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Radiation pressure on the boundary of a fluid and a metal
II. Waves of finite width

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SYNOPSIS

The effect of the finite width of a beam on its radiation pressure is studied. To that end the appropriate Fresnel relations are used in conjunction with the balance equation for the momentum flow at the fluid-metal boundary. An explicit expression is obtained for the radiation pressure that is exerted by a wide beam of Gaussian shape.

1. INTRODUCTION

In the preceding paper [5] we have studied the radiation pressure due to a plane wave, which impinges through an electrically polarized fluid on the boundary with a metal. In the course of the treatment the pressure distribution in the fluid was needed. In particular, its value near the boundary was related to its value at a field-free position . Strictly spoken the latter position does not exist if the incident wave has an infinite width. In practice waves have a finite extension, so that this paradox does not really occur. In a more consistent approach waves of finite width are to be considered from the beginning.

In this paper the effect of the finite extension of the wave on the radiation pressure is studied. For simplicity we confine the discussion to waves of normal incidence.

2. FRESNEL RELATIONS FOR FINITE WAVES AT NORMAL INCIDENCE

An electromagnetic wave, propagating through a polarizable medium of refractive index $n=\sqrt{\varepsilon}$, may be described by a Hertz vector $\mathbf{\Pi}(\mathbf{R},t)$ from which the fields follow as [4]:

$$(1)\qquad \begin{cases} \mathbf{E} = \nabla \wedge (\nabla \wedge \mathbf{\Pi}), \\ \mathbf{B} = n^2 \dfrac{\partial}{\partial t} \nabla \wedge \mathbf{\Pi}. \end{cases}$$

The refractive index n is assumed to be constant, so that electrostriction effects are neglected. The Hertz vector satisfies the wave equation

$$(2)\qquad \Delta \mathbf{\Pi} - n^2 \partial^2 \mathbf{\Pi} / \partial t^2 = 0.$$

A solution of this equation, which describes a cylindrically symmetric wave of finite width is

$$(3)\qquad \mathbf{\Pi} = \mathbf{e}_x J_0(kr \sin\theta) e^{ikz\cos\theta - i\omega t}.$$

It propagates in the z-direction with wave number $k \cos\theta$ and frequency $\omega = k/n$. The Bessel function J_0 confines the wave effectively to finite values of $r = (x^2 + y^2)^{\frac{1}{2}}$; the width of the wave is of order $(k \sin\theta)^{-1}$. The polarization of the Hertz vector has been chosen in the x-direction as denoted by the unit vector $\mathbf{e}_x$. A general cylindrically symmetric solution of (2) with finite width, frequency ω and polarization in the x-direction is obtained from (3) by multiplication with a weight function $g(\sin\theta)$ and integration over $\sin\theta \equiv \xi$:

$$(4)\qquad \mathbf{\Pi} = \mathbf{e}_x \int_0^1 d\xi g(\xi) J_0(kr\xi) e^{ikz\sqrt{1-\xi^2} - i\omega t}.$$

An alternative form of (3) follows from the integral representation of the Bessel function

$$(5)\qquad J_0(kr \sin\theta) = \frac{1}{2\pi} \int_0^{2\pi} d\varphi e^{ikr \sin\theta \cos\varphi}.$$

Substitution into (3) gives, with the change of variable $\varphi \to \varphi - \arctan(y/x)$:

$$(6)\qquad \Pi = \mathbf{e}_x \frac{1}{2\pi} \int_0^{2\pi} d\varphi e^{i\mathbf{k}\cdot\mathbf{R} - i\omega t},$$

with $\mathbf{k}$ a vector with spherical polar coordinates (k, θ, φ).

The fields (1) corresponding to (6) have the form:

$$(7)\qquad \begin{cases} \mathbf{E} = -\dfrac{1}{2\pi} \displaystyle\int_0^{2\pi} d\varphi \mathbf{k} \wedge (\mathbf{k} \wedge \mathbf{e}_x) e^{i\mathbf{k}\cdot\mathbf{R} - i\omega t}, \\ \mathbf{B} = \dfrac{nk}{2\pi} \displaystyle\int_0^{2\pi} d\varphi \mathbf{k} \wedge \mathbf{e}_x e^{i\mathbf{k}\cdot\mathbf{R} - i\omega t}. \end{cases}$$

We shall utilize fields of this form for the description of an electromagnetic wave of finite width, which impinges perpendicularly on a metal surface with normal $\mathbf{n}$ pointing into the metal in the z-direction. The reflected and transmitted waves will follow from the Fresnel relations for the plane waves out of which the fields are composed according to (7). These Fresnel relations are different for transverse electric and transverse magnetic waves (called E- and M-waves, respectively, in the following). The plane of incidence of an elementary wave as given by the integrands of (7) is determined by the vectors $\mathbf{k}$ and $\mathbf{n}$. Since the angle between these vectors is θ, the normal to the plane of incidence is $\mathbf{k}\wedge\mathbf{n}/k \sin\theta$. A unit vector orthogonal to $\mathbf{k}$ in the plane of incidence is $\mathbf{k}\wedge(\mathbf{k}\wedge\mathbf{n})/(k^2 \sin\theta)$. As a consequence the E- and M-components of the electric field of the elementary wave, contained in (7), have the forms

$$\frac{\mathbf{k}\wedge\mathbf{n}}{\sin^2\theta}(\mathbf{k}\wedge\mathbf{n})\cdot\mathbf{e}_x e^{i\mathbf{k}\cdot\mathbf{R}-i\omega t}, \tag{8}$$

$$\frac{\mathbf{k}\wedge(\mathbf{k}\wedge\mathbf{n})}{k^2\sin^2\theta}\mathbf{n}\cdot\mathbf{k}\mathbf{k}\cdot\mathbf{e}_x e^{i\mathbf{k}\cdot\mathbf{R}-i\omega t}. \tag{9}$$

For the E-component the incident electromagnetic fields are according to (7) and (8):

$$\left\{\begin{aligned} \mathbf{E}_i^E &= \frac{1}{2\pi}\int_0^{2\pi} d\varphi\,\mathbf{n}_0 E_{i0} e^{ik\mathbf{n}_i\cdot\mathbf{R}-i\omega t},\\ \mathbf{B}_i^E &= \frac{k}{\omega}\frac{1}{2\pi}\int_0^{2\pi} d\varphi\,\mathbf{n}_i\wedge\mathbf{n}_0 E_{i0} e^{ik\mathbf{n}_i\cdot\mathbf{R}-i\omega t}, \end{aligned}\right. \tag{10}$$

where $\mathbf{n}_i$ is a unit vector with polar angles (θ, φ) (and hence in the direction $\mathbf{k}$) and $\mathbf{n}_0$ the unit vector $\mathbf{n}_i\wedge\mathbf{n}/\sin\theta$, with polar angles $(\frac{1}{2}\pi, \varphi-\frac{1}{2}\pi)$. Furthermore E_{i0} is the amplitude of the wave given by

$$E_{i0} = k^2\mathbf{n}_0\cdot\mathbf{e}_x = k^2\sin\varphi. \tag{11}$$

The integrands of (10) are of the general form of an incident plane E-wave as given in (I.1). The reflected and transmitted waves can therefore be deduced from (I.2) and (I.3):

$$\left\{\begin{aligned} \mathbf{E}_r^E &= \frac{1}{2\pi}\int_0^{2\pi} d\varphi\,\mathbf{n}_0 E_{r0} e^{ik\mathbf{n}_r\cdot\mathbf{R}-i\omega t-i\psi_r},\\ \mathbf{B}_r^E &= \frac{k}{\omega}\frac{1}{2\pi}\int_0^{2\pi} d\varphi\,\mathbf{n}_r\wedge\mathbf{n}_0 E_{r0} e^{ik\mathbf{n}_r\cdot\mathbf{R}-i\omega t-i\psi_r}, \end{aligned}\right. \tag{12}$$

$$\left\{\begin{aligned} \mathbf{E}_t^E &= \frac{1}{2\pi}\int_0^{2\pi} d\varphi\,\mathbf{n}_0 E_{r0} e^{ik'\mathbf{n}_t\cdot\mathbf{R}-k''\mathbf{n}\cdot\mathbf{R}-i\omega t-i\psi_t},\\ \mathbf{B}_t^E &= \frac{1}{2\pi}\int_0^{2\pi} d\varphi\left(\frac{k'\mathbf{n}_t+ik''\mathbf{n}}{\omega}\right)\wedge\mathbf{n}_0\mathbf{E}_{t0} e^{ik'\mathbf{n}_t\cdot\mathbf{R}-k''\mathbf{n}\cdot\mathbf{R}-i\omega t-i\psi_t}, \end{aligned}\right. \tag{13}$$

where the unit vectors $\mathbf{n}_r$ and $\mathbf{n}_t$ have the polar angles $(\pi-\theta, \varphi)$ and (θ_t, φ), respectively. The angle θ_t and the wave numbers k' and k'' follow from (I.4) and (I.5) (with θ_i replaced by θ). The amplitudes $E_{\mu 0}$ and the phases ψ_μ, with $\mu = r$ or t, have been given in (I.7) and (I.8) as the combinations $f_\mu^E \equiv (E_{\mu 0}/E_{i0}) \exp(-i\psi_\mu)$. In the integrals (10, 12, 13) only the unit vectors $\mathbf{n}_0$, $\mathbf{n}_r$ and $\mathbf{n}_t$ depend on the angle φ.

The integrals over φ can be evaluated by means of the identities of the appendix. As an example we consider $\mathbf{E}_i^E$ which reads according to (10):

$$\mathbf{E}_i^E = k^2 e^{ikz\cos\theta - i\omega t} \frac{1}{2\pi} \int_0^{2\pi} d\varphi \sin\varphi [\sin\varphi,\ -\cos\varphi,\ 0] e^{ik\sin\theta(x\cos\varphi + y\sin\varphi)}; \tag{14}$$

the parentheses [, ,] enclose the three cartesian components. The right-hand side follows from (A6) and (A7). The other electromagnetic fields may be obtained likewise. The results are

$$\begin{cases} \mathbf{E}_i^E = k^2 e^{ikz\cos\theta - i\omega t}[F_1^E, F_2^E, 0], \\ \mathbf{B}_i^E = nk^2 e^{ikz\cos\theta - i\omega t}[-F_2^E\cos\theta, F_1^E\cos\theta, iF_3^E\sin\theta], \end{cases} \tag{15}$$

$$\begin{cases} \mathbf{E}_r^E = k^2 e^{-ikz\cos\theta - i\omega t} f_r^E [F_1^E, F_2^E, 0], \\ \mathbf{B}_r^E = nk^2 e^{-ikz\cos\theta - i\omega t} f_r^E [F_2^E\cos\theta, -F_1^E\cos\theta, iF_3^E\sin\theta], \end{cases} \tag{16}$$

$$\begin{cases} \mathbf{E}_t^E = k^2 e^{ik'z\cos\theta_t - k''z - i\omega t} f_t^E [F_1^E, F_2^E, 0], \\ \mathbf{B}_t^E = nk e^{ik'z\cos\theta_t - k''z - i\omega t} \\ \qquad f_t^E[-F_2^E(k'\cos\theta_t + ik''), F_1^E(k'\cos\theta_t + ik''), iF_3^E k'\sin\theta_t]. \end{cases} \tag{17}$$

The dependence on x and y is contained in the three functions:

$$\begin{cases} F_1^E(x,y) = \dfrac{x^2 - y^2}{2r^2} J_2(kr\sin\theta) + \tfrac{1}{2} J_0(kr\sin\theta), \\ F_2^E(x,y) = \dfrac{xy}{r^2} J_2(kr\sin\theta), \\ F_3^E(x,y) = -\dfrac{y}{r} J_1(kr\sin\theta). \end{cases} \tag{18}$$

The transverse magnetic components of the incident field follow by using (9) instead of (8). For later convenience we define the E- and M-components by writing

$$\begin{cases} \mathbf{E}_\mu = \mathbf{E}_\mu^E + \cos\theta\, \mathbf{E}_\mu^M, \\ \mathbf{B}_\mu = \mathbf{B}_\mu^E + \cos\theta\, \mathbf{B}_\mu^M \end{cases} \tag{19}$$

for the incident, the reflected and the transmitted fields ($\mu = i, r, t$, respectively). On a par with (10) the incident fields are found as

$$
(20) \quad \begin{cases} \mathbf{E}_i^M = \dfrac{\omega}{k}\dfrac{1}{2\pi}\int\limits_0^{2\pi} d\varphi \mathbf{n}_0 \wedge \mathbf{n}_i B_{i0} e^{ik\mathbf{n}_i\cdot\mathbf{R}-i\omega t}, \\[2ex] \mathbf{B}_i^M = \dfrac{1}{2\pi}\int\limits_0^{2\pi} d\varphi \mathbf{n}_0 B_{i0} e^{ik\mathbf{n}_i\cdot\mathbf{R}-i\omega t}. \end{cases}
$$

The amplitude B_{i0} is given by

$$
(21) \qquad B_{i0} = -nk^2 \frac{\mathbf{n}_i\cdot\mathbf{e}_x}{\sin\theta} = -nk^2\cos\varphi.
$$

After integration over φ the complete set of M-fields reads:

$$
(22) \quad \begin{cases} \mathbf{E}_i^M = k^2 e^{ikz\cos\theta-i\omega t}[F_1^M\cos\theta, -F_2^M\cos\theta, iF_3^M\sin\theta], \\ \mathbf{B}_i^M = nk^2 e^{ikz\cos\theta-i\omega t}[F_2^M, F_1^M, 0], \end{cases}
$$

$$
(23) \quad \begin{cases} \mathbf{E}_r^M = k^2 e^{-ikz\cos\theta-i\omega t} f_r^M[-F_1^M\cos\theta, F_2^M\cos\theta, iF_3^M\sin\theta], \\ \mathbf{B}_r^M = nk^2 e^{-ikz\cos\theta-i\omega t} f_r^M[F_2^M, F_1^M, 0], \end{cases}
$$

$$
(24) \quad \begin{cases} \mathbf{E}_t^M = \dfrac{nk^2}{\sqrt{\omega^2+\sigma^2}} e^{ik'z\cos\theta_t-k''z-i\omega t} f_t^M[F_1^M(k'\cos\theta_t+ik''), \\ \qquad\qquad -F_2^M(k'\cos\theta_t+ik''), iF_3^M k'\sin\theta_t], \\[2ex] \mathbf{B}_t^M = \dfrac{nk^2(\omega+i\sigma)}{\sqrt{\omega^2+\sigma^2}} e^{ik'z\cos\theta_t-k''z-i\omega t} f_t^M[F_2^M, F_1^M, 0]. \end{cases}
$$

The functions F_i^M have the form

$$
(25) \quad \begin{cases} F_1^M(x,y) = -\dfrac{x^2-y^2}{2r^2} J_2(kr\sin\theta) + \tfrac{1}{2}J_0(kr\sin\theta), \\[2ex] F_2^M(x,y) = F_2^E(x,y) \equiv F_2(x,y), \\[2ex] F_3^M(x,y) = -\dfrac{x}{r} J_1(kr\sin\theta). \end{cases}
$$

The relative amplitudes f_μ^M, with $\mu = r$ or t, have been given in (I.16) and (I.17).

We have found now the E- and M-components of the incident, the reflected and the transmitted waves, generated by the Hertz vector (3). A generalized cylindrically symmetric solution of the Maxwell equations with boundary conditions follows by multiplying (19) and hence (15–17), (22–24) with a weight function $g(\sin\theta)$ and integrating over $\sin\theta$ as in (4).

A different set-up to find waves with a finite width starts from a different

Hertz vector $\mathbf{\Pi}^*$ which yields the fields [4]:

$$(26)\qquad \begin{cases} \mathbf{E} = -\dfrac{\partial}{\partial t}(\nabla\wedge\mathbf{\Pi}^*), \\ \\ \mathbf{B} = \nabla\wedge(\nabla\wedge\mathbf{\Pi}^*). \end{cases}$$

The Hertz vector $\mathbf{\Pi}^*$ again fulfils the wave equation (2). In analogy with (3) we take as a solution

$$(27)\qquad \mathbf{\Pi}^* = \mathbf{e}_y J_0(kr\sin\theta)e^{ikz\sin\theta - i\omega t}.$$

The corresponding fields are then found as

$$(28)\qquad \begin{cases} \mathbf{E}_\mu = n^{-1}(\cos\theta\,\mathbf{E}_\mu^E + \mathbf{E}_\mu^M), \\ \mathbf{B}_\mu = n^{-1}(\cos\theta\,\mathbf{B}_\mu^E + \mathbf{B}_\mu^M) \end{cases}$$

with the same E- and M-components as in (19). A linear combination of these solutions arises by multiplication with a weight function $g^*(\sin\theta)$ and an integration over $\sin\theta$. Combining the generalized $\mathbf{\Pi}$- and $\mathbf{\Pi}^*$-solutions, obtained in this way, is equivalent, in view of (19) and (28), to combining the E- and M-solutions with weight functions

$$(29)\qquad \begin{cases} g^E = g + n^{-1}\cos\theta\, g^*, \\ g^M = \cos\theta\, g + n^{-1}g^*. \end{cases}$$

The electromagnetic fields describing waves of finite width then have the form:

$$(30)\qquad \begin{cases} \mathbf{E}_\mu = \int\limits_0^1 d\xi[g^E(\xi)\mathbf{E}_\mu^E + g^M(\xi)\mathbf{E}_\mu^M], \\ \\ \mathbf{B}_\mu = \int\limits_0^1 d\xi[g^E(\xi)\mathbf{B}_\mu^E + g^M(\xi)\mathbf{B}_\mu^M] \end{cases}$$

with $\xi = \sin\theta$ and $\mu = i, r, t$ for the incident, the reflected and the transmitted fields, respectively.

3. THE RADIATION PRESSURE FOR FINITE WAVES AT NORMAL INCIDENCE

The radiation pressure may be found from the momentum-balance equation of which the time-averaged form has been given in (I.18) with (I.19). Since the fields are non-uniform, we wish to concentrate on the total radiation force F^{rad}, which is the integral of the radiation pressure over the surface of the boundary between the liquid and the metal. Upon integrating the momentum-balance over a thin slab of infinite width lying symmetrically on both sides of the boundary, we get with Gauss's theorem

$$(31)\qquad \begin{cases} \int dS\,\mathbf{n}\cdot[\overline{\boldsymbol{P}_f} - \overline{\mathbf{D}_f\mathbf{E}_f} - \overline{\mathbf{B}_f\mathbf{B}_f} + \tfrac{1}{2}(\overline{E_f^2} + \overline{B_f^2})\boldsymbol{U}] = \\ = \int dS\,\mathbf{n}\cdot[\overline{\boldsymbol{P}_m} - \overline{\mathbf{E}_m\mathbf{E}_m} - \overline{\mathbf{B}_m\mathbf{B}_m} + \tfrac{1}{2}(\overline{E_m^2} + \overline{B_m^2})\boldsymbol{U}] \end{cases}$$

as a generalization of (I.20). Here $\bar{\boldsymbol{P}}_f$ and $\bar{\boldsymbol{P}}_m$ are time-averaged pressure tensors in the fluid and in the metal, both near the boundary. Likewise the electromagnetic fields pertain to positions near the boundary. As in paper I the isotropic fluid pressure tensor $\bar{\boldsymbol{P}}_f = \bar{p}_f \boldsymbol{U}$ may be connected to the pressure p_{f0} at a position where the fields are negligible. Since the incident wave has a finite width one may choose any position that is sufficiently far from the centre of the beam. Neglecting electrostriction effects as before we have

$$\bar{p}_f = p_{f0} + \tfrac{1}{2}\overline{\mathbf{P}_f \cdot \mathbf{E}_f}. \tag{32}$$

Likewise the metal pressure tensor $\bar{\boldsymbol{P}}_m$ may be related to the pressure $\boldsymbol{P}_{m0}$ deep in the metal, where the transmitted fields have been damped out. In fact the right-hand side of (31) becomes upon applying the momentum-balance equation in the metal and taking account of the finite width of the transmitted wave

$$\int dS\, \mathbf{n} \cdot \boldsymbol{P}_{m0}. \tag{33}$$

The perpendicular radiation force follows now from (31–33) as

$$F^{\text{rad}} = \int dS\, \mathbf{nn} : (\boldsymbol{P}_{m0} - p_{f0}\boldsymbol{U}) = \tfrac{1}{2}\int dS (n^2 \overline{E^2_{f,//}} - n^2 \overline{E^2_{f,\perp}} + \overline{B^2_{f,//}} - \overline{B^2_{f,\perp}}) \tag{34}$$

with // and $\perp$ denoting the components parallel and orthogonal to the boundary. The fields in the fluid, which are the sums of the incident and the reflected fields, have been given in (30) with (15–16) and (22–23). Upon insertion of these expressions into (34) one encounters integrals over products of the functions F_i^m (with $m = E, M$ and $i = 1, 2, 3$), which have been defined in (18) and (25). If use is made of the identity

$$\int_0^\infty dr\, r J_l(kr\xi) J_l(kr\xi') = k^{-2} \xi^{-1} \delta(\xi - \xi') \tag{35}$$

for non-negative integers l, the integrals become

$$\left\{ \begin{array}{l} \int dS\, F_1^\lambda(\xi) F_1^{\lambda'}(\xi') = \tfrac{1}{4}\pi k^{-2}\xi^{-1}(2\delta_{\lambda\lambda'} + 1)\delta(\xi - \xi'), \\ \int dS\, F_2(\xi) F_2(\xi') = \tfrac{1}{4}\pi k^{-2}\xi^{-1}\delta(\xi - \xi'), \\ \int dS\, F_3^\lambda(\xi) F_3^{\lambda'}(\xi') = \pi k^{-2}\xi^{-1}\delta_{\lambda\lambda'}\delta(\xi - \xi'), \end{array} \right. \tag{36}$$

where λ and λ' can stand for E or M. In this way the radiation force (34) gets the form:

$$F^{\text{rad}} = \tfrac{1}{2}\pi n^2 k^2 \int_0^1 d\xi\, \xi^{-1}(1 - \xi^2)[|g^E|^2(1 + |f_r^E|^2) + |g^M|^2(1 + |f_r^M|^2)]. \tag{37}$$

The total energy current impinging on the boundary may likewise be obtained from (30) with (15), (22) and (36):

$$S_{\text{tot}} = \tfrac{1}{2}\pi n k^2 \int_0^1 d\xi\, \xi^{-1}\sqrt{1 - \xi^2}(|g^E|^2 + |g^M|^2), \tag{38}$$

so that the reduced radiation force $\hat{F}^{\mathrm{rad}} = F^{\mathrm{rad}}/S_{\mathrm{tot}}$ is:

$$(39) \qquad \hat{F}^{\mathrm{rad}} = n \frac{\int_0^1 d\xi\, \xi^{-1}(1-\xi^2)[|g^E|^2(1+|f_r^E|^2) + |g^M|^2(1+|f_r^M|^2)]}{\int_0^1 d\xi\, \xi^{-1}\sqrt{1-\xi^2}(|g^E|^2 + |g^M|^2)}.$$

If the width of the incident wave becomes large, the weight functions g^E and g^M strongly emphasize the value $\xi = 0$. Since both $|f_r^E|$ and $|f_r^M|$ then reduce to $|f_r^{\perp}|$ (see (I.31-34)) one ends up with

$$(40) \qquad \hat{F}_{\infty}^{\mathrm{rad}} = n(1 + |f_r^{\perp}|^2),$$

as found in (I.30).

The effect of a finite beam width on the radiation pressure may be studied by adopting as a model for both the weight functions a Gaussian distribution

$$(41) \qquad g^{\lambda} = \gamma^{-1}\xi q^{\lambda}(\xi) e^{-\frac{1}{2}\gamma^{-1}k^2\xi^2}$$

with $\lambda = E$ or M. Here γ is a small width parameter ($\gamma \ll k^2$), while $q^{\lambda}(\xi)$ are analytic functions of ξ^2 in the interval $0 \leq \xi \leq 1$, with Taylor expansions around $\xi = 0$ of the form

$$(42) \qquad q^{\lambda}(\xi) = c_0^{\lambda}(1 + c_1^{\lambda}\xi^2 + \ldots).$$

The expressions for the incident fields corresponding to these weight functions follow from (30) with (15) and (22). For E_x one finds for instance

$$(43) \qquad E_x = \gamma^{-1}k^2 \int_0^1 d\xi\, \xi e^{ikz\sqrt{1-\xi^2} - i\omega t - \frac{1}{2}\gamma^{-1}k^2\xi^2}(q^E F_1^E + q^M\sqrt{1-\xi^2}F_1^M).$$

Introducing the new integration variable $\bar{\xi} = \gamma^{-\frac{1}{2}}k\xi$ and expanding subsequently the function $(1-\xi^2)^{\frac{1}{2}} = (1-\gamma k^{-2}\bar{\xi}^2)^{\frac{1}{2}}$ into powers of γ, one gets

$$(44) \qquad \begin{cases} E_x = \int_0^{\infty} d\bar{\xi}\, \bar{\xi} e^{ikz - i\omega t - \frac{1}{2}\bar{\xi}^2}(1 - \frac{1}{2}i\gamma k^{-1}z\bar{\xi}^2 + \ldots) \\ [q^E F_1^E + q^M(1 - \frac{1}{2}\gamma k^{-2}\bar{\xi}^2 + \ldots)F_1^M]. \end{cases}$$

The functions F_1^{λ} as defined in (18) and (25) contain Bessel functions J_n with arguments $\gamma^{\frac{1}{2}}r\bar{\xi}$. The integration over $\bar{\xi}$ can be carried out by means of (A8) and (A9) of the appendix. The field E_x is then found to be the sum of a function that falls off like a Gaussian for large r and a function that is proportional to r^{-2}. The latter function drops out if $c_0^E = c_0^M \equiv c_0$. For that choice the field E_x – and also the other components of the electromagnetic field – become Gaussian, as is suited for a wave of limited width. In fact the field E_x gets the form

$$(45) \qquad \begin{cases} E_x = e^{ikz - i\omega t - \frac{1}{2}\gamma r^2}c_0\{1 + \gamma k^{-2}[(1 - \frac{1}{2}\gamma r^2)(-ikz - \frac{1}{2} + c_1^E + c_1^M) \\ + \frac{1}{2}\gamma(x^2 - y^2)(\frac{1}{2} + c_1^E - c_1^M)] + \ldots\}. \end{cases}$$

The terms between curly brackets are ordered in increasing powers of γ; the combinations γx^2 and γy^2 are considered to be of order unity, since the width of the wave is $\gamma^{-\frac{1}{2}}$. The other components of the electric field are found to be

$$E_y = e^{ikz - i\omega t - \frac{1}{2}\gamma r^2} c_0[\gamma^2 k^{-2} xy(\tfrac{1}{2} + c_1^E - c_1^M) + \dots], \tag{46}$$

$$E_z = -e^{ikz - i\omega t - \frac{1}{2}\gamma r^2} i c_0 \gamma k^{-1} x[1 - 2\gamma k^{-2}(1 - \tfrac{1}{4}\gamma r^2)(ikz - 2c_1^M) + \dots]. \tag{47}$$

The magnetic fields B_x, B_y, B_z follow from the electric fields E_y, E_x, E_z by means of the substitutions $x \leftrightarrow y$, $c_1^E \leftrightarrow c_1^M$ and addition of a factor n.

The leading terms of the expressions (45–47) for the electric field (and their counterparts for the magnetic field) agree with those given by Peierls [3] for the special case of normal incidence.

The reduced radiation force caused by an incident wave of finite width, as described by the fields (45–47), follows by inserting (41) into (39). If use is made of the expressions (I.31) with (I.37) for $|f_r^\lambda|^2$ at small angles of incidence, the integral in the numerator of (39) becomes

$$4\gamma^{-2}(1 + A_\perp)^{-1} |c_0|^2 \int_0^1 d\xi\, \xi(1 - \xi^2)[1 + (c_1^E + c_1^M)\xi^2 + \dots] e^{-\gamma^{-1}k^2\xi^2}. \tag{48}$$

Here $A_\perp$ has been defined in (I.34). Similarly, the denominator of (39) may be written as

$$2\gamma^{-2} |c_0|^2 \int_0^1 d\xi\, \xi\sqrt{1 - \xi^2}[1 + (c_1^E + c_1^M)\xi^2 + \dots] e^{-\gamma^{-1}k^2\xi^2}. \tag{49}$$

The integrals in (48) and (49) can be evaluated by introducing as before, the new integration variable $\bar{\xi} = \gamma^{-\frac{1}{2}} k\xi$. The reduced radiation force is then found to be

$$\hat{F}^{\mathrm{rad}} = 2n(1 + A_\perp)^{-1}[1 - \tfrac{1}{2}\gamma k^{-2} + \dots], \tag{50}$$

up to first order in the width parameter γ.

The expression (50) may be compared to (I.30) with (I.31) for the reduced radiation pressure of an infinitely wide wave. The effects of the finite width have led to the factor between square brackets in (50). The difference of this factor from unity is determined by the ratio of the wave length and the beam width, as could be expected. In the measurements performed up to now [2] the second term of this expression is of the order 10^{-7}–10^{-8}, and hence within the experimental uncertainty. Nevertheless, the evolution of the radiation pressure for beams of finite width is of interest from a theoretical point of view, for reasons mentioned in the introduction.

APPENDIX. INTEGRAL RELATIONS FOR BESSEL FUNCTIONS

An integral representation for the Bessel function of order zero is:

$$\frac{1}{2\pi} \int_0^{2\pi} d\varphi\, e^{ir\cos\varphi} = J_0(r) \tag{A.1}$$

for $r \geq 0$. Writing $\varphi - \varphi_0$ instead of φ one gets:

$$\frac{1}{2\pi} \int_0^{2\pi} d\varphi\, e^{i(x\cos\varphi + y\sin\varphi)} = J_0(r) \tag{A.2}$$

with $x = r\cos\varphi_0$ and $y = r\sin\varphi_0$, so that $r = (x^2+y^2)^{\frac{1}{2}}$. Upon differentiation with respect to x and y we obtain:

$$\text{(A.3)} \qquad \frac{1}{2\pi}\int_0^{2\pi} d\varphi \cos\varphi \, e^{i(x\cos\varphi + y\sin\varphi)} = \frac{ix}{r} J_1(r),$$

$$\text{(A.4)} \qquad \frac{1}{2\pi}\int_0^{2\pi} d\varphi \sin\varphi \, e^{i(x\cos\varphi + y\sin\varphi)} = \frac{iy}{r} J_1(r).$$

By differentiating twice one finds from (A2):

$$\text{(A.5)} \qquad \frac{1}{2\pi}\int_0^{2\pi} d\varphi \cos^2\varphi \, e^{i(x\cos\varphi + y\sin\varphi)} = -\frac{x^2-y^2}{2r^2} J_2(r) + \tfrac{1}{2}J_0(r),$$

$$\text{(A.6)} \qquad \frac{1}{2\pi}\int_0^{2\pi} d\varphi \sin\varphi\cos\varphi \, e^{i(x\cos\varphi + y\sin\varphi)} = -\frac{xy}{r^2} J_2(r),$$

$$\text{(A.7)} \qquad \frac{1}{2\pi}\int_0^{2\pi} d\varphi \sin^2\varphi \, e^{i(x\cos\varphi + y\sin\varphi)} = \frac{x^2-y^2}{2r^2} J_2(r) + \tfrac{1}{2}J_0(r).$$

In deriving the fields for a Gaussian wave we used integrals over a product of a Gaussian and a Bessel function. These integrals follow from the formulae [1]:

$$\text{(A.8)} \qquad \int_0^\infty dx\, x^{2n+m+1} e^{-ax^2} J_m(bx) = \frac{n!\, b^m}{2^{m+1} a^{n+m+1}} e^{-\frac{1}{4}b^2/a} L_n^m\left(\frac{b^2}{4a}\right),$$

$$\text{(A.9)} \qquad \int_0^\infty dx\, x e^{-ax^2} J_2(bx) = -\left(\frac{1}{2a} + \frac{2}{b^2}\right) e^{-\frac{1}{4}b^2/a} + \frac{2}{b^2},$$

valid for non-negative integers m, n and positive a, b. The functions L_n^m are associated Laguerre polynomials.

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