'^{\perp}$-orbits of E, where $V'^{\perp}$ is the orthocomplement of V' in V. It is an affine Euclidean space with V' the associated space of translations and with norm induced by the scalar product on V'. Let $F \subseteq \widehat{E}$ be the real span of R'. Then $F \xrightarrow{\sim} \widehat{E'}$ with form preserving linear isomorphism $a \mapsto a'$ defined by $a'(e + V'^{\perp}) := a(e)$ for all $a \in F$ and $e \in E$. With this identification R' is an affine root system on E'. Furthermore, the corresponding affine Weyl group $W(R')$ is isomorphic to the subgroup of $W(R) \subset \mathrm{O}(E)$ generated by the orthogonal reflections s_a $(a \in R')$.

We call an affine root system R *irreducible* if it cannot be written as a nontrivial orthogonal disjoint union $R' \cup R''$ (orthogonal meaning that $(a, b) = 0$ for all $a \in R'$ and all $b \in R''$). It is called *reducible* otherwise. In that case both R' and R'' are affine root subsystems of R. Each affine root system is an orthogonal disjoint union of irreducible affine root subsystems (cf. [66, §3]).

Remark A.4 Macdonald's [66, §2] definition of an affine root system is (1)–(4) of Definition A.1. Careful analysis reveals that Macdonald tacitly assumes condition (5), which only follows from the four axioms (1)–(4) if R is irreducible. Through a personal communication I learned that Mark Reeder has independently observed that an extra condition besides (1)–(4) is needed in order to avoid examples of affine root systems given as an orthogonal disjoint union of an affine root system R' and a finite crystallographic root system R'' (compare, on the level of affine Weyl groups, with [5, Chap. V, §3] and [6, §1.3]). Mark Reeder proposes to add to (1)–(4) the axiom that for each $\alpha \in D(R)$ there exists at least two affine roots with gradient α. The resulting definition is equivalent to Definition A.1, as well as to the notion of an *échelonnage* from [6, (1.4.1)] (take here [6, (1.3.2)] into account).

An affine root system R is called *reduced* if $\mathbb{R}a \cap R = \{\pm a\}$ for all $a \in R$, and *nonreduced* otherwise. If R is nonreduced then $R = R^{\mathrm{ind}} \cup R^{\mathrm{unm}}$ with R^{ind} (respectively R^{unm}) the reduced affine root subsystem of R consisting of indivisible (respectively unmultiplyable) affine roots.

If $R \subset \widehat{E}$ is an affine root system then $R_0 := D(R) \subset V$ is a finite crystallographic root system in V, called the *gradient root system* of R. The associated Weyl group $W_0 = W_0(R_0)$ is the subgroup of $\mathrm{O}(V)$ generated by the orthogonal reflections $s_\alpha \in \mathrm{O}(V)$ in the hyperplanes $\alpha^\perp$ $(\alpha \in R_0)$, which are explicitly given by $s_\alpha(v) = v - (\alpha^\vee, v)\alpha$ for $v \in V$. The Weyl group W_0 coincides with the image of $W(R)$ under the differential d.

We now define an appropriate equivalence relation between affine root systems, called *similarity*. It is a slightly weaker notion of similarity compared to the one used in [66, §3]. This is to render affine root systems similar that differ by a rescaling of the underlying gradient root system, see Remark A.6.

Definition A.5 We call two affine root systems $R \subset \widehat{E}$ and $R' \subset \widehat{E'}$ *similar*, $R \simeq R'$, if there exists a linear isomorphism $T\colon \widehat{E} \xrightarrow{\sim} \widehat{E'}$ which restricts to a bijection of R onto R' preserving Cartan integers: $((Ta)^\vee, Tb) = (a^\vee, b)$ for $a, b \in R$.

Similarity respects basic notions such as affine root subsystems and irreducibility. If $R \simeq R'$ is realized by the linear isomorphism $T\colon \widehat{E} \xrightarrow{\sim} \widehat{E'}$ then $T s_a T^{-1} = s_{Ta}$ for all $a \in R$. In particular, $W(R) \simeq W(R')$. Note that T maps constant functions to constant functions. Replacing T by $-T$ if necessary, we may assume without loss of generality that $T(c) \in \mathbb{R}_{>0}c'$, where $c \in \widehat{E}$ and $c' \in \widehat{E'}$ denote the constant functions one on E and E' respectively. With this additional condition we call T a *similarity transformation* between R and R'.

If R is an affine root system and $\lambda \in \mathbb{R}^* := \mathbb{R} \setminus \{0\}$ then $\lambda R := \{\lambda a\}_{a \in R}$ is an affine root system similar to R (the similarity transformation realizing $R \xrightarrow{\sim} \lambda R$ is scalar multiplication by $|\lambda|$). We call λR a *rescaling* of the affine root system R. If two affine root systems R and R' are similar, then a similarity transformation T between R and a rescaling of R' exists such that $T(c) = c'$. In this case T arises as the contragredient of an affine linear isomorphism from E' onto E. For instance, each affine Weyl group element $w \in W(R)$ is a selfsimilarity transformation of R in this way.

If the affine root systems $R \subset \widehat{E}$ and $R' \subset \widehat{E'}$ are similar, then so are their gradients $R_0 \subset V$ and $R_0' \subset V'$ (i.e., there exists a linear isomorphism t of V onto V' restricting to a bijection of $R_0 \xrightarrow{\sim} R_0'$ and preserving Cartan integers). Indeed, if T is a similarity transformation between R and R', then the unique linear isomorphism $t\colon V \xrightarrow{\sim} V'$ such that $D \circ T = t \circ D$ realizes the similarity between R_0 and R_0'.

Remark A.6 Let $R \subset \widehat{E}$ be an irreducible affine root system with associated gradient $R_0 \subset V$. Fix an origin $e \in E$. For $\lambda \in \mathbb{R}^* \setminus \{\pm 1\}$ and $a \in R$ define $a_\lambda \in \widehat{E}$ by $a_\lambda(e+v) := a(e) + \lambda(Da, v)$ for all $v \in V$. Then $R_\lambda := \{a_\lambda\}_{a \in R} \subset \widehat{E}$ is an affine root system similar to R, and $R_{\lambda,0} = \lambda R_0$. The affine root systems R and R_λ are not similar if one uses Macdonald's [66, §3] definition of similarity.

In the remainder of this Appendix we assume that R is an irreducible affine root system of rank n. Since $W(R)$ acts properly on E, the set of *regular* elements $E_{\mathrm{reg}} := E \setminus \bigcup_{a \in R} a^{-1}(0)$ decomposes as the disjoint union of open n-simplices, called *chambers* of R. For a fixed chamber C there exists a unique $\mathbb{R}$-basis $\Delta = \Delta(C, R)$ of $\widehat{E}$ consisting of indivisible affine roots $a_i = a_{i,C}$ $(0 \le i \le n)$ such that $C = \{e \in E \mid a_i(e) > 0 \ \forall i \in \{0, \ldots, n\}\}$. The set of roots Δ is called the *basis* of R associated to the chamber C. The affine roots a_i are called *simple affine roots*. Any affine root $a \in R$ can be uniquely written as $a = \sum_{i=0}^n \lambda_i a_i$ with either all $\lambda_i \in \mathbb{Z}_{\ge 0}$ or all $\lambda_i \in \mathbb{Z}_{\le 0}$. The subset of affine roots of the first type is denoted by R^+ and is called the set of *positive affine roots* with respect to Δ. Then $R = R^+ \cup R^-$ (disjoint union) with $R^- := -R^+$ the subset of *negative affine roots*. The affine Weyl group $W(R)$ is a Coxeter group with the simple reflections s_{a_i} $(0 \le i \le n)$ as Coxeter generators.

Definition A.7 A rank n *affine Cartan matrix* is a rational integral $(n+1) \times (n+1)$-matrix $A = (a_{ij})_{i,j=0}^n$ satisfying the four conditions

1 $a_{ii} = 2$,
2 $a_{ij} \in \mathbb{Z}_{\leq 0}$ if $i \neq j$,
3 $a_{ij} = 0$ implies $a_{ji} = 0$,
4 $\det(A) = 0$ and all the proper principal minors of A are strictly positive,
5 A is indecomposable (i.e., the matrices obtained from A by a simultaneous permutation of its rows and columns are not the direct sum of two nontrivial blocks).

The larger class of rational integral matrices satisfying conditions (1)–(3) are called *generalized Cartan matrices*. They correspond to Kac–Moody Lie algebras, see [49]. The Kac–Moody Lie algebras related to the subclass of affine Cartan matrices are the affine Lie algebras. The affine Cartan matrices have been classified by Kac [49, Ch. 4].

Fix an ordered basis $\Delta = (a_0, a_1, \ldots, a_n)$ of R. The matrix $A = A(R, \Delta) = (a_{ij})_{0 \leq i,j \leq n}$ defined by $a_{ij} := (a_i^\vee, a_j)$ is an affine Cartan matrix. The coefficients a_{ij} $(0 \leq i, j \leq n)$ are called the *affine Cartan integers* of R.

If $R \simeq R'$ with associated similarity transformation T, then the T-image of an ordered basis Δ of R is an ordered basis Δ' of R'. Let R and R' be irreducible affine root systems with ordered bases Δ and Δ' respectively. We say that (R, Δ) is *similar* to (R', Δ'), $(R, \Delta) \simeq (R', \Delta')$, if the irreducible affine root systems R and R' are similar and if there exists an associated similarity transformation T mapping the ordered basis Δ of R to the ordered basis Δ' of R'. For similar pairs the associated affine Cartan matrices coincide. Moreover, the affine Cartan matrix $A(R, \Delta)$ modulo simultaneous permutations of its row and columns does not depend on the choice of ordered basis Δ. This leads to a map from the set of similarity classes of reduced irreducible affine root systems to the set of affine Cartan matrices up to simultaneous permutations of rows and columns. Using Kac' [49, Ch. 4] classification of the affine Cartan matrices and the explicit construction of reduced irreducible affine root systems from [66] (see also the next subsection), it follows that the map is surjective. It is injective by a straightforward adjustment of the proof for finite crystallographic root systems to the present affine setup (see [43, Prop. 11.1]). Hence we obtain the following classification result.

Theorem A.8 *Reduced irreducible affine root systems up to similarity are in bijective correspondence to affine Cartan matrices up to simultaneous permutations of the rows and columns.*

Remark A.9 Affine Cartan matrices also parametrize affine Lie algebras (see [49]). For a given affine Cartan matrix the set of real roots of the associated affine Lie algebra is the associated irreducible reduced affine root system.

Affine Cartan matrices up to simultaneous permutations of the rows and columns (and hence similarity classes of irreducible reduced affine root systems) can be naturally encoded by affine Dynkin diagrams [49, 66]. The *affine Dynkin diagram* associated to an affine Cartan matrix $A = (a_{ij})_{i,j=0}^n$ is the graph with $n+1$ vertices in which we join the i-th and j-th node $(i \neq j)$ by $\max(|a_{ij}|, |a_{ji}|)$ edges. In addition we put an arrow towards the ith node if $|a_{ij}| > 1$. In §A.4 we list all affine Dynkin diagrams and link this to Kac's [49] classification.

A.2 Explicit constructions

The main reference for the results in this subsection is [66]. For an irreducible finite crystallographic root system R_0 of type A, D, E or BC there is exactly one similarity class of reduced irreducible affine root systems whose gradient root system is similar to R_0. For the other types of root systems R_0, there are two such similarity classes of reduced irreducible affine root systems. We now proceed to realize them explicitly.

Let $R_0 \subset V$ be an irreducible finite crystallographic root system (possibly nonreduced). The affine space E is taken to be V with forgotten origin. We will write V for E in the sequel if no confusion is possible.

We identify the space $\widehat{E}$ of affine linear functions on E with $V \oplus \mathbb{R}c$ as real vector space, with c the constant function identically equal to one on V and with $V^* \simeq V$ the linear functionals on V (the identification with V is realized by the scalar product on V). With these identifications,

$$\mathrm{O}_c(\widehat{E}) \simeq \mathrm{O}(E) = \mathrm{O}(V) \ltimes \tau(V),$$

with $\tau(v)(e) = e + v$. Regarding $\tau(v)$ as element of $\mathrm{O}_c(\widehat{V})$ it is given by

$$\tau(v)a = -(Da, v)c + a, \qquad a \in \widehat{V}.$$

Note that the orthogonal reflection $s_a \in \mathrm{O}(E)$ associated to $a = \lambda c + \alpha \in \widehat{E}$ ($\lambda \in \mathbb{R}$, $\alpha \in V \setminus \{0\}$) decomposes as $s_{\lambda c+\alpha} = \tau(-\lambda\alpha^\vee)s_\alpha$.

Consider the subset

$$\mathcal{S}(R_0) := \{mc + \alpha\}_{m\in\mathbb{Z},\, \alpha\in R_0^{\mathrm{ind}}} \cup \{(2m+1)c + \beta\}_{m\in\mathbb{Z},\, \beta\in R_0\setminus R_0^{\mathrm{ind}}}$$

of $\widehat{V}$, where $R_0^{\mathrm{ind}} \subseteq R_0$ is the root subsystem of indivisible roots. Then $\mathcal{S}(R_0)$ and $\mathcal{S}(R_0^\vee)^\vee$ are reduced irreducible affine root systems with gradient root system R_0. We call $\mathcal{S}(R_0)$ (respectively $\mathcal{S}(R_0^\vee)^\vee$) the *untwisted* (respectively *twisted*) reduced irreducible affine root system associated to R_0. Note that $\mathcal{S}(R_0) \simeq \mathcal{S}(R_0^\vee)^\vee$ if R_0 is of type A, D, E or BC.

Proposition A.10 *The following reduced irreducible affine root systems form a complete set of representatives of the similarity classes of reduced irreducible affine root systems:*

1 $\mathcal{S}(R_0)$ *with R_0 running through the similarity classes of reduced irreducible finite crystallographic root systems (i.e., R_0 of type* A, B, …, G*),*

2 $\mathcal{S}(R_0^\vee)^\vee$ *with R_0 running through the similarity classes of reduced irreducible finite crystallographic root systems having two root lengths (i.e., R_0 of type* B_n *($n \geq 2$),* C_n *($n \geq 3$),* F_4 *and* G_2*),*

3 $\mathcal{S}(R_0)$ *with R_0 a nonreduced irreducible finite crystallographic root system (i.e., R_0 of type* BC_n *($n \geq 1$)).*

In view of the above proposition we use the following terminology: a reduced irreducible affine root system R is said to be of *untwisted type* if $R \simeq \mathcal{S}(R_0)$ with R_0 reduced, of *twisted type* if $R \simeq \mathcal{S}(R_0^\vee)^\vee$ with R_0 reduced, and of *mixed type* if $R \simeq \mathcal{S}(R_0)$ with R_0 nonreduced. Note that a reduced irreducible affine root system R with gradient root system of type A, D or E is of untwisted and of twisted type.

Suppose that $\Delta_0 = (\alpha_1, \ldots, \alpha_n)$ is an ordered basis of R_0. Let $\varphi \in R_0$ (respectively $\theta \in R_0$) be the associated highest root (respectively the highest short root). Then

$$\Delta := (a_0, a_1, \ldots, a_n) = (c - \varphi, \alpha_1, \ldots, \alpha_n)$$

is an ordered basis of $\mathcal{S}(R_0)$, while

$$\Delta := (a_0, a_1, \ldots, a_n) = (\tfrac{1}{2}|\theta|^2 c - \theta, \alpha_1, \ldots, \alpha_n)$$

is an ordered basis of $\mathcal{S}(R_0^\vee)^\vee$.

A.3 Nonreduced irreducible affine root systems

If R is an irreducible affine root system with ordered basis Δ, then Δ is also an ordered basis of the affine root subsystem R^{ind} of indivisible roots. Furthermore, $R \simeq R'$ implies $R^{\mathrm{ind}} \simeq R'^{,\mathrm{ind}}$. To classify nonreduced irreducible affine root systems up to similarity, one thus only needs to understand the possible ways to extend reduced irreducible affine root systems to nonreduced ones.

Let R' be a reduced irreducible affine root system with affine Weyl group $W = W(R')$. Choose an ordered basis $\Delta = (a_0, a_1, \ldots, a_n)$ of R'. Set

$$S := \{a \in \Delta \mid (\mathbb{Z}R', a^\vee) = 2\mathbb{Z}\}. \tag{A.1}$$

Let $S_m \subset S$ with $\#S_m = m \in \{0, \ldots, \#S\}$. Then $R^{(m)} := R' \cup \bigcup_{a \in S_m} W(2a)$ is an irreducible affine root system with $R^{(m),\mathrm{ind}} \simeq R'$.

By consideration of the possible affine Dynkin diagrams associated to (R', Δ) (see §A.4), it follows that the set S (see (A.1)) is of cardinality at most two. It is of cardinality two iff $R' \simeq \mathcal{S}(R_0^\vee)^\vee$ with R_0 of type A_1 or with R_0 of type B_n ($n \geq 2$). It is of cardinality one iff $R' \simeq \mathcal{S}(R_0)$ with R_0 of type B_n ($n \geq 2$) or of type BC_n ($n \geq 1$). Hence the similarity class of $R^{(m)}$ does not depend on the choice of subset $S_m \subseteq S$ of cardinality m, and it does not depend on the choice of ordered basis Δ of R'. The number of W-orbits of $R^{(m)}$ equals the number of W-orbits of R' plus m. The number of similarity classes of irreducible affine root systems R satisfying $R^{\mathrm{ind}} \simeq R'$ is $\#S + 1$.

If R_0 is of type A_1 or of type B_n ($n \geq 2$) we thus have a nonreduced irreducible affine root system in which two W-orbits are added to $\mathcal{S}(R_0^\vee)^\vee$. It is labelled as $\mathrm{C}^\vee\mathrm{C}_n$ by Macdonald [66]. In the rank one case it has four W-orbits, otherwise five. A detailed description of this affine root system is given in §9.3.8.

Irreducible affine root subsystems with underlying reduced affine root system $\mathcal{S}(R_0)$ having finite root system R_0 of type BC_n ($n \geq 1$) or of type B_n ($n \geq 3$) can be naturally viewed as affine root subsystems of the affine root system of type $\mathrm{C}^\vee\mathrm{C}_n$. This is not the case for the nonreduced extension of the affine root system $\mathcal{S}(R_0)$ with R_0 of type B_2. It can actually be better viewed as the rank two case of the family $\mathcal{S}(R_0)$ with R_0 of type C_n since, in the corresponding affine Dynkin diagram, the vertex labelled by the affine simple root a_0 is double bonded with the finite Dynkin diagram of R_0. The nonreduced extension of $\mathcal{S}(R_0)$ with R_0 of type C_2 was missing in Macdonald's [66] classification list. It was added in [73, (1.3.17)].

A.4 Affine Dynkin diagrams

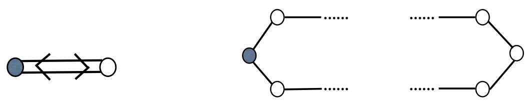
In this subsection we list the connected affine Dynkin diagrams (cf. [66, Appendix 1]) which, as we have seen, are in one-to-one correspondence to similarity classes of irreducible reduced affine root systems. Each similarity class of irreducible reduced affine root systems has a representative of the form $\mathcal{S}(R_0)$ or $\mathcal{S}(R_0^\vee)^\vee$ for a unique irreducible finite crystallographic root system R_0 up to similarity, see §A.2. Recall that $\mathcal{S}(R_0^\vee)^\vee \simeq \mathcal{S}(R_0)$ if R_0 is of type $\mathrm{A}, \mathrm{D}, \mathrm{E}, \mathrm{BC}$.

We label the connected affine Dynkin diagram by $\widehat{X}$ with X the type of the associated finite root system R_0 if $X \in \{\mathrm{A}, \mathrm{D}, \mathrm{E}, \mathrm{BC}\}$. If the associated finite root system R_0 is of type $X \in \{\mathrm{B}, \mathrm{C}, \mathrm{F}, \mathrm{G}\}$ then we label the connected affine Dynkin diagram by $\widehat{X}^u$ (respectively $\widehat{X}^t$) if the associated irreducible reduced affine root system is $\mathcal{S}(R_0)$ (respectively $\mathcal{S}(R_0^\vee)^\vee$). Since $\mathrm{A}_1 \simeq \mathrm{B}_1 \simeq \mathrm{C}_1$ and $\mathrm{B}_2 \simeq \mathrm{C}_2$, there is some redundancy in the notations. We pick the one which is most convenient to fit it into an infinite family of affine Dynkin diagrams. In the terminology of §A.2, the irreducible reduced affine root systems corresponding to affine Dynkin diagrams labelled by $\widehat{X}$ with $X \in \{\mathrm{A}, \mathrm{D}, \mathrm{E}\}$ are of untwisted and of twisted type, labelled by $\widehat{\mathrm{BC}}$ of mixed type, labelled by $\widehat{X}^u$ of untwisted type and labelled by $\widehat{X}^t$ of twisted type. In [66, Appendix 1] the affine Dynkin diagrams labelled $\widehat{\mathrm{B}}_n^t$ and $\widehat{\mathrm{C}}_n^t$ are called of type $\mathrm{C}_n^\vee$ and $\mathrm{B}_n^\vee$ respectively. The remaining relations with the notations and terminologies in [66, Appendix 1] are self-explanatory.

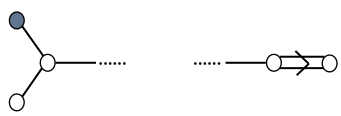
We specify in each affine Dynkin diagram a particular vertex (the grey vertex) which is labelled by the unique affine simple root a_0 in the particular choice of ordered basis Δ of $\mathcal{S}(R_0)$ or $\mathcal{S}(R_0^\vee)^\vee$ as specified in §A.2.

In Kac's notations (see Tables Aff 1–3 in [49, §4.8]) the affine Dynkin diagrams are labelled differently: our label $\widehat{X}$ corresponds to $X^{(1)}$ if $X \in \{\mathrm{A}, \mathrm{D}, \mathrm{E}\}$, and $\widehat{\mathrm{BC}}_n$ corresponds to $\mathrm{A}_{2n}^{(2)}$ ($n \geq 1$). Our label $\widehat{X}^u$ corresponds to $X^{(1)}$ if $X \in \{\mathrm{B}, \mathrm{C}, \mathrm{F}, \mathrm{G}\}$. Finally, $\widehat{\mathrm{B}}_n^t$ corresponds to $\mathrm{D}_{n+1}^{(2)}$ ($n \geq 2$), $\widehat{\mathrm{C}}_n^t$ corresponds to $\mathrm{A}_{2n-1}^{(2)}$ ($n \geq 3$), $\widehat{\mathrm{F}}_4^t$ to $\mathrm{E}_6^{(2)}$ and $\widehat{\mathrm{G}}_2^t$ to $\mathrm{D}_4^{(3)}$.

$\widehat{\mathrm{A}}_1$ and $\widehat{\mathrm{A}}_n$ ($n \geq 2$)

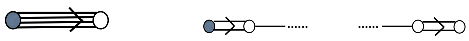


$\widehat{\mathrm{B}}_n^u$ ($n \geq 3$)



$\widehat{\mathrm{B}}_n^t$ ($n \geq 2$)

$\widehat{\mathrm{BC}}_1$ and $\widehat{\mathrm{BC}}_n$ ($n \geq 2$)

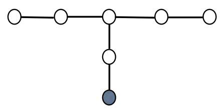


$\widehat{\mathrm{C}}_n^u$ ($n \geq 2$)

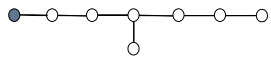
$\widehat{\mathrm{C}}_n^t$ ($n \geq 3$)

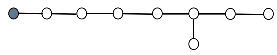
$\widehat{\mathrm{D}}_n$ ($n \geq 4$)

$\widehat{\mathrm{E}}_6$

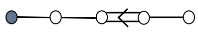


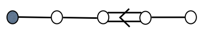
$\widehat{\mathrm{E}}_7$ and $\widehat{E}_8$



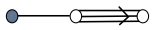


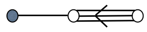
$\widehat{\mathrm{F}}_4^u$ and $\widehat{\mathrm{F}}_4^t$





$\widehat{\mathrm{G}}_2^u$ and $\widehat{\mathrm{G}}_2^t$





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