GQT course "Quantum integrable systems"

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Topics:

1. Integrable lattice models
2. Associated integrable structures:
   - a Quantum Yang-Baxter equations
   - b Scattering matrices
   - c Quantum Knizhnik-Zamolodchikov (KZ) equations

~~[illegible] algebraic approach~~

3 Representation theoretic constructions of solutions of quantum Yang-Baxter equations.
   - a. Affine braid group.
   - b Affine Hecke algebra.
   - c Baxterization
   - d Example: the 6-vertex model or Heisenberg XXZ spin-$\frac{1}{2}$ chain.
   - e. Relation to affine Temperley-Lieb algebras
   - f. Loop models

1. Integrable lattice models

Graphical representation of a linear map

$$R(z): \mathbb{C}^2 \otimes \mathbb{C}^2 \to \mathbb{C}^2 \otimes \mathbb{C}^2$$

depending meromorphically on $z \in \mathbb{C}$, and of its matrixcoefficients.

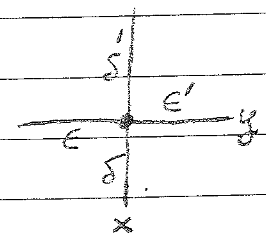
(*) write $\{v_+, v_-\} = \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$.

and $R(z) v_{\delta'} \otimes v_{\epsilon'} = \sum_{\delta, \epsilon} R^{\delta\epsilon}_{\delta'\epsilon'}(z) v_\delta \otimes v_\epsilon$.

for $\delta, \delta', \epsilon, \epsilon' \in \{\pm\}$

Pictorially:

$$R^{\delta\epsilon}_{\delta'\epsilon'}(x-y) =$$



Note: redundancy in assigning "rapidities" to vertical and horizontal directions: only dependence on their difference.

Picking indices $1 \le i \ne j \le M$ numbering the tensor copies of $\mathbb{C}^2$'s in $\underset{1}{\mathbb{C}^2} \otimes \underset{2}{\mathbb{C}^2} \otimes \cdots \otimes \underset{M}{\mathbb{C}^2} = (\mathbb{C}^2)^{\otimes M}$, we can interpret

$$R(z)$$

as linear operator on $(\mathbb{C}^2)^{\otimes M}$ acting as $R(z)$ on the i^{th} and j^{th} tensor component, and as identity on remaining tensor factors; we then write it as $R_{ij}(z)$.

in coordinates (case $i<j$):

$$R_{ij}(z) v_{\epsilon_1} \otimes v_{\epsilon_2} \otimes \cdots \otimes v_{\epsilon_M} =$$

$$= \sum_{\delta_i, \delta_j} R^{\delta_i \delta_j}_{\epsilon_i \epsilon_j}(z) v_{\epsilon_1} \otimes \cdots \otimes v_{\epsilon_{i-1}} \otimes v_{\delta_i} \otimes v_{\epsilon_{i+1}} \otimes \cdots \otimes v_{\epsilon_{j-1}} \otimes v_{\delta_j} \otimes v_{\epsilon_{j+1}} \otimes \cdots \otimes v_{\epsilon_M}$$

Pictorial description of products of $R(z)$'s and their matrix coefficients:

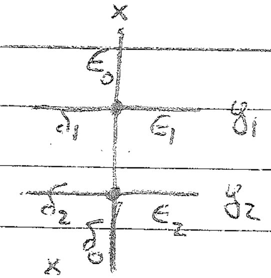
As operators on $(\mathbb{C}^2)^{\otimes 3} = \underset{0}{\mathbb{C}^{2}} \otimes (\underset{1}{\mathbb{C}^{2}} \otimes \underset{2}{\mathbb{C}^{2}})$

$$R_{02}(x-y_2) R_{01}(x-y_1) v_{\epsilon_0} \otimes v_{\epsilon_1} \otimes v_{\epsilon_2} = \Big(\sum_{\xi_0} R^{\delta_0 \delta_2}_{\xi_0 \epsilon_2}(x-y_2) R^{\xi_0 \delta_1}_{\epsilon_0 \epsilon_1}(x-y_1)\Big)$$

$$\times\, v_{\delta_1} \otimes v_{\delta_2} \otimes v_{\delta_3}$$

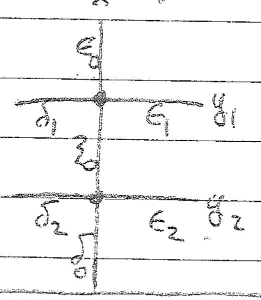
is pictorially represented as:

$$\sum_{\xi_0} R^{\delta_0 \delta_2}_{\xi_0 \epsilon_2}(x-y_2) R^{\xi_0 \delta_1}_{\epsilon_0 \epsilon_1}(x-y_1) =$$



while

$$R^{\delta_0 \delta_2}_{\xi_0 \epsilon_2}(x-y_2) R^{\xi_0 \delta_1}_{\epsilon_0 \epsilon_1}(x-y_1) =$$



Twist operator: $\gamma = (\gamma_+, \gamma_-) \in (\mathbb{C}^*)^2$

$$D_\gamma : \mathbb{C}^2 \to \mathbb{C}^2, \text{ linear: } D_\gamma v_\epsilon := \gamma_\epsilon v_\epsilon$$

Pictorially: $\underset{\epsilon}{\overset{\delta}{\boxed{\gamma}}} = \begin{cases} 0 & \text{if } \delta \ne \epsilon \\ \gamma_\delta & \text{if } \delta = \epsilon. \end{cases}$

Definition ("monodromy matrix"). $\underline{y} = (y_1, \ldots, y_M)$.

$$\mathcal{M}(x; \underline{y}) := R_{0M}(x-y_M) R_{0, M-1}(x-y_{M-1}) \cdots R_{01}(x-y_1) (D_\gamma)_0$$

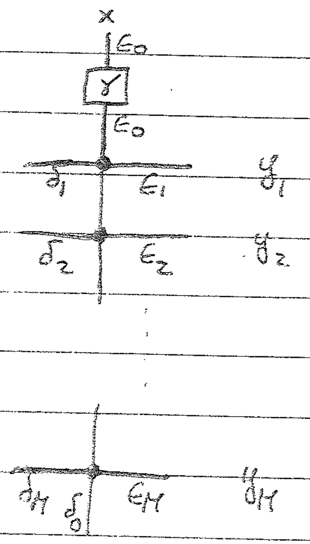
as linear operator on $\underset{0}{\mathbb{C}^2} \otimes (\underset{1}{\mathbb{C}^2} \otimes \cdots \otimes \underset{M}{\mathbb{C}^2}) = (\mathbb{C}^2)^{\otimes (M+1)}$

Exercise 1.1: Writing

$$\mathcal{M}(x;y)\,(v_{\epsilon_0}\otimes v_{\epsilon_1}\otimes\cdots\otimes v_{\epsilon_M}) = \sum_{\delta_0\cdots\delta_M} \mathcal{M}^{\delta_0\cdots\delta_M}_{\epsilon_0\cdots\epsilon_M}(x;y)\, v_{\delta_0}\otimes\cdots\otimes v_{\delta_M}.$$

then:

$$\mathcal{M}^{\delta_0\cdots\delta_M}_{\epsilon_0\cdots\epsilon_M}(x;y) =$$



pictorially. Prove this.

Write: $\mathcal{M}(x;y) = \begin{pmatrix} A(x;y) & B(x;y) \\ C(x;y) & D(x;y) \end{pmatrix}$

with

$$A(x;y),\ B(x;y),\ C(x;y),\ D(x,y) : (\mathbb{C}^2)^{\otimes M} \circlearrowleft$$

the matrix coeff. of $\mathcal{M}(x;y)$ w.r.t. basis $\{v_+, v_-\}$ of $\mathbb{C}^2_0$:

$$\mathcal{M}(x;y)(v_+\otimes\cdot) = v_+\otimes(A(x;y)\cdot) + v_-\otimes(C(x;y)\cdot),$$

$$\mathcal{M}(x;y)(v_-\otimes\cdot) = v_+\otimes(B(x;y)\cdot) + v_-\otimes(D(x;y)\cdot).$$

Then:

Definition: The transfer operator is

$$T(x;y) := \mathrm{Tr}_{\mathbb{C}^2_0}(\mathcal{M}(x;y))$$

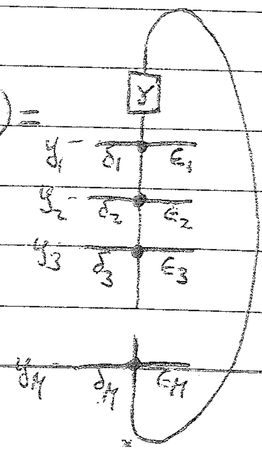
$$= A(x;y) + D(x;y)$$

(as linear operators on $(\mathbb{C}^2)^{\otimes M}$)

In pictures: $T(x;y)\,v_{\epsilon_1}\otimes\cdots\otimes v_{\epsilon_M} = \sum_{\delta_1\cdots\delta_M} T^{\delta_1\cdots\delta_M}_{\epsilon_1\cdots\epsilon_M}(x;y)\, v_{\delta_1}\otimes\cdots\otimes v_{\delta_M}$

with

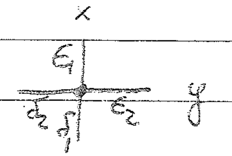
$$T^{\delta_1\cdots\delta_M}_{\epsilon_1\cdots\epsilon_M}(x;y) =$$



Relation to lattice models

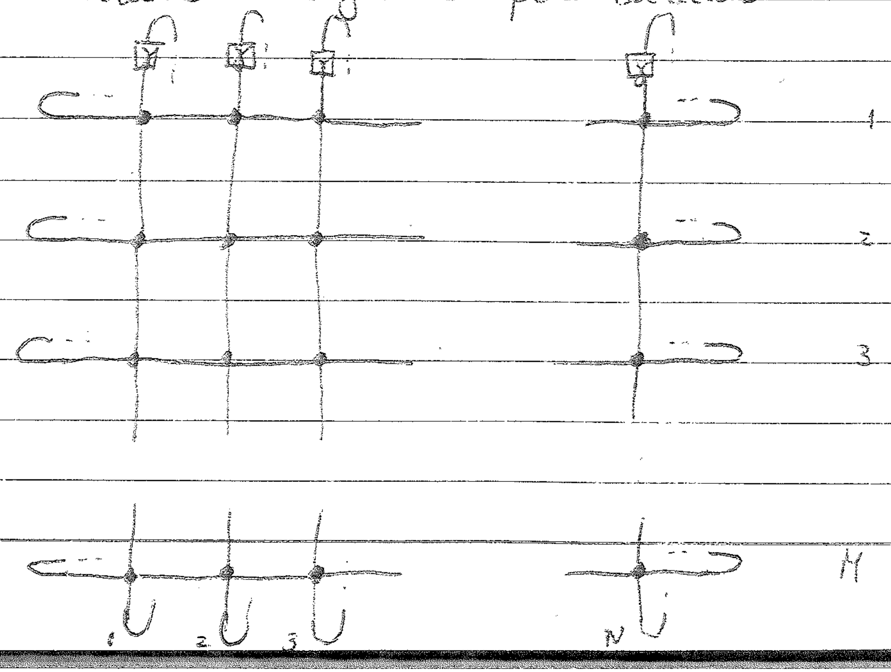
Main idea: consider

$$R^{\delta_1\delta_2}_{\epsilon_1\epsilon_2}(x-y) = $$



as the local weight attached to a vertex in a square lattice with local vertex configuration given by $\delta_1, \delta_2, \epsilon_1, \epsilon_2$.

The lattice we take is the rectangular $M \times N$-lattice wrapped around the torus with twisted cyclic boundary conditions in one of the two period directions



As a graph (forgetting coupons labeled γ):

$$\mathcal{G} = (V, E)$$

with vertex set $V = \{1, \dots, M\} \times \{1, \dots, N\} \cong \mathbb{Z}/M\mathbb{Z} \times \mathbb{Z}/N\mathbb{Z}$ and edges connecting to vertex $(i,j) \in \mathbb{Z}/M\mathbb{Z} \times \mathbb{Z}/N\mathbb{Z}$ given by

$$((i,j),(i\pm1,j)) \;\&\; ((i,j),(i,j\pm1))$$

Def. A <u>vertex configuration</u> $\mathcal{C}$ on $\mathcal{G}$ is a map

$$\mathcal{C}: E \to \{\pm\}, \quad e \mapsto \mathcal{C}_e$$

assigning a sign to each edge of $\mathcal{G}$.

Def. Let $R(z): \mathbb{C}^2 \otimes \mathbb{C}^2 \to \mathbb{C}^2 \otimes \mathbb{C}^2$, $\gamma = (\gamma_+, \gamma_-)$ as before, and two sets of variables

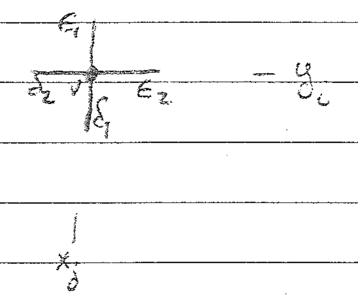
$$\underline{x} = (x_1, \dots, x_N), \quad \underline{y} = (y_1, \dots, y_M).$$

Let $v \in V$ and $\mathcal{C}$ a configuration.

(1) The local vertex Boltzmann weight of $\mathcal{C}$ at v is:

$$W_v(\mathcal{C}) := R^{\delta_1\delta_2}_{\epsilon_1\epsilon_2}(x_j - y_i)$$

if the local configuration is given by: $v=(i,j)$ with $1\le i\le M$, $1\le j\le N$ and



(ii) The total vertex Boltzmann weight of $\mathcal{C}$ is

$$W(\mathcal{C}) := \gamma_{\mathcal{C}} \prod_{v\in V} W_v(\mathcal{C})$$

with $\gamma_{\mathcal{C}} = \gamma_{\mathcal{C}_{e_1}} \gamma_{\mathcal{C}_{e_2}} \cdots \gamma_{\mathcal{C}_{e_N}}$ and $e_\ell \in E$ the edge

$$e_\ell := ((M,\ell),(1,\ell)) \qquad (1\le \ell\le N)$$

(iii) The partition function of $(\mathcal{G}, R, \gamma)$ with rapidities is

$$Z(\underline{x};\underline{y}) := \sum_{\substack{\mathcal{C} \\ \text{configur. on } \mathcal{G}}} W(\mathcal{C}).$$

Formal statistical mechanical interpretation:

* ($\mathcal{G}$, conf.) describes a classical 2-dim. lattice model on $\mathcal{G}$ with configurations as possible states.
* (in case $W(\mathcal{C}) > 0$ for all conf. $\mathcal{C}$): The chance that the model is in state $\mathcal{C}$ is

$$\frac{W(\mathcal{C})}{Z(\underline{x};\underline{y})}.$$

The choice of local weights thus dictates what the probability distribution will be.

Theorem (exercise 1.2).

$$Z(\underline{x};\underline{y}) = \mathrm{Tr}_{(\mathbb{C}^2)^{\otimes M}}\left(T(x_1;\underline{y}) \cdots T(x_N;\underline{y})\right)$$

with $T(x;\underline{y})$ the transfer matrix w.r.t. to the initial data (R,γ).

Remark: Many quantities of physical interest are directly expressible in terms of the partition function. Key ~~one~~ is to "compute" the partition function as explicit as possible.

The theorem gives a natural criterium to relate the computation of partition functions to spectral problems of commuting operators:

Corollary: Suppose that the transfer operators $T(x;y) : (\mathbb{C}^2)^{\otimes M} \circlearrowleft$ ~~pair~~ commute:

$$[T(x;y), T(x';y)] = 0,$$

~~Then~~ and are simultaneously diagonalisable:

$$(\mathbb{C}^2)^{\otimes M} = \bigoplus_{i=1}^{L} W_i, \quad W_i := \{v \in (\mathbb{C}^2)^{\otimes M} \mid T(x;y)v = \lambda_i(x;y)v \ \forall x\}$$

for proper $\lambda_i(x;y) \in \mathbb{C}$.

Then $Z(x;y) = \sum_{i=1}^{L} \mathrm{Dim}(W_i)\lambda_i(x_1;y)\cdots\lambda_i(x_N;y)$

(spectral formula for the partition function).

2. Associated integrable structures

2.a Quantum Yang-Baxter equation

Definition: The vertex model $(\mathcal{G}, R, \gamma)$ is called integrable if:

(1) $R(z)$ is generically invertible and satisfies the quantum Yang-Baxter equation

$$R_{12}(x_1-x_2)R_{13}(x_1-x_3)R_{23}(x_2-x_3) = R_{23}(x_2-x_3)R_{13}(x_1-x_3)R_{12}(x_1-x_2)$$

as linear operators on $\overset{1}{\mathbb{C}^2} \otimes \overset{2}{\mathbb{C}^2} \otimes \overset{3}{\mathbb{C}^2}$;

and (2) $[D_\gamma \otimes D_\gamma, R(z)] = 0$ (ice-rule).

Proposition (exercise 2.1):

Let $M_0(x;y)$ and $M_{0'}(x;y)$ be the monodromy operator assoc. to (R,γ), acting on the tensor legs ~~of~~ $\underset{0}{\mathbb{C}^2} \otimes (\mathbb{C}^2)^{\otimes M}$ (resp. $\underset{0'}{\mathbb{C}^2} \otimes (\mathbb{C}^2)^{\otimes M}$) of $\underset{0}{\mathbb{C}^2} \otimes \underset{0'}{\mathbb{C}^2} \otimes (\mathbb{C}^2)^{\otimes M}$:

$$M_0(x;y) = R_{0M}(x-y_M)R_{0,M-1}(x-y_{M-1})\cdots R_{01}(x-y_1)(D_\gamma)_0$$

$$M_{0'}(x;y) = R_{0'M}(x-y_M)R_{0',M-1}(x-y_{M-1})\cdots R_{0'1}(x-y_1)(D_\gamma)_{0'}$$

Then:

$$R_{00'}(x-x')M_0(x;y)M_{0'}(x';y) = M_{0'}(x';y)M_0(x;y)R_{00'}(x-x')$$

as linear operators on $\underset{0}{\mathbb{C}^2} \otimes \underset{0'}{\mathbb{C}^2} \otimes (\mathbb{C}^2)^{\otimes M}$

Theorem: If $(\mathcal{G}, R, \gamma)$ is integrable, then

$$[T(x;y), T(x';y)] = 0$$

as lin. operators on $(\mathbb{C}^2)^{\otimes M}$.

Proof. $T(x;y)T(x';y) = \mathrm{Tr}_{\mathbb{C}^2_0 \otimes \mathbb{C}^2_{0'}}\left(M_0(x;y)M_{0'}(x';y)\right)$

$= \mathrm{Tr}_{\mathbb{C}^2_0 \otimes \mathbb{C}^2_{0'}}\left(R_{00'}(x-x')M_0(x;y)M_{0'}(x';y)R_{00'}(x-x')^{-1}\right)$

by cyclicity of the trace $\mathrm{Tr}_{\mathbb{C}^2_0 \otimes \mathbb{C}^2_{0'}}$

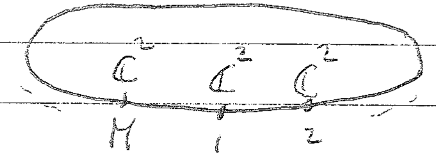
$= \mathrm{Tr}_{\mathbb{C}^2_0 \otimes \mathbb{C}^2_{0'}}\left(M_{0'}(x';y)M_0(x;y)\right)$

by exercise 2.3

$= T(x';y)T(x;y)$. $\boxtimes$

Remarks: (1) Can think of $T(x;y) : (\mathbb{C}^2)^{\otimes M} \circlearrowleft$ $(x \in \mathbb{C})$ as commuting family of quantum Hamiltonians for a one-dimensional quantum spin chain on the circle

(with twisted boundary conditions); Quantum state space is $(\mathbb{C}^2)^{\otimes M}$ with copy of $\mathbb{C}^2$ attached to the M lattice sites on the circle



Interplay: $\{$2-dimensional integrable vertex models$\}$

$\updownarrow$

$\{$1-dimensional quantum integrable spin chains$\}$

Solving the spectral problem of the quantum Hamiltonians is synonymous to solving the quantum integrable spin chain.

Remark: Trivial solutions of the quantum Yang-Baxter equations are $R(z) = \mathrm{Id}_{\mathbb{C}^2 \otimes \mathbb{C}^2}$ and $R(z) = P : \mathbb{C}^2 \otimes \mathbb{C}^2 \circlearrowleft$ permutation operator (constant solutions). We say that a solution $R(z)$ is regular if

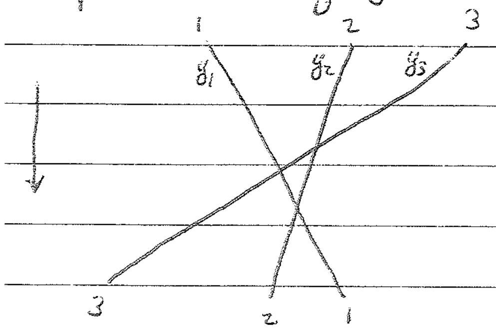
$$R(0) = P.$$

If $R(z)$ is a solution of qYB eqn (invertible), then so is $R_{21}(-z)^{-1}$. We say that $R(z)$ is unitary if $R(z) = R_{21}(-z)^{-1}$.

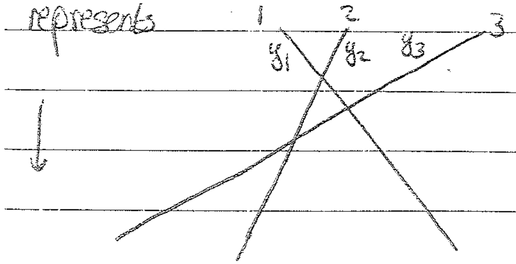
2.6. Scattering matrices: $(\mathcal{G}, R, \gamma)$ integrable

Think of
$$R_{ij}(y_i - y_j) \text{ as scattering data}$$
of particle i and j colliding (i with rapidity y_i, j with rapidity y_j), with the particles on the line.

Then $R_{12}(y_1-y_2)R_{13}(y_1-y_3)R_{23}(y_2-y_3)$ represents scattering of three collisions,



while $R_{23}(y_2-y_3)R_{13}(y_1-y_3)R_{12}(y_1-y_2)$
represents



Quantum Yang-Baxter equation then encodes the fact that scattering data does not depend on the

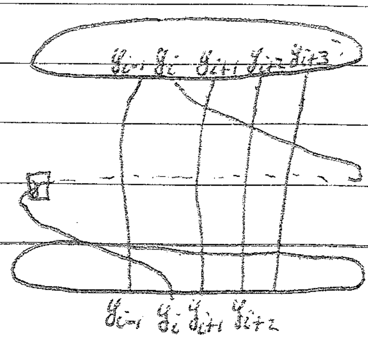
distance between the particles (no ambiguity if three particles collide at same time).

Def (Scattering matrices) (R, γ) integrable data
For $1 \le i \le n$,
$$S_i(y) := R_{i,i-1}(y_i - y_{i-1}) \cdots R_{i1}(y_i - y_1)(D_\gamma)_i \times R_{iM}(y_i - y_M) \cdots R_{i,i+1}(y_i - y_{i+1})$$
are called scattering matrices associated to (R, γ).

Proposition: If (R, γ) integrable data (so $R(z)$ satisfies quantum Yang-Baxter equation and $[D_\gamma \otimes D_\gamma, R(z)] = 0$) then
$$[S_i(y), S_j(y)] = 0 \quad (1 \le i, j \le M),$$
as linear operators on $(\mathbb{C}^2)^{\otimes M}$ if $R(z)$ is unitary.

Proof (exercise). Simple (topological) proof later on! $\boxtimes$

Scattering picture of $S_i(y)$:



Proposition: If (R,γ) integrable and $R(z)$ is regular, $R(0)=P$, then

$$S_i(y) = T(y_i; y), \qquad 1\le i\le M,$$

so commutativity of scattering matrices then consequence of commutativity of transfer operators $T(x;y)$ $(x\in\mathbb{C})$.

Proof: $T(x;y) = \mathrm{Tr}_{\mathbb{C}^2_0}\left(R_{0M}(x-y_M)\cdots R_{01}(x-y_1)(D_y)_0\right)$

so since $R(0)=P$,

$$T(y_i;y) = \mathrm{Tr}_{\mathbb{C}^2_0}\Big(R_{0M}(y_i-y_M)\cdots R_{0,i+1}(y_i-y_{i+1})P_{0i} \times R_{0,i-1}(y_i-y_{i-1})\cdots R_{01}(y_i-y_1)(D_y)_0\Big)$$

$$= \mathrm{Tr}_{\mathbb{C}^2_0}\Big(P_{0i}R_{iM}(y_i-y_M)\cdots R_{i,i+1}(y_i-y_{i+1}) \times R_{0,i-1}(y_i-y_{i-1})\cdots R_{01}(y_i-y_1)(D_y)_0\Big)$$

$$= \Big(\mathrm{Tr}_{\mathbb{C}^2_0}\big(P_{0i}R_{0,i-1}(y_i-y_{i-1})\cdots R_{01}(y_i-y_1)(D_y)_0\big) \times R_{iM}(y_i-y_M)\cdots R_{i,i+1}(y_i-y_{i+1})\Big)$$

$$= R_{i,i-1}(y_i-y_{i-1})\cdots R_{i1}(y_i-y_1)(D_y)_i\left(\mathrm{Tr}_{\mathbb{C}^2_0}(P_{0i})\right) \times R_{iM}(y_i-y_M)\cdots R_{i,i+1}(y_i-y_{i+1})$$

$$= S_i(y). \qquad \boxtimes$$

Exercise 2.3: For $1\le i<M$,

$$P_{i,i+1}R_{i,i+1}(y_i-y_{i+1})T(x;y) = T(x;s_iy)P_{i,i+1}R_{i,i+1}(y_i-y_{i+1})$$

where $s_iy = (y_1,\ldots,y_{i-1},y_{i+1},y_i,y_{i+2},\ldots,y_M)$.

Remark: The scattering interpretation of the quantum Yang-Baxter equation is used extensively in integrable massive quantum field theories. From a mathematical perspective, it can be thought of as analogue of monodromy.

2C Quantum Knizhnik-Zamolodchikov equations

Generalized scattering matrices: $\tau\in\mathbb{C}$; (R,γ) integrable data

$$A_i(y) := R_{i,i-1}(y_i-y_{i-1})\cdots R_{i1}(y_i-y_1)(D_y)_i \times R_{iM}(y_i-y_M+\tau)\cdots R_{i,i+1}(y_i-y_{i+1}+\tau)$$

$(1\le i\le M)$ as linear operators on $(\mathbb{C}^2)^{\otimes M}$. We call them ~~unitary~~ transport matrices. Note that

$$A_i(y)|_{\tau=0} = S_i(y) \qquad (1\le i\le M).$$

Proposition: $\forall 1\le i,j\le M$, if $R(z)$ is unitary, then

$$A_i(y)A_j(y_1,\ldots,y_i+\tau,\ldots,y_M) = A_j(y)A_i(y_1,\ldots,y_j+\tau,\ldots,y_M)$$

Proof. This is a longish computation, similar to exercise 2.7. We will give a simple proof later on. $\boxtimes$

Definition We say that a meromorphic function

$$f : \mathbb{C}^M \longrightarrow (\mathbb{C}^2)^{\otimes M}$$

taking values in $(\mathbb{C}^2)^{\otimes M}$, is a solution of the quantum Knizhnik-Zamolodchikov equations if

$$(*) \quad A_i(y) f(y_1, \ldots, y_{i-1}, y_i + \tau, y_{i+1}, \ldots, y_M) = f(y)$$

for all $1 \leq i \leq M$.

Remark: Proposition of page 18 shows that $(*)$ is a compatible system of equations: $\forall 1 \leq i < j \leq M$, if $f(y)$ is a solution:

$$A_i(y) A_j(\ldots, y_i + \tau, \ldots) f(\ldots, y_i + \tau, \ldots, y_j + \tau, \ldots)$$
$$= f(y)$$
$$= A_j(y) A_i(\ldots, y_j + \tau, \ldots) f(\ldots, y_i + \tau, \ldots, y_j + \tau, \ldots)$$

is resolved in the obvious way since

$$A_i(y) A_j(\ldots, y_i + \tau, \ldots) = A_j(y) A_i(\ldots, y_j + \tau, \ldots).$$

Remarks: (i) One can also think of $(*)$ as defining/characterize an integrable difference connection. Integrability is prop. p.8, and $(*)$ characterizes the corresponding space of ~~[illegible]~~ horizontal sections.

(ii) A key example of quantum KZ equations is provided by solutions of quantum Yang-Baxter equations coming from the braiding of quantum affine algebras. The associated quantum KZ equations are natural deformations of Knizhnik-Zamolodchikov equations arising naturally as consistency conditions for conformal blocks in Wess-Zumino-Witten conformal field theories. A quantum analog of the construction of solutions of KZ equations using vertex operators is derived in work of I. Frenkel and N. Reshetikhin (CMP, '90).

(iii) Correlation functions of integrable vertex models satisfy compatibility conditions that can be seen as special cases of qKZ equations.
(Jimbo, Miwa, "Algebraic analysis of solvable lattice models", CBMS no. 85, AMS, 1993)

Representation theoretic construction of solutions of quantum Yang-Baxter equations.

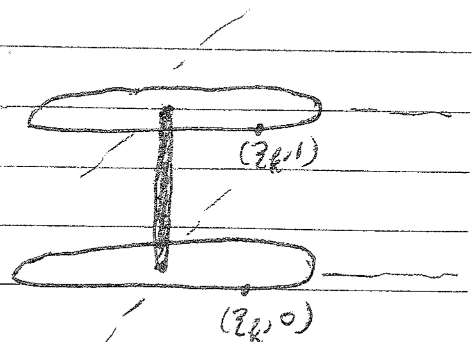
Two methods:

(i) Braiding of quantum affine algebras.
(ii) Baxterizations of affine Hecke algebra modules.

In this course discuss (ii).

a. Affine braid group

$X := \mathbb{C}^* \times [0,1] =$

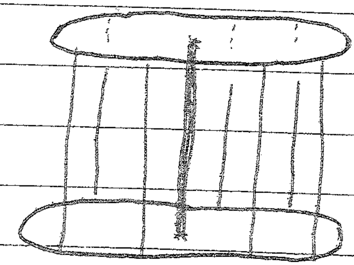


Fix M points on top and bottom of X:

$\{(z_k, 1)\}_{k=1}^{M}$ and $\{(z_k, 0)\}_{k=1}^{M}$ with $z_k := e^{2\pi i k/M}$.

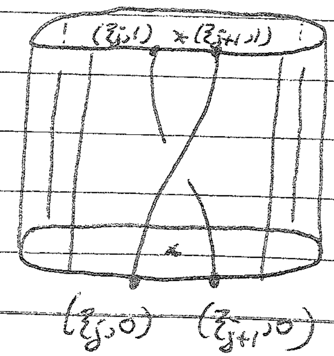
Definition: The extended affine braid group $\hat{\mathcal{B}}$ is the collection of M-braids in X connecting $\{(z_k, 1)\}_{k=1}^{M}$ to $\{(z_k, 0)\}_{k=1}^{M}$.

Is a group: $\sigma \cdot \tau =$ putting σ on top of τ and shrinking the height;
neutral element:



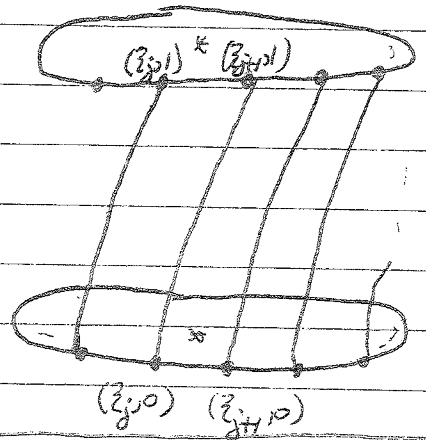
Special M-braids: $j \in \mathbb{Z}/M\mathbb{Z}$:

$\sigma_j :=$



nd

$C =$



Theorem: $\hat{B}$ is generated as group by σ_j $(j \in \mathbb{Z}/M\mathbb{Z})$ and c, with defining relations:

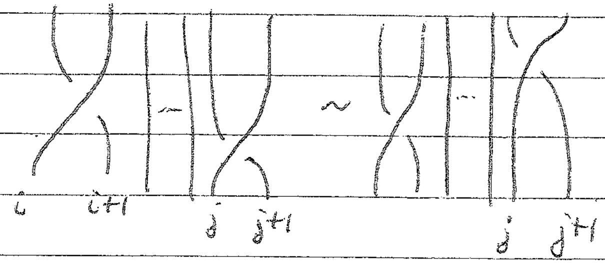
1. $\sigma_i \sigma_j = \sigma_j \sigma_i$ if $|\bar{i} - \bar{j}| \neq 1$ for all representatives $\bar{i}, \bar{j} \in \mathbb{Z}$ of $i, j \in \mathbb{Z}/M\mathbb{Z}$;
2. $\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}$;
3. $c\sigma_j = \sigma_{j+1} c$.

Proof: Reduce it to application of Reidemeister's theorem, using the projection of the affine M-braids on the cylinder

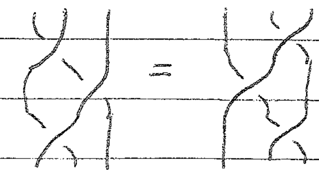
$$S^1 \times [0,1].$$

Using ambient isotopy one can arrange that the crossings on the cylinder happen at different heights; each of such crossing can be taken care of by a σ_j or σ_j^{-1}; Moving all crossings to top by ambient isotopies one can write the braid as a finite product of $\sigma_j^{\pm 1}$'s and a remaining braid without crossings; but such a braid without crossing is c^k for some $k \in \mathbb{Z}$.

Relation 1. signifies that distant crossings at different heights can be moved up & down interchanging their relative height:



2. is Reidemeister III move:



Using Reidemeister's theorem one can then show that 1-3 are the defining relations for $\hat{B}$ in terms of the group generators $c, \sigma_1, \ldots, \sigma_M$ ☒

Exercise 3.1: Let $\mathcal{G}$ be the group generated by $s_0, s_1, \ldots, s_{M-1}$ with relations

$$s_0 s_1 s_0 s_1 = s_1 s_0 s_1 s_0,$$
$$s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1},$$
$$s_i s_j = s_j s_i \quad \text{if } |i-j|>1$$

Prove that $\hat{\mathcal{B}} \simeq \mathcal{G}$ as groups with isomorphism determined by $\sigma_i \longleftrightarrow s_i \quad (1 \le i < M)$

$\sigma_1^{-1} \cdots \sigma_{M-2}^{-1} \sigma_{M-1}^{-1} c^{-1} \longleftrightarrow s_0$

Remark: $\{e_1, \ldots, e_M\}$ standard basis of $\mathbb{C}^M$,

$(\cdot,\cdot)$ standard bilin. form on $\mathbb{C}^M$, $(e_i, e_j) = \delta_{i,j}$

$R^+ := \{e_j\}_{j=1}^M \cup \{e_i \pm e_j\}_{1 \le i < j \le M}$ (positive roots of root system of type B_M).

$\alpha \in R^+$: $H_\alpha = \{z \in \mathbb{C}^M \mid (\alpha, z) = 0\}$

$$\mathbb{C}^M_{reg} := \mathbb{C}^M \setminus \bigcup_{\alpha \in R^+} H_\alpha = \{z \in (\mathbb{C}^*)^M \mid z_i^2 \neq z_j^2 \ (i \neq j)\}$$

$W := S_M \ltimes (\pm 1)^M$ hyperoctahedral group, acting faithfully on $\mathbb{C}^M_{reg}$. Theorem of Brieskorn (for arbitrary root systems, but applied here to type B):

$$\pi_1(\mathbb{C}^M_{reg}/W) \simeq \mathcal{G}.$$

Since

$$\mathbb{C}^M_{reg}/W \simeq (\mathbb{C}^M_{reg}/\{\pm 1\}^M)/S_M \simeq Y/S_M$$

with $Y = \{y = (y_1, \ldots, y_M) \in (\mathbb{C}^*)^M \mid y_i \neq y_j, \ i \neq j\}$

and $\pi_1(Y/S_M) \simeq \hat{\mathcal{B}}$ the affine braid group

hence, combined with Brieskorn's theorem,

$$\hat{\mathcal{B}} \simeq \pi_1(Y/S_M) \simeq \pi_1(\mathbb{C}^M_{reg}/W) \simeq \mathcal{G},$$

reproving, in combination with exercise 3.1, Thm. p. 23.

Exercise 3.2: Define for $1 \le j \le M$,

$$g_j := \sigma_{j-1}^{-1} \cdots \sigma_1^{-1} c\, \sigma_{M-1} \cdots \sigma_j \in \hat{\mathcal{B}}.$$

Show that $g_i g_j = g_j g_i$ for $1 \le i, j \le M$.

Theorem: Let (R, γ) be integrable data, so $R(z): \mathbb{C}^2 \otimes \mathbb{C}^2 \circlearrowleft$ invertible and satisfying the quantum Yang-Baxter equation, and $[D_\gamma \otimes D_\gamma, R(z)] = 0$. Then $\hat{\mathcal{B}}$ acts on the space of $(\mathbb{C}^2)^{\otimes M}$-valued meromorphic functions on $\mathbb{C}^M$ by:

$$(\sigma_j f)(y) := P_{j,j+1} R_{j,j+1}(y_j - y_{j+1}) f(\ldots, y_{j+1}, y_j, \ldots)$$
$$(c f)(y) := P_{12} P_{23} \cdots P_{M-1,M} (D_\gamma)_M f(y_2, \ldots, y_M, y_1 - \tau)$$

for $1 \le j < M$, with fixed step-size τ.

Proof. $N \trianglelefteq \hat{\mathcal{B}}$ normal subgroup generated by σ_j^2 $(1\le j<M)$.
Write $\tau(\lambda)$ for $\lambda \in \mathbb{Z}^M$ viewed as element of $S_M \ltimes \mathbb{Z}^M$.
Then

$$S_M \ltimes \mathbb{Z}^M \xrightarrow{\sim} \hat{\mathcal{B}}/N$$
$$\tau(e_i) \longmapsto g_i N \quad (1\le i\le M)$$
$$s_j \longmapsto \sigma_j N \quad (1\le j<M)$$

with $s_j \in S_M$ the transposition $j \leftrightarrow j+1$.

Writing $\xi \in S_M \ltimes \mathbb{Z}^M$ for the element corresponding to $cN \in \hat{\mathcal{B}}/N$ we have

$$\xi = s_1 \cdots s_{M-1}\tau(e_M).$$

$S_M \ltimes \mathbb{Z}^M$ acts on $\mathbb{C}^M$ by:

$$\tau(\lambda)y = (y_1+\tau\lambda_1, \ldots, y_M+\tau\lambda_M), \quad \lambda=(\lambda_1,\ldots,\lambda_M)\in\mathbb{Z}^M,$$
$$s_j y = (\ldots, y_{j-1}, y_{j+1}, y_j, y_{j+2}, \ldots), \quad 1\le j<M$$

Thus

$$\xi\cdot y = (y_M+\tau, y_1, \ldots, y_{M-1}).$$

Through

$$\hat{\mathcal{B}} \twoheadrightarrow \hat{\mathcal{B}}/N \simeq S_M \ltimes \mathbb{Z}^M$$

we lift it to an action of $\hat{\mathcal{B}}$ on $\mathbb{C}^M$. Then

$$(w\cdot f)(y) := A_w(y) f(w^{-1}y)$$

for a family $\{A_w(y)\}_{w\in\hat{\mathcal{B}}}$ of invertible linear operators on $(\mathbb{C}^2)^{\otimes M}$, depending meromorphically on $y\in\mathbb{C}^M$, defines a representation of $\hat{\mathcal{B}}$ on the space of $(\mathbb{C}^2)^{\otimes M}$-valued meromorphic functions $f(y)$ in $y\in\mathbb{C}^M$ iff

$$A_1(y) = \mathrm{Id}_{(\mathbb{C}^2)^{\otimes M}} \quad \& \quad A_{uv}(y) = A_u(y)A_v(u^{-1}y)$$

for all $u,v\in\hat{\mathcal{B}}$.

Such a cocycle is uniquely determined by $A_c(y)$ and $A_{\sigma_j}(y)$ $(1\le j<M)$, which need to satisfy the following analogs of the defining relations of $\hat{\mathcal{B}}$ in terms of the generators $c, \sigma_1, \ldots, \sigma_{M-1}$:

$$(*)\begin{cases} A_{\sigma_i}(y)A_{\sigma_j}(s_i y) = A_{\sigma_j}(y)A_{\sigma_i}(s_j y), & |i-j|>1, \\ A_{\sigma_i}(y)A_{\sigma_{i+1}}(s_i y)A_{\sigma_i}(s_{i+1}s_i y) = A_{\sigma_{i+1}}(y)A_{\sigma_i}(s_{i+1}y)A_{\sigma_{i+1}}(s_i s_{i+1} y) & \text{for } 1\le i<M-1, \\ A_c(y)A_{\sigma_i}(\xi^{-1}y) = A_{\sigma_{i+1}}(y)A_c(s_{i+1}y) & (1\le i<M-1), \\ A_c(y)A_c(\xi^{-1}y)A_{\sigma_{M-1}}(\xi^{-2}y) = A_{\sigma_1}(y)A_c(s_1 y)A_c(\xi^{-1}s_1 y) & \end{cases}$$

(the last equation is needed because we have not added σ_M to our generators of $\hat{\mathcal{B}}$ (which is $c\sigma_{M-1}c^{-1}$)).
The following exercise now completes the proof of the theorem. ☒

Exercise 3.3 If (R,γ) is an integrable datum, then

$$A_{\sigma_j}(y) := P_{j,j+1} R_{j,j+1}(y_j - y_{j+1}) \qquad (1\le j< M)$$
$$A_c(y) := P_{12}P_{23}\cdots P_{M-1,M}(D_\gamma)_M$$

gives rise to a $\hat{B}$-cocycle $\{A_u(y)\}_{u\in\hat{B}}$, i.e. they satisfy the identities (*) on p.28. Prove this.

Exercise 3.4 Show that, in setting of exercise 3.3,

$$A_{g_i}(y) = R_{i-1,i}(y_i - y_{i-1})^{-1} \cdots R_{1i}(y_i - y_1)^{-1}(D_\gamma)_i \times R_{iM}(y_M - y_i + \tau)\cdots R_{i,i+1}(y_{i+1} - y_i + \tau).$$

By the cocycle conditions for $\{A_u(y)\}_{u\in\hat{B}}$:

$$A_{g_i}(y)A_{g_j}(y-\tau e_i) = A_{g_j}(y)A_{g_i}(y-\tau e_j)$$

for $1\le i,j\le M$, which are the integrability conditions for transport matrices of system of difference equations

$$A_{g_i}(y)f(y-\tau e_i) = f(y) \qquad (1\le i\le M).$$

The $A_{g_i}(y)$ are essentially the transport operators $A_i(y)$ encountered before in the context of vertex models. Indeed, if $R(z)$ is unitary (which is equivalent to the condition that the $\hat{B}$-representation of theorem p.26 factorizes to a representation of the affine symmetric group $S_M \ltimes \mathbb{Z}^M$) then

$$A_{g_i}(y) = R_{i,i-1}(y_{i-1} - y_i)\cdots R_{i1}(y_1 - y_i)(D_\gamma)_i \times R_{iM}(y_M - y_i + \tau)\cdots R_{i,i+1}(y_{i+1} - y_i + \tau)$$

which is the transport operator $A_i(-y)$ from page 18. So the above arguments based on the affine braid group $\hat{B}$ gives a proof of propositions p.18 & p.16.

b Affine Hecke algebra

Special case of theorem p.26 when $R(z)$ is a constant solution of the quantum Yang-Baxter equation gives:

Lemma: Suppose $\mathbb{B} : \mathbb{C}^2\otimes\mathbb{C}^2 \circlearrowleft$ is an invertible linear operator and $D : \mathbb{C}^2 \circlearrowleft \mathbb{C}^2$, such that

$$\mathbb{B}_{12}\mathbb{B}_{13}\mathbb{B}_{23} = \mathbb{B}_{23}\mathbb{B}_{13}\mathbb{B}_{12}$$
$$[D\otimes D, \mathbb{B}] = 0.$$

Then: $\Pi_{B,D} : \hat{B} \longrightarrow gl((\mathbb{C}^2)^{\otimes M})$ representation, defined by:

$$\Pi_{B,D}(\sigma_j) = P_{j,j+1} B_{j,j+1} \qquad (1 \leq j < M),$$
$$\Pi_{B,D}(c) = P_{12} P_{23} \cdots P_{M-1,M} D_M$$

The construction of constant solutions $B : \mathbb{C}^2 \otimes \mathbb{C}^2 \circlearrowleft$ of quantum Yang-Baxter equations can be done in various ways; the most well-known case (with $\mathbb{C}^2$ replaced by arbitrary vectorspace) uses quantum groups. The following important example arises in this way:

$$r := P \circ B_k = \begin{pmatrix} k^{-1} & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & k-k^{-1} & 0 \\ 0 & 0 & 0 & -k^{-1} \end{pmatrix}$$

with respect to the ordered basis $(v_+ \otimes v_+, v_+ \otimes v_-, v_- \otimes v_+, v_- \otimes v_-)$ of $\mathbb{C}^2 \otimes \mathbb{C}^2$, i.e.

$$B_k = \begin{pmatrix} k^{-1} & 0 & 0 & 0 \\ 0 & & k-k^{-1} & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -k^{-1} \end{pmatrix}$$

It satisfies $[D_\gamma \otimes D_\gamma, B_k] = 0 \quad \forall \gamma \in (\mathbb{C}^*)^2$ as well as the Hecke-relation: in terms of $r = P \circ B_k$,

$$(r-k)(r+k^{-1}) = 0.$$

Definition: The extended affine Hecke algebra (of type A) H is the unital assoc. complex algebra $/\mathbb{C}$ generated by $T_\zeta^{\pm 1}, T_1, \ldots, T_M$ and with defining relations:

(i) $T_i T_j = T_j T_i$ if $|\bar{i} - \bar{j}| \neq 1$ for all representatives $\bar{i}, \bar{j} \in \mathbb{Z}$ of $i, j \in \mathbb{Z}/M\mathbb{Z}$;

(ii) $T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1}$;

(iii) $T_\zeta T_i = T_{i+1} T_\zeta$;

(iv) $(T_i - k)(T_i + k^{-1}) = 0$;

(v) $T_\zeta T_\zeta^{-1} = 1 = T_\zeta^{-1} T_\zeta$.

Remarks: (i) For $k=1$; $H \simeq \mathbb{C}[S_M \ltimes \mathbb{Z}^M]$ the group algebra of the affine symmetric group.

(ii) For all k, surjective algebra map

$$\mathbb{C}[\hat{B}] \overset{\phi}{\twoheadrightarrow} H$$
$$\sigma_i \longmapsto k^{-\frac{1}{2}} T_i, \qquad i \in \mathbb{Z}/M\mathbb{Z}$$
$$c \longmapsto T_\zeta$$

(iii) Bernstein–Zelevinsky decomposition:

Set:

$$Y_i := \phi(g_i) = k^{\frac{1}{2}(2i-M-1)} T_{i-1}^{-1} \cdots T_1^{-1} T_2 T_{M-1} \cdots T_i$$

(note that $T_j \in H$ is invertible with inverse $T_j - k + k^{-1}$).

Then: $[Y_i, Y_j] = 0$ in H $\forall 1 \le i, j \le M$
and H is generated as algebra by
$T_1, \dots, T_{M-1}$ & $Y_1^{\pm 1}, \dots, Y_M^{\pm 1}$.

It is possible to describe the defining commutation relations of H in terms of the algebraic generators $T_1, \dots, T_{M-1}, Y_1^{\pm 1}, \dots, Y_M^{\pm 1}$ explicitly.
(Bernstein–Zelevinsky–Lusztig).

Question:
How to promote the constant solution B of the quantum Yang–Baxter equation to a solution with spectral parameter z?

c Baxterization V. Jones gave first examples of "Baxterizations" of constant solutions B of quantum Yang–Baxter equations,

i.e. solutions $R(z)$ of the quantum Yang–Baxter equation with spectral parameter such that

$$\lim_{z \to \infty} R(z) \sim B$$

Cherednik vastly generalized this procedure, by in fact achieving a notion of Baxterization of affine Hecke algebra modules

Theorem (Cherednik)
Let $\pi: H \to \mathrm{End}(V)$ be a representation of the affine Hecke algebra H. Define

$$b(y) := \frac{k^{-1} - k e^{y}}{1 - e^{y}}$$

The following formulas,

$$(\nabla^{\pi}(s_j) f)(y) := \left(\frac{\pi(T_j) + b(y_j - y_{j+1}) - k}{b(y_j - y_{j+1})} \right) f(s_j y), \quad 1 \le j < M,$$

$$(\nabla^{\pi}(\xi) f)(y) := \pi(T_\xi) f(\xi^{-1} y),$$

define an action of $S_M \ltimes \mathbb{Z}^M$ on the space of $(\mathbb{C}^2)^{\otimes M}$-valued meromorphic functions $f(y)$ in $y \in \mathbb{C}^M$.

<u>Remark</u> (i) The Baxterization thus depends on a choice of step-size τ, since $\xi^{-1}y = (y_2, \ldots, y_M, y_1 - \tau)$.

(ii) Formally the representation is recovered in the asymptotic sector where $|y_i - y_{i+1}| \to \infty$ $\forall i$, since $\lim_{|y|\to\infty} b(y) = k$.

<u>Proof</u>. It can be verified by direct but tedious computations that the operators $\nabla^\pi(s_j)$ $(1 \le j < M)$ and $\nabla^\pi(\xi)$ respect the defining relations of $S_M \ltimes \mathbb{Z}^M$ with respect to the generators s_j $(1 \le j < M)$, ξ.

These defining relations are:

(a) $s_j^2 = 1$ $\quad (1 \le j < M)$

(b) $s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}$ $\quad (1 \le i < M-1)$

(c) $s_i s_j = s_j s_i$ $\quad (|i-j| > 1)$

(d) $c s_i = s_{i+1} c$ $\quad (1 \le i < M-1)$

(e) $c^2 s_{M-1} = s_1 c^2$

For instance, for (a):

$$\nabla^\pi(s_i)^2 = \left(\frac{\pi(T_i) + b(y_i - y_{i+1}) - k}{b(y_i - y_{i+1})}\right)\left(\frac{\pi(T_i) + b(y_{i+1} - y_i) - k}{b(y_{i+1} - y_i)}\right)$$

$$= \frac{1}{b(y_i - y_{i+1})\, b(y_{i+1} - y_i)}\Big((\pi(T_i) - k) + b(y_i - y_{i+1})\Big)\Big(\pi(T_i) + k^{-1} + (b(y_{i+1} - y_i) - k - k^{-1})\Big)$$

$$\underset{\boxed{(T_i - k)(T_i + k^{-1}) = 0}}{=} \frac{1}{b(y_i - y_{i+1})\, b(y_{i+1} - y_i)}$$

$$\times \Big(\big(b(y_i - y_{i+1}) + b(y_{i+1} - y_i) - k - k^{-1}\big)\pi(T_i)$$

$$+ k^{-1}\big(b(y_i - y_{i+1}) - k\big)\big(b(y_{i+1} - y_i) - k - k^{-1}\big)$$

$$+ b(y_i - y_{i+1})\big(b(y_{i+1} - y_i) - k - k^{-1}\big)\Big)$$

$$= \frac{1}{b(y_i - y_{i+1})\, b(y_{i+1} - y_i)}$$

$$\times\Big(\big(b(y_{i+1} - y_i) + b(y_i - y_{i+1}) - k - k^{-1}\big)\big(\pi(T_i) - k\big)$$

$$+ b(y_i - y_{i+1})\, b(y_{i+1} - y_i)\Big)$$

$$= 1$$

since $b(u) + b(-u) = k + k^{-1}$.

Others can be derived directly by similar (tedious) computations. There is a different, conceptual proof that uses the double affine Hecke algebra of Cherednik. We sketch it in the following remark $\boxtimes$

Remark (double affine Hecke algebra approach).
Exploring the structure of the affine Hecke algebra in terms of the generators $T_1, \dots, T_{M-1}, Y_1^{\pm 1}, \dots, Y_M^{\pm 1}$, Cherednik proved the existence of an action of H on $\mathbb{C}[e^{\pm y_1}, \dots, e^{\pm y_M}]$ determined by difference reflection equations (Demazure–Lusztig type operators): $f(y) \in \mathbb{C}[e^{\pm y_1}, \dots, e^{\pm y_M}] =: \mathbb{C}[e^{\pm y}]$,

$$(\Delta)\begin{cases} 1 \le j < M: & (T_j f)(y) := k f(y) + b(y_j - y_{j+1})(f(s_j y) - f(y)) \\ & (T_\xi f)(y) := f(\xi^{-1} y) = f(y_2, \dots, y_M, y_1 - \tau) \end{cases}$$

(Cherednik's polynomial/basic representation).
The double affine Hecke algebra (of type A) is the subalgebra of $\mathrm{End}(\mathbb{C}[e^{\pm y}])$ generated by the affine Hecke algebra H and $\mathbb{C}[e^{\pm y}]$ (viewed as multiplication operators on $\mathbb{C}[e^{\pm y}]$). Localizing $\mathbb{H}$ in $\mathbb{C}[e^{\pm y}]$ we can now see from (Δ) that $\mathbb{H}$ contains the extended affine symmetric group element s_j $(1 \le j < M)$, ξ: indeed, (Δ) says

$$T_j = k + b(y_j - y_{j+1})(s_j - 1)$$
$$T_\xi = \xi$$

so formally rewriting,

$$s_j = b(y_j - y_{j+1})^{-1}(T_j - k) + 1$$
$$= b(y_j - y_{j+1})^{-1}(T_j + b(y_j - y_{j+1}) - k)$$

which is essentially the Baxterization formula for the action of T_j (see Thm 34), hence automatically satisfies the appropriate braid relations.

Example: the 6-vertex model or Heisenberg XXZ spin-$\frac{1}{2}$ chain

Corollary: For the constant solution

$$B_k = \begin{pmatrix} k^{-1} & 0 & 0 & 0 \\ 0 & 1 & k-k^{-1} & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -k^{-1} \end{pmatrix}$$

of the quantum Yang–Baxter equation, define

$$R(z) = b(z)^{-1}(B_k + (b(z)-k)\mathbb{1}) : \mathbb{C}^2 \otimes \mathbb{C}^2 \circlearrowleft.$$

Then for any $\gamma \in (\mathbb{C}^*)^2$,
(i) $R(z)$ is a unitary solution of the qYB equation.
(ii) $[D_\gamma \otimes D_\gamma, R(z)] = 0$.
(iii) The Baxterization of the representation $\pi_{B,D}$ (see lemma p.30/31 & theorem p.34) is given by

$$(\nabla^{\pi_{B,D}}(s_j) f)(y) = P_{j,j+1} R_{j,j+1}(y_j - y_{j+1}) f(s_j y)$$

for $1 \le j < M$ and

$$(\nabla^{\pi_{B,D}}(\xi) f)(y) = P_{12} P_{23} \cdots P_{M-1,M} (D_\gamma)_M f(\xi^{-1} y)$$

which is exactly of the form as on page 26.

Proof: (i) follows from (iii), also (ii) but this is immediate by direct inspection.

Finally for (iii),

$$\frac{\pi_{B,0}(T_j)+b(y_j-y_{j+1})-k}{b(y_j-y_{j+1})} = \frac{P_{j,j+1}B_{j,j+1}+b(y_j-y_{j+1})-k}{b(y_j-y_{j+1})}$$

$$= P_{j,j+1}R_{j,j+1}(y_j-y_{j+1})$$

from which the result immediately follows. $\boxtimes$

Explicitly, the Baxterization $R(z)$ of B_k (see (o) p 38) thus is

$$R(z) = \frac{1}{b(z)}\left(B_k + (b(z)-k)P\right)$$

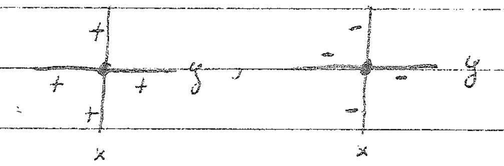
$$= \begin{pmatrix} \frac{b(z)-k-k^{-1}}{b(z)} & 0 & 0 & 0 \\ 0 & \frac{1}{b(z)} & \frac{b(z)-k^{-1}}{b(z)} & 0 \\ 0 & \frac{b(z)-k}{b(z)} & \frac{1}{b(z)} & 0 \\ 0 & 0 & 0 & \frac{b(z)-k-k^{-1}}{b(z)} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{k^{-1}e^{z}-k}{k^{-1}-ke^{z}} & 0 & 0 & 0 \\ 0 & \frac{1-e^{z}}{k^{-1}-ke^{z}} & \frac{(k^{-1}-k)e^{z}}{k^{-1}-ke^{z}} & 0 \\ 0 & \frac{k^{-1}-k}{k^{-1}-ke^{z}} & \frac{1-e^{z}}{k^{-1}-ke^{z}} & 0 \\ 0 & 0 & 0 & \frac{k^{-1}e^{z}-k}{k^{-1}-ke^{z}} \end{pmatrix}$$

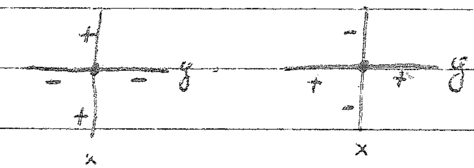
Note that $R(z)$ is regular, $R(0)=P$, and unitary, $R(z)^{-1}=R_{21}(-z)$.

Remark: (i) The vertex model associated to (R, γ) is the so-called 6-vertex model with quasi-periodic boundary conditions.

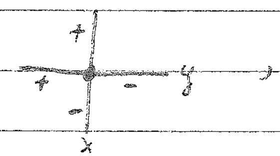
The only vertex configurations with nonzero local Boltzmann weights are



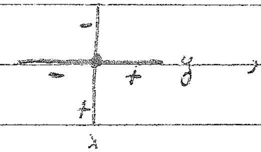
weight 1,



weight $\frac{1}{b(x-y)}$



weight $\frac{b(x-y)-k^{-1}}{b(x-y)}$



weight $\frac{b(x-y)-k}{b(x-y)}$

It is a simple model of two-dimensional ice.

(ii) The associated one-dimensional quantum spin chain, with quantum Hamiltonians encoded by the transfer operator

$$T(x;\underline{y}) := \mathrm{Tr}_{\mathbb{C}_0^2}\left(R_{0M}(x-y_M) \cdots R_{01}(x-y_1)(D_\gamma)_0\right)$$

is the Heisenberg XXZ spin-$\frac{1}{2}$ chain with inhomogeneties $y_1, \ldots, y_M$ (corresponding quantum state space is $(\mathbb{C}^2)^{\otimes M}$).

The homogeneous case ($\underline{y} = (0, \ldots, 0)$) has the famous quantum Hamiltonian:

$$(D_\gamma^{-1})_M h_{M1} (D_\gamma)_M + \sum_{j=2}^{M} h_{j-1,j},$$

with $h = \sigma^x \otimes \sigma^x + \sigma^y \otimes \sigma^y - \left(\frac{k+k^{-1}}{2}\right)\sigma^z \otimes \sigma^z$ and $\sigma^x, \sigma^y, \sigma^z$ the Pauli matrices: $\sigma^x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\sigma^y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$, $\sigma^z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, as quantum conserved quantity. The following exercise shows this in a couple of steps.

Exercise 3.5 Show that

(i) $T(0;\underline{0}) = \mathrm{Tr}_{\mathbb{C}_0^2}\left(P_{0M} \cdots P_{02} P_{01} (D_\gamma)_0\right)$

$$= P_{12} P_{23} \cdots P_{M-1,M} (D_\gamma)_M;$$

(ii) For $2 \leq j < M$,

$$\mathrm{Tr}_{\mathbb{C}_0^2}\left(P_{0M} \cdots P_{0,j+1} P_{0,j-1} \cdots P_{01} (D_\gamma)_0\right) = T(0;\underline{0}) P_{j-1,j}$$

(iii)

$$\mathrm{Tr}_{\mathbb{C}_0^2}\left(P_{0M} \cdots P_{03} P_{02} (D_\gamma)_0\right) = T(0;\underline{0}) (D_\gamma^{-1})_M (D_\gamma)_1 P_{1M}$$

(iv) $\left.\frac{d}{dx}\right|_{x=0} T(x;\underline{0}) =$

$$= T(0;\underline{0}) \Big(P_{1M} (D_\gamma^{-1})_1 R'_{M1}(0) (D_\gamma)_M + \sum_{j=2}^{M} P_{j-1,j} R'_{j-1,j}(0) \Big).$$

(v) $T(0;\underline{0})^{-1} \left.\frac{d}{dx}\right|_{x=0} T(x;\underline{0}) =: H_{XXZ}$

$$= P_{1M} (D_\gamma^{-1})_1 R'_{M1}(0) (D_\gamma)_M + \sum_{j=2}^{M} P_{j-1,j} R'_{j-1,j}(0)$$

$$= (D_\gamma^{-1})_M P_{1M} R'_{M1}(0) (D_\gamma)_M + \sum_{j=2}^{M} P_{j-1,j} R'_{j-1,j}(0)$$

(vi) $PR'(0) = \frac{1}{2(k-k^{-1})}\left(\sigma^x \otimes \sigma^x + \sigma^y \otimes \sigma^y - \frac{k+k^{-1}}{2}\sigma^z \otimes \sigma^z + (k^{-1}-k)\mathcal{E}\right)$

$$- \frac{3}{4}\left(\frac{k+k^{-1}}{k-k^{-1}}\right) \mathrm{Id}_{\mathbb{C}^2} \otimes \mathrm{Id}_{\mathbb{C}^2}$$

with $\mathcal{E}$ the linear operator on $\mathbb{C}^2 \otimes \mathbb{C}^2$ defined by $\mathcal{E}(v_+ \otimes v_+) = 0 = \mathcal{E}(v_- \otimes v_-)$, $\mathcal{E}(v_+ \otimes v_-) = v_+ \otimes v_-$, $\mathcal{E}(v_- \otimes v_+) = -v_- \otimes v_+$

(v) $(D_\gamma^{-1})_M \mathcal{E}_{M1} (D_\gamma)_M + \sum_{j=2}^{M} \mathcal{E}_{j-1,j} \equiv 0.$

(vi.) $H_{XXZ} = -\frac{3}{4} M \left(\frac{k+k^{-1}}{k-k^{-1}}\right) \mathrm{Id}_{(\mathbb{C}^2)^{\otimes M}}$

$$+ \frac{1}{2(k-k^{-1})}\Big((D_\gamma^{-1})_M h_{M1} (D_\gamma)_M + \sum_{j=2}^{M} h_{j-1,j}\Big).$$

Remark: The quantum KZ equations associated to the integrable data $(\mathcal{R}(z), \gamma)$ of the 6-vertex model (Heisenberg XXZ spin-$\frac{1}{2}$ chain), which amounts to the equations

(*) $(\nabla^{\pi_{\delta,0}}(\tau(\xi_j))f)(y) = f(y) \quad \forall 1 \le j \le M$,

are well-studied using:

(i) "Algebraic Bethe ansatz" techniques: explicit construction of integral solutions (Tarasov, Varchenko, ...)

(ii) Representation theory of quantum affine algebras (Frenkel, Reshetikhin, ...)

(iii) Vertex operator approaches (Jimbo, Miwa, ...)

(iv) Representation theory of affine Hecke algebras and theory of Macdonald polynomials (Cherednik, ...)

Key point: solutions of (*) give rise to correlation functions (form factors) of the 6-vertex model.

e. Relation to affine Temperley-Lieb algebra

Def: The extended affine Temperley-Lieb algebra TL, depending on parameter $d \in \mathbb{C}$, is the unital assoc. algebra$/\mathbb{C}$ with generators e_i $(i \in \mathbb{Z}/M\mathbb{Z})$ and $\zeta^{\pm 1}$ with relations

$$e_i^2 = d e_i$$

$$e_i e_{i\pm 1} e_i = e_i \qquad \text{(indices modulo } M\text{)}$$

$$e_i e_j = e_j e_i \quad \text{if } |\tilde{i} - \tilde{j}| \neq 1 \text{ for all representatives } \tilde{i}, \tilde{j} \in \mathbb{Z} \text{ of } i, j \in \mathbb{Z}/M\mathbb{Z}$$

$$\zeta e_i = e_{i+1} \zeta$$

$$\zeta \zeta^{-1} = 1 = \zeta^{-1} \zeta$$

Proposition: There exists a unique unit-preserving surjective algebra map $H \to TL$ given by

$$T_i \longmapsto e_i - k^{-1}$$

$$T_\zeta \longmapsto \zeta$$

when $d = k + k^{-1}$.

Proof: Set $\tilde{T}_i := e_i - k^{-1}$, $\tilde{T}_\zeta := \zeta \in TL$. To show that they satisfy the defining relations of H in terms of T_i $(i \in \mathbb{Z}/M\mathbb{Z})$ and T_ζ.

$$(\tilde{T}_i - k)(\tilde{T}_i + k^{-1}) = (e_i - k - k^{-1})e_i = e_i^2 - de_i = 0.$$

$$\begin{aligned}\tilde{T}_i \tilde{T}_{i+1} \tilde{T}_i &= (e_i - k^{-1})(e_{i+1} - k^{-1})(e_i - k^{-1}) \\ &= e_i e_{i+1} e_i - k^{-1}(e_i e_{i+1} + e_i^2 + e_{i+1} e_i) \\ &\quad + k^{-2}(2e_i + e_{i+1}) - k^{-3} \\ &= e_i - k^{-1}(e_i e_{i+1} + (k+k^{-1})e_i + e_{i+1} e_i) \\ &\quad + k^{-2}(2e_i + e_{i+1}) - k^{-3} \\ &= -k^{-1}(e_i e_{i+1} + e_{i+1} e_i) + k^{-2}(e_i + e_{i+1}) - k^{-3}.\end{aligned}$$

On the other hand, by exactly the same computation but with $i \leftrightarrow i+1$,

$$\tilde{T}_{i+1} \tilde{T}_i \tilde{T}_{i+1} = -k^{-1}(e_i e_{i+1} + e_{i+1} e_i) + k^{-2}(e_i + e_{i+1}) - k^{-3},$$

hence

$$\tilde{T}_i \tilde{T}_{i+1} \tilde{T}_i = \tilde{T}_{i+1} \tilde{T}_i \tilde{T}_{i+1}.$$

Other relations are obvious. $\boxtimes$

Proposition ~~Corollary~~ The spin representation

$$\pi_{B_k, D_\gamma} : H \longrightarrow \mathrm{End}((\mathbb{C}^2)^{\otimes M})$$

with

$$B_k = \begin{pmatrix} -k^{-1} & 0 & 0 & 0 \\ 0 & 1 & k-k^{-1} & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -k^{-1} \end{pmatrix}$$

and $\gamma = (\gamma_+, \gamma_-) \in (\mathbb{C}^*)^2$ factorizes through TL.

Proof: set $e_i := PB_k + k^{-1}$

$$= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & k^{-1} & 1 & 0 \\ 0 & 1 & k & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

It suffices to show that

$$(\Delta) \begin{cases} \rho(e_i) := e_{i,i+1} & (1 \le i < M) \\ \rho(\zeta) := P_{12} P_{23} \cdots P_{M-1,M} (D_\gamma)_M \end{cases}$$

uniquely defines a representation

$$\rho : TL \longrightarrow \mathrm{End}((\mathbb{C}^2)^{\otimes M}).$$

Since we take indices in (Δ) between 1 and $M-1$, the defining relations involving ζ in TL are

$$\zeta e_i = e_{i+1} \zeta \qquad (1 \le i < M-1)$$

$$\zeta^2 e_{M-1} = e_1 \zeta^2$$

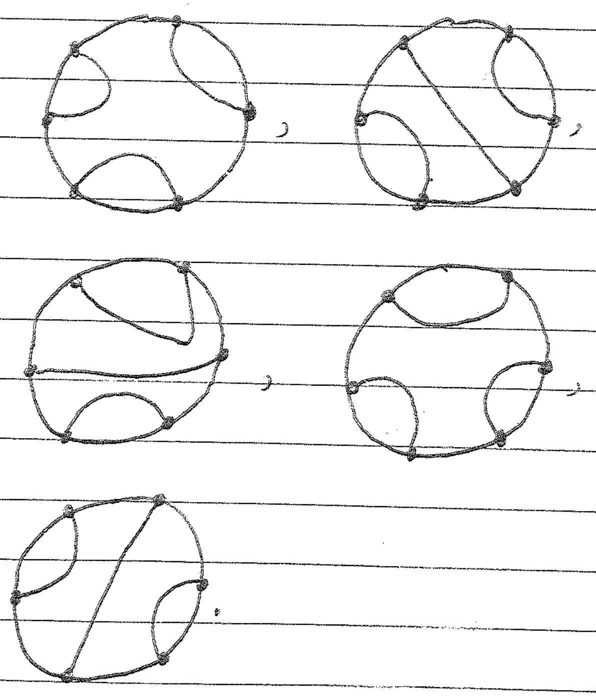
By a direct computation one shows (Δ) to define a TL-representation. $\boxtimes$

§ Loop models

Take M even and consider the set L of perfect matchings of $\{k\}_{k=1}^{M}$ within the

disc $D := \{z \in \mathbb{C} \mid |z| \leq 1\}$. In other words, $L \in \mathcal{L}$ is a collection of noncrossing paths in D with: (i) each path has endpoints in $\{z_k\}_{k=1}^{M}$
(ii) each z_k is the endpoint of exactly one path (and paths are not loops).

For instance, for $M = 6$ the collection $\mathcal{L}$ ~~[illegible]~~ has elements



Write $\mathbb{C}[\mathcal{L}]$ be the formal vector space $/\mathbb{C}$ with basis $\mathcal{L}$. We take $d = k + k^{-1}$ in the rest of the text.

<u>Proposition</u> (matchmaker representation)

There exists a unique representation

$$\sigma : TL \to \mathrm{End}(\mathbb{C}[h])$$

with

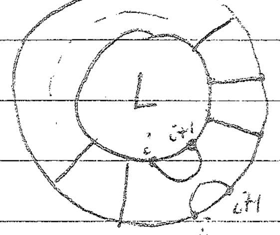
$$\sigma(e_i)L = \begin{cases} L' & \text{if } i \text{ is not matched to } i+1 \text{ in } L \\ dL & \text{otherwise} \end{cases}$$

where

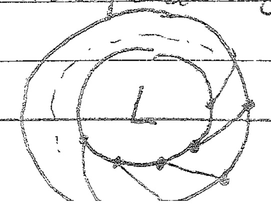
L' is obtained by removing the paths γ_1, γ_2 with endpoint i and $i+1$ and replace them by paths connecting i and $i+1$ resp. the two other endpoints of γ_1 & γ_2

$\sigma(\xi)L =$ rotating L by angle $\frac{2\pi}{N}$.

In pictures:

$\sigma(e_i)L =$ 

with the convention that closed loops are removed on the cost of a multiplicative factor d.

$\sigma(\xi)L =$ 

<u>Proof</u>: Exercise 3.6 ☒

The matchmaker representation σ can be lifted to a repr of the affine Hecke algebra using canonical map $H \twoheadrightarrow TL$

$$T_i \mapsto e_i - k^{-1}, \quad T_\xi \mapsto \xi$$

and corr. "matchmaker" repr

$$\tilde{\sigma} : H \to \mathrm{End}(\mathbb{C}[h]).$$

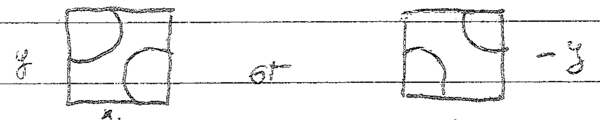
can be Baxterized:

$$(\nabla^{\tilde{\sigma}}(s_j)f)(y) = R_j(y) f(s_j y), \quad 1 \le j < M,$$
$$(\nabla^{\tilde{\sigma}}(\xi)f)(y) = \sigma(\xi) f(\xi^{-1} y)$$

with
$$R_j(y) = \frac{\sigma(e_j) + b(y_j - y_{j+1}) - k - k^{-1}}{b(y_j - y_{j+1})} : \mathbb{C}[h] \to \mathbb{C}[h]$$
$$= \left(\frac{1 - e^{y_j - y_{j+1}}}{k^{-1} - ke^{y_j - y_{j+1}}}\right)\sigma(e_j) + \left(\frac{k^{-1}e^{y_j - y_{j+1}} - k}{k^{-1} - ke^{y_j - y_{j+1}}}\right)$$

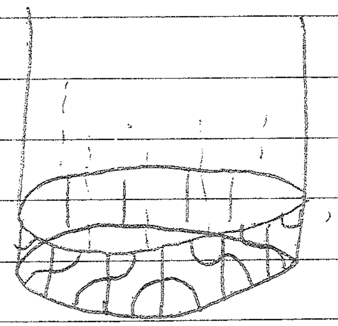
There are two ways to assign an integrable statistical mechanical model to it:

Classical type: consider as configurations of the (half) cylinder, the tiling by plaquettes



local weight $\frac{1-e^{x-y}}{k^{-1}-ke^{x-y}}$ local weight $\frac{k^{-1}e^{x-y}-k}{k^{-1}-ke^{x-y}}$

and consider corresponding partition function.



Completely packed loop model. Plays an important role in percolation theory.

Quantum type: Quantum state space $\mathbb{C}[h]$, with quantum integrability (and quantum Hamiltonians) arising from ∇^{σ}.

Homogeneous case, the quantum Hamiltonian simply becomes

$$H_{Loop} := \sum_{j\in\mathbb{Z}/n\mathbb{Z}} \sigma(e_j) : \mathbb{C}[h] \to \mathbb{C}[h],$$

defining the dense $O(k+k^{-1})$-loop model.

Proposition: Let $k = e^{\pi i/3}$ (primitive sixth root of unity). Then H_{Loop} has a unique eigenvector v (up to scaling) with eigenvalue n. It can be normalized such that

$$v = \sum_{L\in h} c_L L \quad \text{with } c_L \in \mathbb{Z}_{>0}.$$

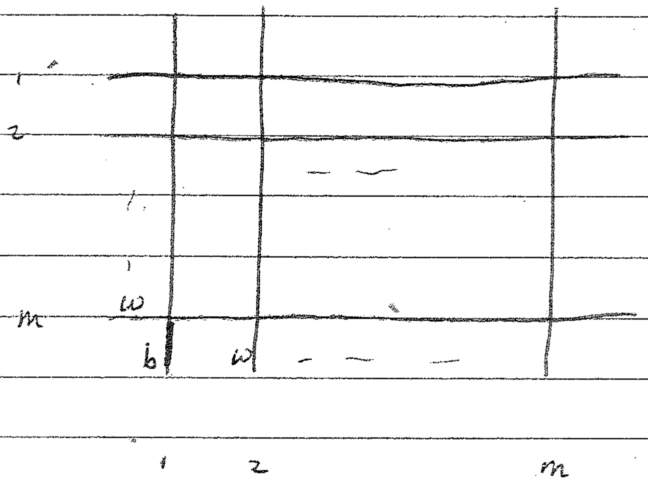
Proof: $k = e^{\pi i/3}$ means $k + k^{-1} = 1$, so the matchmaker representation $\sigma : T_n \to \mathrm{End}(\mathbb{C}[h])$ arises from linearizing the matchmaker operators

$$e_i : h \to h$$
$$\xi : h \to h$$

(which indeed maps a matching to a matching since closed loops are removed by a factor one). Hence H_{Loop}, w.r.t. standard basis h of $\mathbb{C}[h]$, is a nonnegative irreducible matrix with spectral radius n. Now use Perron-Frobenius theorem. $\boxtimes$

Razumov-Stroganov gave a combinatorial interpretation of c_L (as a conjecture), which was finally proved by L. Cantini and A. Sportiello:

set $m = \frac{M}{2}$, and consider the square of size m:



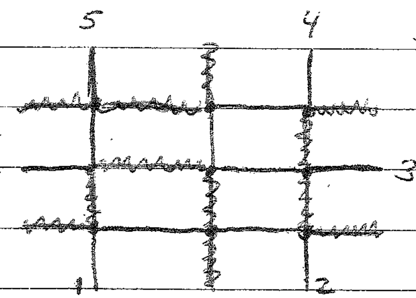
we colour the boundary edges black - white (checkerboard).

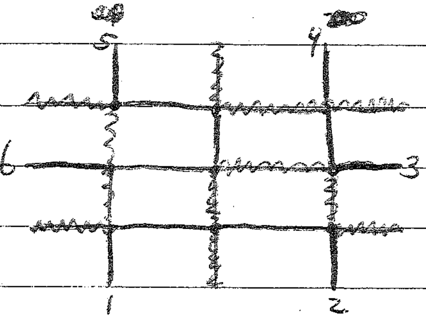
A fully packed loop is a further colouring of the edges by black & white such that at each vertex, two black and two white edges meet.

We number the black sticky ends by $1, .., M$ counter-clockwise, starting at the south-west corner.

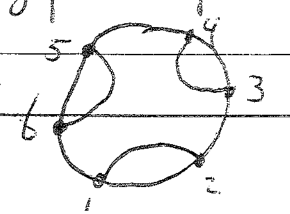
Each fully packed loop K now gives rise to a matching $L_K \in \mathcal{L}_L$, by looking at the black paths in the fully packed loop.

Example: $M=6$, $m=3$, we use two colours for edges, — = black, ww = white

$K_1 =$ 

$K_2 =$ 

are the two fully packed loops with associated perfect matching



Theorem (Cantini & Sportiello's Razumov–Stroganov conjecture)

Define for $L \in P_n$, $c_L := \#\{\text{fully packed loops } K \text{ on } m\text{-square} \mid L_K = L\}$ $(m = \frac{M}{2})$. Then $v := \sum_{L \in P_n} c_L L$ is "the" eigenvector of the $O(1)$-periodic loop Hamiltonian

$$H_{Loop} \text{ with eigenvalue } M.$$

Remark: The proof is entirely combinatorial, although it is believed that the deeper understanding should come from the quantum integrability of the dense $O(k+k^{-1})$ loop model (and the representation theoretic subtleties arising from the choice $k = e^{\pi i/3}$ of non-generic value of the free parameter).

The quantum integrability (of $O(k+k^{-1})$ dense loop model) can be understood from the quantum integrability of the Heisenberg XXZ spin-$\frac{1}{2}$ chain with quantum Hamiltonian

$$H_{XXZ} = (D_\gamma^{-1})_M h_{M1} (D_\gamma)_M + \sum_{j=2}^{M} h_{j-1,j} \circlearrowright (\mathbb{C}^2)^{\otimes M}$$

where $h = \sigma^x \otimes \sigma^x + \sigma^y \otimes \sigma^y - \frac{k+k^{-1}}{2} \sigma^z \otimes \sigma^z$ and

$$\gamma = (k, k^{-1}),$$

of prop 4. The identification is realized as follows.

Set: $\vec{P}_n = \{\text{oriented perfect matchings of } \{\xi_k\}_{k=1}^{M} \text{ within } D\}$

$\mathrm{Forg}: \vec{P}_n \to P_n$ forgetting the orientation

For $\vec{L} \in \vec{P}_n$, $r_j(\vec{L}) = \begin{cases} + & \text{if strand at } j \text{ outgoing} \\ - & \text{if strand at } j \text{ incoming} \end{cases}$

$(j = 1, \ldots, M)$ and

$$\sigma(\vec{L}) := \#\{\text{strands from } \xi_k \text{ to } \xi_l \text{ with } 1 \le l < k \le M\} - \#\{\text{strands from } \xi_k \text{ to } \xi_l \text{ with } 1 \le k < l \le M\}.$$

Define $\psi : \mathbb{C}[P_n] \to (\mathbb{C}^2)^{\otimes M}$, linear,

$$\psi(L) := \sum_{\vec{L} \in \mathrm{Forg}^{-1}(L)} k^{\sigma(\vec{L})/2} \, v_{r_1(\vec{L})} \otimes \cdots \otimes v_{r_M(\vec{L})}$$

View $\mathbb{C}[P_n]$ as TL-module w.r.t. the matchmaker representation σ, and $(\mathbb{C}^2)^{\otimes M}$ as TL-module w.r.t. the spin representation ρ with $\gamma = (\gamma_+, \gamma_-) = (k, k^{-1})$.

Proposition: ψ is an intertwiner of TL-modules, injective for generic k.

Appendix: More on Perron-Frobenius.

Theorem (Perron-Frobenius)
Let A be a real $n \times n$-matrix with nonnegative coeff., which is irreducible (not rewriteable by reshuffling of indices as $\begin{pmatrix} E & F \\ 0 & G \end{pmatrix}$). Let $r = \rho(A) = \max\{|\lambda| \mid \lambda \text{ eigenvalue of } A\}$ be the spectral radius of A. Then:

(i) $r > 0$ and r is an eigenvalue of A (the Perron-Frobenius eigenvalue).

(ii) The PF-eigenvalue is simple.

(iii) A has a left & right eigenvector with eigenvalue r having positive entries.

Application:

$k = e^{\pi i/3}$, $H_{loop} = \sum_{i=1}^{M} \sigma(e_i) : \mathbb{C}[\mathcal{L}] \to \mathbb{C}[\mathcal{L}]$.

With respect to the basis $\mathcal{L}$ of $\mathbb{C}[\mathcal{L}]$, H_{loop} is represented by $\#\mathcal{L} \times \#\mathcal{L}$-matrix A with nonnegative integer coefficients, and each column has the property that the sum of its entries add up to M (since $\sigma(e_i) L \in \mathcal{L}$ for all i and all $L \in \mathcal{L}$). Here we use that closed loops are removed by a multiplicative factor 1.

Now given L and L' in $\mathcal{L}$, $\exists\, i_1, \ldots, i_s \in \{1, \ldots, M\}$ such that $\sigma(e_{i_1}) \ldots \sigma(e_{i_s}) L = L'$. Hence the matrix coeff $(H^s_{loop}) L \big|_{L'} > 0 \Rightarrow$

There exists a $S > 0$ such that all entries of A^S are > 0, i.e. A is a primitive matrix, hence irreducible.

Note that

$$\rho(A) \leq M$$

Indeed, for any matrix norm $\|\cdot\|$, we have

$$\rho(A) = \lim_{k \to \infty} \|A^k\|^{1/k} \quad \text{(Gelfand).}$$

Take $\|\cdot\| = \|\cdot\|_\infty$ then

$$\rho(A) = \lim_{k \to \infty} \|A^k\|_\infty^{1/k} \leq \|A\|_\infty \leq M$$

since each entry in A is a nonnegative integer $\leq M$.

But: $\rho(A) = \rho(A^t) = M$, since A^t has eigenvector $\begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}$ with eigenvalue M.

Since A^t has now rows with the sum of entries of each row equal to M. The theorem of Perron-Frobenius thus says that $\exists\, v = \sum_{L \in \mathcal{L}} c_L L$, $c_L \in \mathbb{R}_{>0}$, unique up to $\mathbb{R}_{>0}$-scalar multiplication, such that

$$H_{loop} v = M v;$$

(and $\rho(H_{loop}) = M$, so M largest eigenvalue)

Since A has integer coeff., the normalization of v can we chosen such that $c_L \in \mathbb{Z}_{>0}$.

This fills in the details of the proposition on p.50.

Appendix B: More on relation XXZ spin chain and dense $O(k+k^{-1})$ loop model.

Sketch of proof of proposition p.54:

Let $L \in h$, $i \in \{1, \dots, M\}$. To check:

$$\psi^T(\sigma(e_i)L) = \mathfrak{z}(e_i)\psi^T(L)$$
$$\psi^T(\sigma(\mathfrak{z})L) = \mathfrak{z}(\mathfrak{z})\psi^T(L)$$

This involves many case-by-case checks. We give two examples:

1: $1 \le i < M$ and i is connected to $i+1$ in L:

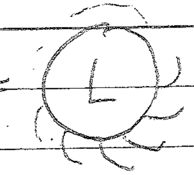
Then $\psi^T(\sigma(e_i)L) = (k+k^{-1})\psi^T(L)$

while

$$\mathfrak{z}(e_i)\psi^T(L) = \sum_{\substack{\vec{L} \in \mathrm{Forg}^{-1}(L):\\ r_i(\vec{L}) = +}} k^{\mathrm{or}(\vec{L})/2} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & k^{-1} & 1 & 0 \\ 0 & 1 & k & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}_{i,i+1} (\cdots \otimes \overset{i^{th}\text{ spot}}{v_+} \otimes v_- \otimes \cdots)$$

$$+ \sum_{\substack{\vec{L} \in \mathrm{Forg}^{-1}(L)\\ r_i(\vec{L}) = -}} k^{\mathrm{or}(\vec{L})/2} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & k^{-1} & 1 & 0 \\ 0 & 1 & k & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}_{i,i+1} (\cdots \otimes \overset{i^{th}\text{ spot}}{v_-} \otimes v_+ \otimes \cdots)$$

$$= \sum_{\substack{\vec{L} \in \mathrm{Forg}^{-1}(L):\\ r_i(\vec{L}) = +}} k^{\mathrm{or}(\vec{L})/2} (k+k^{-1}) v_{r_1(\vec{L})} \otimes \cdots \otimes v_{r_M(\vec{L})} + \sum_{\substack{\vec{L} \in \mathrm{Forg}^{-1}(L):\\ r_i(\vec{L}) = -}} k^{\mathrm{or}(\vec{L})/2} (k+k^{-1}) v_{r_1(\vec{L})} \otimes \cdots \otimes v_{r_M(\vec{L})} = (k+k^{-1})\psi^T(L)$$

Set $L' = \sigma(\mathfrak{z})L =$ 

Note: if $r_M(\vec{L}) = +$ then $\mathrm{or}(\vec{L'}) = \mathrm{or}(\vec{L}) - 2$ (where L' is given the canonical orientation inherited from $\vec{L}$), and if $r_M(\vec{L}) = -$ then $\mathrm{or}(\vec{L'}) = \mathrm{or}(\vec{L}) + 2$.

So:

$$\psi^T(\sigma(\mathfrak{z})L) = \sum_{\substack{\vec{L} \in \mathrm{Forg}^{-1}(L):\\ r_M(\vec{L}) = +}} k^{(\mathrm{or}(\vec{L})+2)/2} v_+ \otimes v_{r_1(\vec{L})} \otimes \cdots \otimes v_{r_{M-1}(\vec{L})}$$
$$+ \sum_{\substack{\vec{L} \in \mathrm{Forg}^{-1}(L):\\ r_M(\vec{L}) = -}} k^{(\mathrm{or}(\vec{L})-2)/2} v_- \otimes v_{r_1(\vec{L})} \otimes \cdots \otimes v_{r_{M-1}(\vec{L})}$$

while

$$\mathfrak{z}(\mathfrak{z})\psi^T(L) = \sum_{\vec{L} \in \mathrm{Forg}^{-1}(L)} k^{\mathrm{or}(\vec{L})/2} P_{12} P_{23} \cdots P_{M-1,M} (D_\gamma)_M (v_{r_1(\vec{L})} \otimes \cdots \otimes v_{r_M(\vec{L})})$$

$$= \sum_{\substack{\vec{L} \in \mathrm{Forg}^{-1}(L):\\ r_M(\vec{L}) = +}} \gamma_+ k^{\mathrm{or}(\vec{L})/2} v_+ \otimes v_{r_1(\vec{L})} \otimes \cdots \otimes v_{r_{M-1}(\vec{L})}$$
$$+ \sum_{\substack{\vec{L} \in \mathrm{Forg}^{-1}(L):\\ r_M(\vec{L}) = -}} \gamma_- k^{\mathrm{or}(\vec{L})/2} v_- \otimes v_{r_1(\vec{L})} \otimes \cdots \otimes v_{r_{M-1}(\vec{L})}$$

Hence $\psi^T(\sigma(\mathfrak{z})L) = \mathfrak{z}(\mathfrak{z})\psi^T(L)$ for $\gamma = (k, k^{-1})$

For injectivity (generic) note that ψ, as a linear map, depends rationally on k. Hence it suffices to show that it is injective for some fixed value of k.

$L \in \mathcal{L}$, let $\vec{T}_+$ be the unique orientation such that all strands are oriented from z_k to z_ℓ with $k > \ell$.

Then

$\cdot\ \mathrm{or}(\vec{T}_+) = M$

and $\mathrm{or}(\vec{T}) < M$ for any other $\vec{T} \in \mathrm{Forg}^{-1}(L)$,

so

$$\lim_{k\to\infty} k^{-m}\psi =: \psi_0 : \mathbb{C}[\mathcal{L}] \longrightarrow (\mathbb{C}^2)^{\otimes M}$$

is the injective linear map defined by

$$\psi_0(L) = v_{T_1(\vec{T}_+)} \otimes - \otimes v_{T_M(\vec{T}_+)}$$

for all $L \in \mathcal{L}$. $\boxtimes$

Remark: Identifying $\mathbb{C}[\mathcal{L}]$ with the subspace
$\psi(\mathbb{C}[\mathcal{L}]) \subseteq (\mathbb{C}^2)^{\otimes M}$,
the lin. operators $P_{i,i+1} R_{i,i+1}(y_i - y_{i+1})$ on $(\mathbb{C}^2)^{\otimes M}$
(see p. 39)
thus restrict to the "loop R-matrices" $R_i(y)$
on $\mathbb{C}[\mathcal{L}]$ (see p. 48).

Exercises:

* Exercise 1.1 (page 5)
* Exercise 1.2 (page 10)
* Exercise 2.1 (page 12)
* Exercise 2.2 (page 16)
* Exercise 2.3 (page 18)
* Exercise 3.1 (page 25)
* Exercise 3.2 (page 26)
* Exercise 3.3 (page 29)
* Exercise 3.4 (page 29)
* Exercise 3.5 (page 41)