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ON THE EQUIVALENCE OF CONVERGENT KINETIC EQUATIONS FOR HOT DILUTE PLASMAS

I. TRANSPORT COEFFICIENTS IN FIRST CHAPMAN–COWLING APPROXIMATION

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Two competitive convergent kinetic equations for a one-component Coulomb plasma with small plasma parameter ϵ are discussed. It is shown how a Chapman–Enskog scheme can be set up to calculate the heat conductivity and the viscosity. Both kinetic equations are found to yield identical transport coefficients for small ϵ, at least in lowest order of the Chapman–Cowling truncation procedure.

1. Introduction

In the kinetic description of hot dilute plasmas the long-range character of the electrostatic interaction plays an essential role. In fact, not only close binary collisions, with impact parameter near the Landau length r_L, but also collective screening interactions, at interparticle distances of the order of the Debye length r_D, must be taken into account. If either of these is neglected unphysical infinities in the transport coefficients show up. To a crude approximation this defect may be remedied by introducing a cut-off in the divergent integrals[1]).

In a consistent approach the transport coefficients are derived from a kinetic equation with a convergent collision term which treats all kinds of interactions in the plasma on an equal footing. A method to construct such a convergent collision term consists in taking a linear combination of the well-known divergent collision terms of Boltzmann, Balescu–Guernsey–Lenard and Landau in such a way that the divergencies cancel[2–4]). The ensuing composite collision term has been used[5,6]) to evaluate the transport properties in leading order in the plasma parameter $\epsilon = r_L/r_D$.

Recently an alternative convergent collision term has been put forward by several authors[7–9]). Its derivation has been based on an approximation for the nonequilibrium pair correlation function that is uniformly valid in the phase space of two interacting particles. The physical consequences of this new collision term, which is more complicated than the earlier one, have not been investigated systematically. The only transport property that has been evaluated is the high-frequency limit of the electrical conductivity for a plasma mixture[9,10]). It turns out that in this case the leading terms in ϵ are independent of the adopted kinetic equation.

In view of the equivalence of the kinetic expressions for the conductivity it may be surmised that the two convergent equations only differ in their formal appearance, but not in their physical consequences. To verify this supposition we study in this and in the following[11]) paper the transport properties of a hot dilute plasma on the basis of both equations. We shall calculate the leading terms in the ϵ expansions of the heat conductivity and the viscosity for a one-component plasma with a neutralizing background.

In section 2 of this paper a review of the convergent kinetic equations for a hot dilute plasma is given. These equations are linearized in section 3. The ensuing collision brackets are discussed in section 4. In particular we show how the usual Chapman–Enskog scheme can be applied to obtain the transport coefficients from these brackets. Finally, in section 5 we evaluate the heat conductivity and the viscosity in the lowest order of the Chapman–Cowling truncation procedure.

The results of this paper lead to the conclusion that the two convergent kinetic equations are equivalent in so far as the associated transport coefficients are concerned, at least in lowest Chapman–Cowling approximation. In the next paper this result will be generalized to arbitrary order of the truncation procedure.

2. Convergent kinetic equations for a hot dilute plasma

The general form of the kinetic equation for the one-particle distribution function $f(\boldsymbol{v}, \boldsymbol{x}, t)$ describing the non-equilibrium properties of a dilute system of particles can be derived from the hierarchy equations due to Bogoliubov, Born, Green, Kirkwood and Yvon[1]). For a homogeneous system the kinetic equation reads

$$\frac{\partial f(\boldsymbol{v}, t)}{\partial t} = C(\boldsymbol{v}, t), \tag{2.1}$$

with the collision term

$$C(\boldsymbol{v}, t) = m^{-1}(\partial/\partial \boldsymbol{v}) \cdot \int \mathrm{d}\boldsymbol{r} \boldsymbol{\nabla}\varphi(r) \int \mathrm{d}\boldsymbol{v}' g(\boldsymbol{r}, \boldsymbol{v}, \boldsymbol{v}', t). \tag{2.2}$$

Here φ is the interaction potential and $g(\boldsymbol{r}, \boldsymbol{v}, \boldsymbol{v}', t)$ the two-particle correlation function, which is a functional of f.

For a dilute system of particles with a short-range interaction the correlation function g may be written in the form[12])

$$g_{\mathrm{B}}(\boldsymbol{r}, \boldsymbol{v}, \boldsymbol{v}', t) = f(\boldsymbol{v}_0, t) f(\boldsymbol{v}_0', t) - f(\boldsymbol{v}, t) f(\boldsymbol{v}', t), \tag{2.3}$$

where $\boldsymbol{v}_0$ and $\boldsymbol{v}_0'$ are the initial velocities in the infinite past of two particles which move under the influence of their mutual interaction to a configuration with relative position $\boldsymbol{r}$ and velocities $\boldsymbol{v}$ and $\boldsymbol{v}'$. Insertion of (2.3) into (2.2) leads to the well-known Boltzmann collision term:

$$C_{\mathrm{B}}(\boldsymbol{v}, t) = \int \mathrm{d}\boldsymbol{v}' \, \mathrm{d}\Omega \, \frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} |\boldsymbol{v} - \boldsymbol{v}'| [f(\boldsymbol{v}_1, t) f(\boldsymbol{v}_1', t) - f(\boldsymbol{v}, t) f(\boldsymbol{v}', t)], \tag{2.4}$$

with $\mathrm{d}\sigma/\mathrm{d}\Omega$ the cross-section for a collision of the type $(\boldsymbol{v}, \boldsymbol{v}') \to (\boldsymbol{v}_1, \boldsymbol{v}_1')$ under the influence of the potential φ.

The above derivation of the Boltzmann collision term breaks down if the interaction potential has a long range, as is the case for a plasma. In a plasma, soft collisions with large impact parameters give rise to collective interactions that are not taken into account in (2.4). Such screening effects are adequately treated in the collision term of Balescu, Guernsey and Lenard (BGL)[1]), which follows from the hierarchy equations by a suitable ordering with respect to powers of the plasma parameter. The expression for the correlation function g in this case is rather complicated. However, in view of (2.2) we are interested only in the integral $\int \mathrm{d}\boldsymbol{v}' g$. Its Fourier transform (which for general $F(\boldsymbol{r})$ is defined as $\hat{F}(\boldsymbol{q}) = \int \mathrm{d}\boldsymbol{r} \exp(-\mathrm{i}\boldsymbol{q} \cdot \boldsymbol{r}) F(\boldsymbol{r})$) is found as

$$\begin{aligned}\int \mathrm{d}\boldsymbol{v}' \hat{g}_{\mathrm{BGL}}(\boldsymbol{q}, \boldsymbol{v}, \boldsymbol{v}', t) = &- |1 - \epsilon^{-1}(\boldsymbol{q}, \boldsymbol{q} \cdot \boldsymbol{v})|^2 f(\boldsymbol{v}) \\ &+ \int \mathrm{d}\boldsymbol{v}' \frac{\hat{\varphi}(q)}{m[\boldsymbol{q} \cdot (\boldsymbol{v} - \boldsymbol{v}') - \mathrm{i}\delta]} \boldsymbol{q} \cdot \left[\frac{1}{|\epsilon(\boldsymbol{q}, \boldsymbol{q} \cdot \boldsymbol{v}')|^2} \frac{\partial}{\partial \boldsymbol{v}} \right. \\ &\left. - \frac{1}{|\epsilon(\boldsymbol{q}, \boldsymbol{q} \cdot \boldsymbol{v})|^2} \frac{\partial}{\partial \boldsymbol{v}'} \right] [f(\boldsymbol{v}, t) f(\boldsymbol{v}', t)].\end{aligned} \tag{2.5}$$

The dielectric permeability appearing here is given by

$$\epsilon(\boldsymbol{q}, \omega) = 1 + \frac{\hat{\varphi}(q)}{m} \int \mathrm{d}\boldsymbol{v} \, \frac{\boldsymbol{q} \cdot \partial f / \partial \boldsymbol{v}}{\omega - \boldsymbol{q} \cdot \boldsymbol{v} + \mathrm{i}\delta}, \tag{2.6}$$

with $\hat{\varphi}(q) = e^2/q^2$ for a plasma. The collision term that follows from (2.2) with (2.5) is

$$C_{\mathrm{BGL}}(\boldsymbol{v}, t) = \frac{1}{8\pi^2 m^2} \frac{\partial}{\partial \boldsymbol{v}} \cdot \int \mathrm{d}\boldsymbol{q}\, \mathrm{d}\boldsymbol{v}' \delta[\boldsymbol{q} \cdot (\boldsymbol{v} - \boldsymbol{v}')] \frac{|\hat{\varphi}(q)|^2}{|\epsilon(\boldsymbol{q}, \boldsymbol{q} \cdot \boldsymbol{v})|^2} \times \boldsymbol{q}\boldsymbol{q} \cdot \left(\frac{\partial}{\partial \boldsymbol{v}} - \frac{\partial}{\partial \boldsymbol{v}'}\right)[f(\boldsymbol{v}, t) f(\boldsymbol{v}', t)]. \tag{2.7}$$

For large $|\boldsymbol{q}|$-values, which correspond to close binary encounters, the collision term (2.7) diverges. Hence the Boltzmann and BGL approaches are complementary in their description of collisions with large and with small impact parameter.

The Landau kinetic equation for a plasma may be considered as a hybrid of the Boltzmann and BGL equations. Its correlation function is obtained from (2.3) by taking the large-impact-parameter approximation. In Fourier space one gets

$$\hat{g}_{\mathrm{L}}(\boldsymbol{q}, \boldsymbol{v}, \boldsymbol{v}', t) = \frac{\hat{\varphi}(q)}{m[\boldsymbol{q} \cdot (\boldsymbol{v} - \boldsymbol{v}') - \mathrm{i}\delta]} \boldsymbol{q} \cdot \left(\frac{\partial}{\partial \boldsymbol{v}} - \frac{\partial}{\partial \boldsymbol{v}'}\right)[f(\boldsymbol{v}, t) f(\boldsymbol{v}', t)]. \tag{2.8}$$

Alternatively, the integral $\int \mathrm{d}\boldsymbol{v}' \hat{g}_{\mathrm{L}}$ is found from (2.5) by neglecting the screening effects and putting ϵ equal to 1. Substitution of (2.8) into (2.2) gives the Landau collision term:

$$C_{\mathrm{L}}(\boldsymbol{v}, t) = \frac{1}{8\pi^2 m^2} \frac{\partial}{\partial \boldsymbol{v}} \cdot \int \mathrm{d}\boldsymbol{q}\, \mathrm{d}\boldsymbol{v}' \delta[\boldsymbol{q} \cdot (\boldsymbol{v} - \boldsymbol{v}')] |\hat{\varphi}(q)|^2 \times \boldsymbol{q}\boldsymbol{q} \cdot \left(\frac{\partial}{\partial \boldsymbol{v}} - \frac{\partial}{\partial \boldsymbol{v}'}\right)[f(\boldsymbol{v}, t) f(\boldsymbol{v}', t)]. \tag{2.9}$$

The Landau collision term diverges both for small and large $|\boldsymbol{q}|$ so that neither hard nor soft collisions are treated adequately. In other words, the Landau equation combines the deficiencies of both the Boltzmann and the BGL equations.

Neither of the three collision terms given above leads to a consistent description of a plasma. Often an ad hoc cut-off procedure has been employed to avoid the infinities in the kinetic integrals. In a more systematic approach one wants to start from a convergent expression for the collision term. A possible method to obtain a convergent collision term consists in modifying and combining the three divergent collision terms in such a way that the divergencies drop out. In fact, one writes for the plasma collision term[2–4])

$$C_{\mathrm{P}}(\boldsymbol{v}, t) = C_{\mathrm{B}'}(\boldsymbol{v}, t) + C_{\mathrm{BGL}}(\boldsymbol{v}, t) - C_{\mathrm{L}'}(\boldsymbol{v}, t). \tag{2.10}$$

Here the Boltzmann collision term $C_{\mathrm{B}'}$ and the Landau collision term $C_{\mathrm{L}'}$ are modifications of (2.4) and (2.9), obtained by inserting for φ the screened

Coulomb potential:

$$\varphi'(r) = \frac{e^2\,\mathrm{e}^{-k_\mathrm{D} r}}{4\pi r}, \quad \hat{\varphi}'(q) = \frac{e^2}{q^2 + k_\mathrm{D}^2}. \tag{2.11}$$

The parameter $k_\mathrm{D} = r_\mathrm{D}^{-1} = e(\beta n)^{1/2}$ contains the density n and the kinetic temperature $T = k_\mathrm{B}^{-1}\beta^{-1}$, which is defined in terms of the average kinetic energy per particle. The collision term $C_{\mathrm{B}'}$ is convergent now since the potential has lost its long-range character. Likewise, the modified Landau term $C_{L'}$ is no longer divergent for small $|\boldsymbol{q}|$. Its divergence for large $|\boldsymbol{q}|$ is cancelled by that of C_BGL (2.7), since the dielectric permeability $\epsilon(\boldsymbol{q}, \boldsymbol{q}\cdot\boldsymbol{v})$ tends to 1 as $|\boldsymbol{q}| \to \infty$.

Recently[7–9]) an alternative convergent collision term has been derived from the hierarchy equations. It has a form that follows from (2.2) by inserting as the two-particle correlation function the combination $g_\mathrm{B} g_\mathrm{BGL}/g_\mathrm{L}$ of the three correlation functions considered before:

$$C_{\hat{\mathrm{P}}}(\boldsymbol{v}, t) = m^{-1}\frac{\partial}{\partial \boldsymbol{v}}\cdot \int \mathrm{d}\boldsymbol{r} \nabla\varphi(r) \int \mathrm{d}\boldsymbol{v}'\, \frac{g_\mathrm{B}(\boldsymbol{r}, \boldsymbol{v}, \boldsymbol{v}', t) g_\mathrm{BGL}(\boldsymbol{r}, \boldsymbol{v}, \boldsymbol{v}', t)}{g_\mathrm{L}(\boldsymbol{r}, \boldsymbol{v}, \boldsymbol{v}', t)}. \tag{2.12}$$

The potential φ occurring both explicitly and implicitly in the correlation functions is the ordinary Coulomb potential; screened potentials are not introduced in this approach.

The competitive convergent collision terms (2.10) and (2.12) are at first sight rather different although they are built up of closely related ingredients. As a consequence they may be expected to lead to different physical predictions, in particular with respect to the plasma transport coefficients. In the next sections we shall show how explicit expressions for these coefficients may be obtained both from (2.10) and from (2.12).

3. Linearization of the convergent kinetic equations

In the previous section two convergent collision terms for a homogeneous dilute plasma have been introduced. We may use the same expressions for the inhomogeneous case, provided the variations of the distribution functions do not become appreciable over distances of the order of the Debye length k_D^{-1}. Under these circumstances the convergent kinetic equations become

$$\left(\frac{\partial}{\partial t} + \boldsymbol{v}\cdot\frac{\partial}{\partial \boldsymbol{x}}\right) f(\boldsymbol{v}, \boldsymbol{x}, t) = C_\mathrm{P}(\boldsymbol{v}, \boldsymbol{x}, t) \tag{3.1a}$$

$$= C_{\hat{\mathrm{P}}}(\boldsymbol{v}, \boldsymbol{x}, t). \tag{3.1b}$$

At the left-hand side we neglected the consistent-field term, since the relaxation time associated with the neutralization of local space charges is short compared to the characteristic time of changes in f as a result of collisions.

The right-hand sides (2.10) and (2.12) of (3.1a) and (3.1b), which seem to be rather different, will become more alike after we have linearized around local equilibrium. To that end we write

$$f(\boldsymbol{v}, \boldsymbol{x}, t) = f^{(0)}(\boldsymbol{v}, \boldsymbol{x}, t)[1 + h(\boldsymbol{v}, \boldsymbol{x}, t)], \tag{3.2}$$

with the local equilibrium distribution

$$f^{(0)}(\boldsymbol{v}, \boldsymbol{x}, t) = n\left(\frac{\beta m}{2\pi}\right)^{3/2} \exp[-\tfrac{1}{2}\beta m(\boldsymbol{v} - \boldsymbol{V})^2] \equiv n\bar{f}_{\mathrm{M}}(\boldsymbol{v}, \boldsymbol{x}, t). \tag{3.3}$$

Here the density n, the inverse temperature β and the hydrodynamical velocity $\boldsymbol{V}$ are all local quantities depending on $\boldsymbol{x}$ and t. In the following we omit the arguments $\boldsymbol{x}$ and t in the distribution functions.

The collision term C_{P} as given by (2.10) vanishes upon insertion of the local equilibrium distribution function $f^{(0)}$. For (2.4) this is obvious, while for (2.7) and (2.9) one should use that $\boldsymbol{q} \cdot (\partial/\partial\boldsymbol{v} - \partial/\partial\boldsymbol{v}')f^{(0)}(\boldsymbol{v})f^{(0)}(\boldsymbol{v}')$ is proportional to the argument $\boldsymbol{q} \cdot (\boldsymbol{v} - \boldsymbol{v}')$ of the δ-function. Hence, if (3.2) is substituted into (2.10) the leading contribution is linear in h. We may write $C_{\mathrm{P}} = -n^2 I_{\mathrm{P}} h$, with

$$I_{\mathrm{P}} = I_{\mathrm{B}'} + I_{\mathrm{BGL}} - I_{\mathrm{L}'}. \tag{3.4}$$

The operators I_{i}, with $\mathrm{i} = \mathrm{B}'$, BGL and L′, will now be considered in succession.

The linearized Boltzmann operator is defined as

$$[I_{\mathrm{B}'}h](\boldsymbol{v}) = \int \mathrm{d}\boldsymbol{v}'\,\mathrm{d}\Omega\,\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}\,|\boldsymbol{v} - \boldsymbol{v}'|\bar{f}_{\mathrm{M}}(\boldsymbol{v})\bar{f}_{\mathrm{M}}(\boldsymbol{v}')[h(\boldsymbol{v}) + h(\boldsymbol{v}') - h(\boldsymbol{v}_1) - h(\boldsymbol{v}_1')], \tag{3.5}$$

where $\mathrm{d}\sigma/\mathrm{d}\Omega$ is the differential cross-section corresponding to the Debye potential.

The linearized BGL operator has the form

$$[I_{\mathrm{BGL}}h](\boldsymbol{v}) = -\frac{e^4}{8\pi^2 m^2}\frac{\partial}{\partial\boldsymbol{v}} \cdot \int \mathrm{d}\boldsymbol{q}\,\mathrm{d}\boldsymbol{v}'\delta[\boldsymbol{q}\cdot(\boldsymbol{v} - \boldsymbol{v}')]\,\frac{\boldsymbol{q}}{q^4|\epsilon^{(0)}(\boldsymbol{q}, \boldsymbol{q}\cdot\boldsymbol{v})|^2} \\ \times \bar{f}_{\mathrm{M}}(\boldsymbol{v})\bar{f}_{\mathrm{M}}(\boldsymbol{v}')\boldsymbol{q} \cdot \left(\frac{\partial}{\partial\boldsymbol{v}} - \frac{\partial}{\partial\boldsymbol{v}'}\right)[h(\boldsymbol{v}) + h(\boldsymbol{v}')]. \tag{3.6}$$

In the integrand the term in which the differential operators act on the Maxwell–Boltzmann distributions has been omitted on account of the argument mentioned above (3.4). For the same reason the dielectric permeability ϵ has been replaced by its zeroth-order contribution $\epsilon^{(0)}$.

Finally the linearized Landau operator is defined as

$$[I_{\mathrm{L}}h](\boldsymbol{v}) = -\frac{e^4}{8\pi^2 m^2}\frac{\partial}{\partial \boldsymbol{v}} \cdot \int \mathrm{d}\boldsymbol{q}\, \mathrm{d}\boldsymbol{v}'\delta[\boldsymbol{q}\cdot(\boldsymbol{v}-\boldsymbol{v}')]\frac{\boldsymbol{q}}{(q^2+k_{\mathrm{D}}^2)^2}$$

$$\times \bar{f}_{\mathrm{M}}(\boldsymbol{v})\bar{f}_{\mathrm{M}}(\boldsymbol{v}')\boldsymbol{q}\cdot\left(\frac{\partial}{\partial \boldsymbol{v}}-\frac{\partial}{\partial \boldsymbol{v}'}\right)[h(\boldsymbol{v})+h(\boldsymbol{v}')]. \tag{3.7}$$

Similarly a linearized version of the collision integral (2.12) can be obtained[9]). Firstly we consider the zeroth-order expressions $g_{\mathrm{i}}^{(0)}$ for the correlation functions that appear in (2.12). Substitution of $f^{(0)}$ into (2.3) yields:

$$g_{\mathrm{B}}^{(0)}(r,\boldsymbol{v},\boldsymbol{v}') = [\mathrm{e}^{-\beta\varphi(r)}-1]f^{(0)}(\boldsymbol{v})f^{(0)}(\boldsymbol{v}'). \tag{3.8}$$

For the BGL and Landau cases one finds:

$$g_{\mathrm{BGL}}^{(0)}(r,\boldsymbol{v},\boldsymbol{v}') = -\beta\varphi(r)\,\mathrm{e}^{-k_{\mathrm{D}}r}f^{(0)}(\boldsymbol{v})f^{(0)}(\boldsymbol{v}'), \tag{3.9}$$

$$g_{\mathrm{L}}^{(0)}(r,\boldsymbol{v},\boldsymbol{v}') = -\beta\varphi(r)f^{(0)}(\boldsymbol{v})f^{(0)}(\boldsymbol{v}'). \tag{3.10}$$

All three correlation functions $g_{\mathrm{i}}^{(0)}$ are isotropic in $\boldsymbol{r}$. Hence (2.12) vanishes in zeroth order.

In first order we have

$$\Delta\,\frac{g_{\mathrm{B}}g_{\mathrm{BGL}}}{g_{\mathrm{L}}} = \frac{g_{\mathrm{BGL}}^{(0)}}{g_{\mathrm{L}}^{(0)}}\Delta g_{\mathrm{B}} + \frac{g_{\mathrm{B}}^{(0)}}{g_{\mathrm{L}}^{(0)}}\Delta g_{\mathrm{BGL}} - \frac{g_{\mathrm{B}}^{(0)}g_{\mathrm{BGL}}^{(0)}}{g_{\mathrm{L}}^{(0)2}}\Delta g_{\mathrm{L}}, \tag{3.11}$$

where the operator Δ selects the contributions that are linear in h. From (2.12) and (3.11) we get $C_{\tilde{\mathrm{P}}} = -n^2 I_{\tilde{\mathrm{P}}}h$, with

$$I_{\tilde{\mathrm{P}}} = I_{\tilde{\mathrm{B}}} + I_{\mathrm{B\tilde{G}L}} - I_{\tilde{\mathrm{L}}}; \tag{3.12}$$

note that this form is analogous to (3.4). We now proceed by specifying the partial operators I_{i}, with $\mathrm{i} = \tilde{\mathrm{B}}$, $\mathrm{B\tilde{G}L}$ and $\tilde{\mathrm{L}}$.

The modified Boltzmann operator $I_{\tilde{\mathrm{B}}}$ follows from (2.12) with (2.3) and (3.9)–(3.11) as

$$[I_{\tilde{\mathrm{B}}}h](\boldsymbol{v}) = -m^{-1}\frac{\partial}{\partial \boldsymbol{v}}\cdot\int \mathrm{d}\boldsymbol{r}\boldsymbol{\nabla}\varphi_{\tilde{\mathrm{B}}}\int \mathrm{d}\boldsymbol{v}'\{\bar{f}_{\mathrm{M}}(\boldsymbol{v}_0)\bar{f}_{\mathrm{M}}(\boldsymbol{v}_0')[h(\boldsymbol{v}_0)+h(\boldsymbol{v}_0')]$$

$$-\bar{f}_{\mathrm{M}}(\boldsymbol{v})\bar{f}_{\mathrm{M}}(\boldsymbol{v}')[h(\boldsymbol{v})+h(\boldsymbol{v}')]\}. \tag{3.13}$$

Here the effective potential $\varphi_{\tilde{\mathrm{B}}}$ is defined through

$$\boldsymbol{\nabla}\varphi_{\tilde{\mathrm{B}}}(r) = \frac{g_{\mathrm{BGL}}^{(0)}}{g_{\mathrm{L}}^{(0)}}\boldsymbol{\nabla}\left(\frac{e^2}{4\pi r}\right) = \mathrm{e}^{-k_{\mathrm{D}}r}\boldsymbol{\nabla}\left(\frac{e^2}{4\pi r}\right); \tag{3.14}$$

the potential $\varphi_{\tilde{\mathrm{B}}}$ is uniquely determined if we put it equal to 0 for $r\to\infty$.

In (3.13) $\boldsymbol{v}_0$ and $\boldsymbol{v}_0'$ are the velocities of the incoming particles in the infinite past; under the influence of the Coulomb potential the particles move towards the relative position $\boldsymbol{r}$ with velocities $\boldsymbol{v}$, $\boldsymbol{v}'$, so that $\boldsymbol{v}_0$ and $\boldsymbol{v}_0'$ are both functions of $\boldsymbol{r}$, $\boldsymbol{v}$ and $\boldsymbol{v}'$. These functions can be determined explicitly in terms of the conserved quantities: total energy E, centre-of-mass velocity $\boldsymbol{U}$, angular momentum $\boldsymbol{L}$ and Runge–Lenz vector $\boldsymbol{N}$. In fact, one has

$$E = \frac{1}{4} m |\boldsymbol{v} - \boldsymbol{v}'|^2 + \frac{e^2}{4\pi r}, \tag{3.15a}$$

$$\boldsymbol{U} = \tfrac{1}{2}(\boldsymbol{v} + \boldsymbol{v}'), \tag{3.15b}$$

$$\boldsymbol{L} = \tfrac{1}{2} m \boldsymbol{r} \wedge (\boldsymbol{v} - \boldsymbol{v}'), \tag{3.15c}$$

$$\boldsymbol{N} = (\boldsymbol{v} - \boldsymbol{v}') \wedge \boldsymbol{L} + \frac{e^2}{4\pi} \hat{\boldsymbol{r}}, \tag{3.15d}$$

with $\hat{\boldsymbol{r}} = \boldsymbol{r}/|\boldsymbol{r}|$. If we put

$$\boldsymbol{w} = \frac{2}{N^2} \left(\frac{E}{m}\right)^{1/2} \left[-\frac{e^2}{4\pi} \boldsymbol{N} + 2 \left(\frac{E}{m}\right)^{1/2} (\boldsymbol{L} \wedge \boldsymbol{N}) \right], \tag{3.16}$$

then we have

$$\boldsymbol{v}_0 = \boldsymbol{U} + \tfrac{1}{2}\boldsymbol{w}, \quad \boldsymbol{v}_0' = \boldsymbol{U} - \tfrac{1}{2}\boldsymbol{w}. \tag{3.17}$$

Furthermore it will prove useful to introduce $\boldsymbol{u} = \boldsymbol{v} - \boldsymbol{v}'$, so that $\boldsymbol{v} = \boldsymbol{U} + \frac{1}{2}\boldsymbol{u}$, $\boldsymbol{v}' = \boldsymbol{U} - \frac{1}{2}\boldsymbol{u}$. Conservation of energy implies the relation:

$$w^2 = u^2 + \frac{e^2}{\pi m r}, \tag{3.18}$$

so that one has

$$\bar{f}_{\mathrm{M}}(\boldsymbol{v}_0)\bar{f}_{\mathrm{M}}(\boldsymbol{v}_0') = \mathrm{e}^{-r_{\mathrm{L}}/r}\bar{f}_{\mathrm{M}}(\boldsymbol{v})\bar{f}_{\mathrm{M}}(\boldsymbol{v}'). \tag{3.19}$$

The second term between the curly brackets in (3.13) does not depend on $\boldsymbol{r}$. Hence its contribution to the integral vanishes. Using furthermore (3.17) and (3.19) in (3.13) we obtain for the modified Boltzmann operator:

$$\begin{aligned}[I_{\bar{\mathrm{B}}} h](\boldsymbol{v}) = &- m^{-1} \frac{\partial}{\partial \boldsymbol{v}} \cdot \int \mathrm{d}\boldsymbol{r} (\boldsymbol{\nabla} \varphi_{\bar{\mathrm{B}}}) \int \mathrm{d}\boldsymbol{v}' \bar{f}_{\mathrm{M}}(\boldsymbol{v}) \bar{f}_{\mathrm{M}}(\boldsymbol{v}') \\ &\times \mathrm{e}^{-r_{\mathrm{L}}/r} [h(\boldsymbol{U} + \tfrac{1}{2}\boldsymbol{w}) + h(\boldsymbol{U} - \tfrac{1}{2}\boldsymbol{w})]. \end{aligned} \tag{3.20}$$

The modified BGL operator $I_{\mathrm{B\bar{G}L}}$ in (3.12) is found from (2.12) with (2.5) and (3.11) as

$$[I_{\mathrm{B\tilde{G}L}}h](v) = \frac{\mathrm{i}e^2}{8\pi^3 m^2}\frac{\partial}{\partial v}\cdot\int \mathrm{d}q\,\mathrm{d}v'\frac{q\hat{\varphi}_{\mathrm{B\tilde{G}L}}(q)}{q^2[q\cdot(v-v')-\mathrm{i}\delta]}$$
$$\times\Delta\left(q\cdot\left[\frac{1}{|\epsilon(q,q\cdot v')|^2}\frac{\partial}{\partial v}-\frac{1}{|\epsilon(q,q\cdot v)|^2}\frac{\partial}{\partial v'}\right]\right.$$
$$\left.\times\{\bar{f}_{\mathrm{M}}(v)\bar{f}_{\mathrm{M}}(v')[1+h(v)+h(v')]\}\right). \tag{3.21}$$

Here $\hat{\varphi}_{\mathrm{B\tilde{G}L}}$ denotes the Fourier transform of the effective potential which is determined by

$$\nabla\varphi_{\mathrm{B\tilde{G}L}}(r) = \frac{g_{\mathrm{B}}^{(0)}}{g_{\mathrm{L}}^{(0)}}\nabla\left(\frac{e^2}{4\pi r}\right) = \frac{1-\mathrm{e}^{-r_{\mathrm{L}}/r}}{r_{\mathrm{L}}/r}\nabla\left(\frac{e^2}{4\pi r}\right), \tag{3.22}$$

with $\lim_{r\to\infty}\varphi_{\mathrm{B\tilde{G}L}}(r)=0$; we used (3.8) and (3.10) with $\beta\varphi = \beta e^2/(4\pi r) = r_{\mathrm{L}}/r$ to obtain the explicit expression in the third member of (3.22). By substitution of the Rayleigh expansion for $\mathrm{e}^{-\mathrm{i}q\cdot r}$ into the Fourier transform of (3.22) we obtain

$$\hat{\varphi}_{\mathrm{B\tilde{G}L}}(q) = \frac{e^2}{r_{\mathrm{L}}q}\int_0^\infty \mathrm{d}r(1-\mathrm{e}^{-r_{\mathrm{L}}/r})rj_1(qr). \tag{3.23}$$

Owing to the symmetry of the integrand in (3.21) under the reflection $q\to -q$ we are allowed to replace the distribution $[q\cdot(v-v')-\mathrm{i}\delta]^{-1}$ by $\pi\,\mathrm{i}\delta[q\cdot(v-v')]$. Then $\epsilon(q,q\cdot v')$ is equal to $\epsilon(q,q\cdot v)$. With a similar argument as used above (3.4) we arrive at

$$[I_{\mathrm{B\tilde{G}L}}h](v) = -\frac{e^2}{8\pi^2 m^2}\frac{\partial}{\partial v}\cdot\int \mathrm{d}q\,\mathrm{d}v'\delta[q\cdot(v-v')]\frac{q\hat{\varphi}_{\mathrm{B\tilde{G}L}}(q)}{q^2|\epsilon^{(0)}(q,q\cdot v)|^2}$$
$$\times\bar{f}_{\mathrm{M}}(v)\bar{f}_{\mathrm{M}}(v')q\cdot\left(\frac{\partial}{\partial v}-\frac{\partial}{\partial v'}\right)[h(v)+h(v')]. \tag{3.24}$$

It should be noted that the difference between I_{BGL} (3.6) and $I_{\mathrm{B\tilde{G}L}}$ amounts to the replacement of e^2q^{-2} by $\hat{\varphi}_{\mathrm{B\tilde{G}L}}(q)$.

Finally, the modified Landau operator $I_{\tilde{\mathrm{L}}}$ in (3.12) is obtained from (2.12) with (2.8) and (3.11):

$$[I_{\tilde{\mathrm{L}}}h](v) = -\frac{e^2}{8\pi^2 m^2}\frac{\partial}{\partial v}\cdot\int \mathrm{d}q\,\mathrm{d}v'\delta[q\cdot(v-v')]\frac{q\hat{\varphi}_{\tilde{\mathrm{L}}}(q)}{q^2}$$
$$\times\bar{f}_{\mathrm{M}}(v)\bar{f}_{\mathrm{M}}(v')q\cdot\left(\frac{\partial}{\partial v}-\frac{\partial}{\partial v'}\right)[h(v)+h(v')]. \tag{3.25}$$

The effective potential $\varphi_{\tilde{\mathrm{L}}}$ satisfies the relation

$$\nabla\varphi_{\tilde{\mathrm{L}}}(r) = \frac{g_{\mathrm{B}}^{(0)}g_{\mathrm{BGL}}^{(0)}}{g_{\mathrm{L}}^{(0)2}}\nabla\left(\frac{e^2}{4\pi r}\right) = \frac{1-\mathrm{e}^{-r_{\mathrm{L}}/r}}{r_{\mathrm{L}}/r}\mathrm{e}^{-\kappa_{\mathrm{D}}r}\nabla\left(\frac{e^2}{4\pi r}\right), \tag{3.26}$$

with $\lim_{r\to\infty} \varphi_{\tilde{L}}(r) = 0$; we used (3.8)–(3.10) for the correlation functions. Similarly to (3.23) one easily obtains

$$\hat{\varphi}_{\tilde{L}}(q) = \frac{e^2}{r_L q} \int_0^\infty dr(1 - e^{-r_L/r})\, e^{-k_D r} r j_1(qr). \tag{3.27}$$

Having calculated the partial collision operators I_i we conclude this section by mentioning an important property they have in common. Let Ψ be a collision invariant, i.e. $\Psi = 1$, $\boldsymbol{v}$ or v^2. Then one has

$$I_i \Psi = 0. \tag{3.28}$$

The proof of this property is trivial for all i except i = $\tilde{B}$; the latter case must be considered separately. When we put $h = 1$ in (3.20) the $\boldsymbol{r}$ integration gives a vanishing result owing to rotational symmetry. A similar argument holds for the case $h(\boldsymbol{v}) = \boldsymbol{v}$. For $h(\boldsymbol{v}) = v^2$ we use that according to (3.18) w^2 is independent of the direction of $\boldsymbol{r}$, so that again rotational symmetry leads to (3.28).

4. Collision brackets

The linearized collision operators will be used in the following to evaluate the transport coefficients of a plasma. These can be expressed in terms of collision brackets which for an arbitrary I are defined in the usual way[12]):

$$[h, k] = \int d\boldsymbol{v} h(\boldsymbol{v})[Ik](\boldsymbol{v}). \tag{4.1}$$

Obviously the bracket is linear in I, so that the brackets of the composite collision operators (3.4) and (3.12) follow directly from those of its constituents. In fact, one has

$$[h, k]_P = [h, k]_{B'} + [h, k]_{BGL} - [h, k]_{L'} \tag{4.2}$$

and a similar relation for $[h, k]_{\tilde{P}}$. We shall calculate now the explicit expressions for the brackets associated with the six partial collision operators.

The bracket corresponding to the Boltzmann collision operator has the well-known form

$$[h, k]_{B'} = \tfrac{1}{4} \int d\boldsymbol{v}\, d\boldsymbol{v}'\, d\Omega \frac{d\sigma}{d\Omega} |\boldsymbol{v} - \boldsymbol{v}'| \bar{f}_M(\boldsymbol{v}) \bar{f}_M(\boldsymbol{v}')$$
$$\times [h(\boldsymbol{v}) + h(\boldsymbol{v}') - h(\boldsymbol{v}_1) - h(\boldsymbol{v}_1')][k(\boldsymbol{v}) + k(\boldsymbol{v}') - k(\boldsymbol{v}_1) - k(\boldsymbol{v}_1')]. \tag{4.3}$$

For the BGL operator we obtain from (3.6) by partial integration and symmetrizing in $\boldsymbol{v}$ and $\boldsymbol{v}'$:

$$[h, k]_{\mathrm{BGL}} = \frac{e^4}{16\pi^2 m^2} \int \mathrm{d}\boldsymbol{v}\,\mathrm{d}\boldsymbol{v}'\,\mathrm{d}\boldsymbol{q}\,\delta[\boldsymbol{q}\cdot(\boldsymbol{v}-\boldsymbol{v}')]\frac{1}{q^4|\epsilon^{(0)}(\boldsymbol{q}, \boldsymbol{q}\cdot\boldsymbol{v})|^2}\bar{f}_{\mathrm{M}}(\boldsymbol{v})\bar{f}_{\mathrm{M}}(\boldsymbol{v}')$$
$$\times\,\boldsymbol{q}\cdot\left(\frac{\partial}{\partial\boldsymbol{v}}-\frac{\partial}{\partial\boldsymbol{v}'}\right)[h(\boldsymbol{v})+h(\boldsymbol{v}')]\boldsymbol{q}\cdot\left(\frac{\partial}{\partial\boldsymbol{v}}-\frac{\partial}{\partial\boldsymbol{v}'}\right)[k(\boldsymbol{v})+k(\boldsymbol{v}')]. \tag{4.4}$$

The brackets $[h, k]_{\mathrm{L}'}$ can easily be obtained from (4.4) by the replacements $q^{-4}\to(q^2+k_{\mathrm{D}}^2)^{-2}$ and $\epsilon^{(0)}\to 1$, as follows by comparing (3.6) and (3.7).

The bracket associated with the modified Boltzmann collision operator $I_{\bar{\mathrm{B}}}$ (3.20) becomes upon partial integration with respect to $\boldsymbol{v}$:

$$[h, k]_{\bar{\mathrm{B}}} = m^{-1}\int \mathrm{d}\boldsymbol{v}\,\mathrm{d}\boldsymbol{v}'\,\mathrm{d}\boldsymbol{r}\,\bar{f}_{\mathrm{M}}(\boldsymbol{v})\bar{f}_{\mathrm{M}}(\boldsymbol{v}')\exp(-r_{\mathrm{L}}/r)$$
$$\times\,[k(\boldsymbol{U}+\tfrac{1}{2}\boldsymbol{w})+k(\boldsymbol{U}-\tfrac{1}{2}\boldsymbol{w})](\nabla\varphi_{\bar{\mathrm{B}}})\cdot\frac{\partial}{\partial\boldsymbol{v}}h(\boldsymbol{v}). \tag{4.5}$$

We now introduce as new integration variables the centre-of-mass and the relative velocities $\boldsymbol{U}=\frac{1}{2}(\boldsymbol{v}+\boldsymbol{v}')$ and $\boldsymbol{u}=\boldsymbol{v}-\boldsymbol{v}'$, so that $\partial h(\boldsymbol{v})/\partial\boldsymbol{v} = 2\partial h(\boldsymbol{U}+\frac{1}{2}\boldsymbol{u})/\partial\boldsymbol{u}$. Furthermore we symmetrize the integrand with respect to the transformation $\boldsymbol{u}\to-\boldsymbol{u}$, $\boldsymbol{r}\to-\boldsymbol{r}$, which implies $\boldsymbol{w}\to-\boldsymbol{w}$. Then (4.5) becomes

$$[h, k]_{\bar{\mathrm{B}}} = m^{-1}\int \mathrm{d}\boldsymbol{U}\,\mathrm{d}\boldsymbol{u}\,\mathrm{d}\boldsymbol{r}\,\bar{f}_{\mathrm{M}}(\boldsymbol{U}+\tfrac{1}{2}\boldsymbol{u})\bar{f}_{\mathrm{M}}(\boldsymbol{U}-\tfrac{1}{2}\boldsymbol{u})\exp(-r_{\mathrm{L}}/r)$$
$$\times\,[k(\boldsymbol{U}+\tfrac{1}{2}\boldsymbol{w})+k(\boldsymbol{U}-\tfrac{1}{2}\boldsymbol{w})](\nabla\varphi_{\bar{\mathrm{B}}})\cdot\frac{\partial}{\partial\boldsymbol{u}}[h(\boldsymbol{U}+\tfrac{1}{2}\boldsymbol{u})+h(\boldsymbol{U}-\tfrac{1}{2}\boldsymbol{u})]. \tag{4.6}$$

From (3.18) we observe that

$$\bar{f}_{\mathrm{M}}(\boldsymbol{U}+\tfrac{1}{2}\boldsymbol{u})\bar{f}_{\mathrm{M}}(\boldsymbol{U}-\tfrac{1}{2}\boldsymbol{u})\exp(-r_{\mathrm{L}}/r) \tag{4.7}$$

is a function of $\boldsymbol{U}$ and $\boldsymbol{w}$ only. Let us perform now in (4.6) a partial integration with respect to $\boldsymbol{u}$. Since $\boldsymbol{w}$ (3.16) is a function of conserved quantities only, the Poisson bracket of E (3.15a) and an arbitrary function $G(\boldsymbol{w})$ of $\boldsymbol{w}$ vanishes. Hence we infer:

$$\boldsymbol{r}\cdot(\partial/\partial\boldsymbol{u})G(\boldsymbol{w}) = -2\pi e^{-2}mr^3\boldsymbol{u}\cdot\nabla G(\boldsymbol{w}). \tag{4.8}$$

Substituting moreover the explicit expression (3.14) for $\nabla\varphi_{\bar{\mathrm{B}}}$ we arrive at:

$$[h, k]_{\bar{\mathrm{B}}} = -\tfrac{1}{2}\int \mathrm{d}\boldsymbol{U}\,\mathrm{d}\boldsymbol{u}\,\mathrm{d}\boldsymbol{r}\,\bar{f}_{\mathrm{M}}(\boldsymbol{U}+\tfrac{1}{2}\boldsymbol{u})\bar{f}_{\mathrm{M}}(\boldsymbol{U}-\tfrac{1}{2}\boldsymbol{u})\exp(-k_{\mathrm{D}}r)$$
$$\times\,[h(\boldsymbol{U}+\tfrac{1}{2}\boldsymbol{u})+h(\boldsymbol{U}-\tfrac{1}{2}\boldsymbol{u})]\boldsymbol{u}\cdot\nabla\{\exp(-r_{\mathrm{L}}/r)$$
$$\times\,[k(\boldsymbol{U}+\tfrac{1}{2}\boldsymbol{w})+k(\boldsymbol{U}-\tfrac{1}{2}\boldsymbol{w})]\}. \tag{4.9}$$

One more partial integration, this time with respect to r, finally yields

$$[h, k]_{\tilde{\mathrm{B}}} = -\tfrac{1}{2} \int \mathrm{d}U\, \mathrm{d}u\, \mathrm{d}r\, \bar{f}_{\mathrm{M}}(U + \tfrac{1}{2}u)\bar{f}_{\mathrm{M}}(U - \tfrac{1}{2}u)\exp(-k_{\mathrm{D}}r - r_{\mathrm{L}}/r)$$
$$\times k_{\mathrm{D}}u \cdot \hat{r}[h(U + \tfrac{1}{2}u) + h(U - \tfrac{1}{2}u)][k(U + \tfrac{1}{2}w) + k(U - \tfrac{1}{2}w)]. \quad (4.10)$$

Next we consider the bracket pertaining to the modified B$\tilde{\mathrm{G}}$L collision operator. It is easily obtained from (4.4) by the replacement of a factor e^2q^{-2} by $\hat{\varphi}_{\mathrm{B\tilde{G}L}}(q)$, as follows from the remark below (3.24). Consequently we have

$$[h, k]_{\mathrm{B\tilde{G}L}} = \frac{e^2}{16\pi^2 m^2} \int \mathrm{d}v\, \mathrm{d}v'\, \mathrm{d}q\, \delta[q \cdot (v - v')] \frac{\hat{\varphi}_{\mathrm{B\tilde{G}L}}(q)}{q^2|\epsilon^{(0)}(q, q \cdot v)|^2} \bar{f}_{\mathrm{M}}(v)\bar{f}_{\mathrm{M}}(v')$$
$$\times q \cdot \left(\frac{\partial}{\partial v} - \frac{\partial}{\partial v'}\right)[h(v) + h(v')]q \cdot \left(\frac{\partial}{\partial v} - \frac{\partial}{\partial v'}\right)[k(v) + k(v')]. \quad (4.11)$$

Finally, the modified Landau bracket $[h, k]_{\tilde{\mathrm{L}}}$ is straightforwardly obtained from (4.11) by replacing $\varphi_{\mathrm{B\tilde{G}L}}$ by $\varphi_{\tilde{\mathrm{L}}}$ and letting $\epsilon^{(0)} \to 1$, as a comparison of (3.24) and (3.25) shows.

Let us turn now to the discussion of some general properties of the collision brackets. In the first place we note the symmetry property:

$$[h, k]_{\mathrm{i}} = [k, h]_{\mathrm{i}}, \quad \mathrm{i} \neq \tilde{\mathrm{B}} \quad (4.12)$$

for arbitrary functions h and k. The proof trivially follows on inspection. In the case $\mathrm{i} = \tilde{\mathrm{B}}$ the symmetry is violated. In fact, in (4.10) the relative velocities u and w, which appear in the arguments of h and k, respectively, have a different norm, as is apparent from (3.18). However, we shall show in a following paper[11]) that in leading order with respect to the plasma parameter ϵ the symmetry of the modified Boltzmann bracket is restored.

Secondly, the collision brackets vanish if at least one of the functions h and k is a collision invariant Ψ:

$$[h, \Psi]_{\mathrm{i}} = 0, \quad (4.13\mathrm{a})$$

$$[\Psi, k]_{\mathrm{i}} = 0, \quad (4.13\mathrm{b})$$

for all i. The first line is a direct consequence of (3.28) and (4.1). For $\mathrm{i} \neq \tilde{\mathrm{B}}$ the second line follows immediately from (4.12). The case $\mathrm{i} = \tilde{\mathrm{B}}$ will be discussed separately with the help of (4.6). For $h(v) = 1$ or v (4.13b) is trivial. When we put $h(v) = v^2$ the integrand of (4.6) contains the factor $u \cdot \nabla\varphi_{\tilde{\mathrm{B}}}$. After a partial integration with respect to r and the use of (4.8) we arrive at an integral over u which vanishes by Gauss's theorem.

The properties discussed in the preceding paragraphs suggest that a straightforward application of the usual methods[12]) to solve the linear kinetic equations (3.1) with (3.4) and (3.12) is feasible. The transport coefficients then follow by the truncation of an infinite set of linear equations that contain collision brackets of Sonine polynomials as coefficients. In the Boltzmann theory of dilute neutral gases the convergence of this Chapman–Cowling procedure is established by making use of a variational principle that is based on the positive-definiteness of the collision brackets. In the present case, however, the definiteness of the sign of the collision brackets associated with the collision terms in (3.1) is not readily established. The convergence of the procedure can therefore only be verified empirically.

The Chapman–Enskog procedure yields for the heat conductivity in lowest Chapman–Cowling order[12]):

$$\lambda = 1/\Lambda^{11}, \tag{4.14}$$

where Λ^{pq} is proportional to a collision bracket of Sonine polynomials:

$$\Lambda^{pq} = \frac{4m^2\beta^2}{75k_B}[S_p^{(3/2)}\{\tfrac{1}{2}\beta m(\boldsymbol{v}-\boldsymbol{V})^2\}(\boldsymbol{v}-\boldsymbol{V})\cdot, S_q^{(3/2)}\{\tfrac{1}{2}\beta m(\boldsymbol{v}-\boldsymbol{V})^2\}(\boldsymbol{v}-\boldsymbol{V})]. \tag{4.15}$$

For the viscosity one finds similarly

$$\eta = 1/H^{00}, \tag{4.16}$$

with

$$\begin{aligned} H^{pq} = \tfrac{1}{10}m^2\beta^3[&S_p^{(5/2)}\{\tfrac{1}{2}\beta m(\boldsymbol{v}-\boldsymbol{V})^2\}(\boldsymbol{v}-\boldsymbol{V})^0(\boldsymbol{v}-\boldsymbol{V}):,\\ &S_q^{(5/2)}\{\tfrac{1}{2}\beta m(\boldsymbol{v}-\boldsymbol{V})^2\}(\boldsymbol{v}-\boldsymbol{V})^0(\boldsymbol{v}-\boldsymbol{V})], \end{aligned} \tag{4.17}$$

where the symbol ^{0} denotes the traceless part of a tensor.

In the remaining part of this paper we shall evaluate the lowest-order collision brackets of Sonine polynomials. In a following paper the higher-order collision brackets, as given by (4.15) and (4.17) for general p and q, will be studied by making use of their generating functions.

5. Transport coefficients in lowest Chapman–Cowling order

In this section we calculate the quantities Λ^{11} and H^{00} which give the inverse heat-conduction coefficient and the inverse viscosity, respectively, in lowest order of approximation. In particular, the leading terms in an asymptotic expansion for small plasma parameter ϵ will be derived.

Firstly we discuss the Boltzmann-like contributions to the composite collision brackets. In the ordinary Boltzmann case B′ the brackets follow by employing (4.3) in (4.15) and (4.17). A well-known calculation yields[12])

$$\Lambda_{\mathrm{B}'}^{11} = \frac{32m\beta}{75k_{\mathrm{B}}}\,\Omega^{(2,2)}, \tag{5.1}$$

$$H_{\mathrm{B}'}^{00} = \frac{8\beta}{5}\,\Omega^{(2,2)} = \frac{15k_{\mathrm{B}}}{4m}\,\Lambda_{\mathrm{B}'}^{11}, \tag{5.2}$$

where the Ω-integrals are defined as usual:

$$\Omega^{(l,r)} = (\pi m\beta)^{-1/2} \int_0^\infty \mathrm{d}\bar{u}\; \mathrm{e}^{-\bar{u}^2}\bar{u}^{2r+3} \int \mathrm{d}\Omega\, \frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}\,(1-\cos^l\theta), \tag{5.3}$$

with $\bar{u} = \frac{1}{2}(m\beta)^{1/2}|\boldsymbol{u}|$. The cross-section $\mathrm{d}\sigma/\mathrm{d}\Omega$ corresponds to the Debye potential (2.11). For small ϵ the integral $\Omega^{(2,2)}$ has the asymptotic expression[13])

$$\Omega^{(2,2)} \simeq \frac{e^4\beta^{3/2}}{16\pi^{3/2}m^{1/2}}\,(-\log\epsilon - 2\gamma + 2\log 2), \tag{5.4}$$

where $\gamma = 0.5772\ldots$ is Euler's constant and $\epsilon = r_{\mathrm{L}}/r_{\mathrm{D}} = e^3 n^{1/2}\beta^{3/2}/(4\pi)$ the plasma parameter. From (5.1) and (5.4) we conclude:

$$\Lambda_{\mathrm{B}'}^{11} \simeq \lambda_0^{-1}(-\log\epsilon - 2\gamma + 2\log 2), \tag{5.5}$$

while (5.2) and (5.4) yield:

$$H_{\mathrm{B}'}^{00} \simeq \eta_0^{-1}(-\log\epsilon - 2\gamma + 2\log 2). \tag{5.6}$$

In (5.5) and (5.6) we introduced the abbreviations:

$$\lambda_{\mathrm{D}} = \frac{75\pi^{3/2}k_{\mathrm{B}}}{2e^4 m^{1/2}\beta^{5/2}}, \quad \eta_0 = \frac{10\pi^{3/2}m^{1/2}}{e^4\beta^{5/2}}. \tag{5.7}$$

In the modified Boltzmann case $\tilde{\mathrm{B}}$ we infer from (4.10), (4.13) and (4.15), with $S_1^{(3/2)}(x) = 5/2 - x$:

$$\begin{aligned} \Lambda_{\tilde{\mathrm{B}}}^{11} = -\frac{2m^4\beta^4}{75k_{\mathrm{B}}} \int & \mathrm{d}\boldsymbol{U}\,\mathrm{d}\boldsymbol{u}\,\mathrm{d}\boldsymbol{r}\,\bar{f}_{\mathrm{M}}(\boldsymbol{U}+\tfrac{1}{2}\boldsymbol{u})\bar{f}_{\mathrm{M}}(\boldsymbol{U}-\tfrac{1}{2}\boldsymbol{u})\exp(-k_{\mathrm{D}}r - r_{\mathrm{L}}/r) \\ & \times k_{\mathrm{D}}\boldsymbol{u}\cdot\hat{\boldsymbol{r}}[(\boldsymbol{U}-\boldsymbol{V})^2(\boldsymbol{U}-\boldsymbol{V}) + \tfrac{1}{2}\boldsymbol{u}\cdot(\boldsymbol{U}-\boldsymbol{V})\boldsymbol{u} + \tfrac{1}{4}u^2(\boldsymbol{U}-\boldsymbol{V})] \\ & \cdot[(\boldsymbol{U}-\boldsymbol{V})^2(\boldsymbol{U}-\boldsymbol{V}) + \tfrac{1}{2}\boldsymbol{w}\cdot(\boldsymbol{U}-\boldsymbol{V})\boldsymbol{w} + \tfrac{1}{4}w^2(\boldsymbol{U}-\boldsymbol{V})]. \end{aligned} \tag{5.8}$$

The contraction and the $\boldsymbol{U}$-integration are straightforwardly performed; the two factors between square brackets then reduce to a linear combination of u^2, w^2 and $(\boldsymbol{u}\cdot\boldsymbol{w})^2$. Subsequently, we note that under the reflection $\boldsymbol{r}\to-\boldsymbol{r}$ the factor $\boldsymbol{u}\cdot\hat{\boldsymbol{r}}$ changes sign, while u^2 and w^2 are invariant. Hence, the only term that yields a non-vanishing contribution is proportional to $(\boldsymbol{u}\cdot\boldsymbol{w})^2$:

$$\Lambda_{\tilde{\mathrm{B}}}^{11} = -\frac{m^{9/2}\beta^{9/2}}{2400\pi^{3/2}k_{\mathrm{B}}} \int \mathrm{d}\boldsymbol{u}\,\mathrm{d}\boldsymbol{r}\,k_{\mathrm{D}}\boldsymbol{u}\cdot\hat{\boldsymbol{r}}\,\exp(-\tfrac{1}{4}\beta m u^2 - k_{\mathrm{D}}r - r_{\mathrm{L}}/r)(\boldsymbol{u}\cdot\boldsymbol{w})^2. \tag{5.9}$$

The integration variables in this expression are $|\boldsymbol{u}|$, $|\boldsymbol{r}|$ and $\zeta = \hat{\boldsymbol{u}} \cdot \hat{\boldsymbol{r}}$. Instead of the former two one may take the energy E (3.15a) and the potential $\varphi = \beta^{-1} r_{\mathrm{L}}/r$ or rather[10])

$$\xi = \beta E, \quad \eta = \varphi/E. \tag{5.10}$$

Since $E \geqslant \varphi \geqslant 0$ we have $\xi \geqslant 0$ and $0 \leqslant \eta \leqslant 1$. All constituents of the integrand in (5.9) can be expressed now in terms of ξ, η and ζ. In particular we find:

$$\boldsymbol{u} \cdot \boldsymbol{w} = \frac{e^4}{4\pi^2 m\beta} \frac{\xi(1-\eta)^{1/2}}{N^2\eta^2} [2 - \eta + 2\zeta(1-\eta)^{1/2}][2(1-\eta)^{1/2} - \zeta(2-\eta)], \tag{5.11}$$

where N^2 is given by

$$N^2 = \frac{e^4}{16\pi^2\eta^2}[(2-\eta)^2 - 4\zeta^2(1-\eta)]. \tag{5.12}$$

The expression (5.9) becomes in this way

$$\begin{aligned} \Lambda_{\mathrm{B}}^{11} = &- \lambda_0^{-1}\epsilon \int_0^\infty \mathrm{d}\xi \int_0^1 \mathrm{d}\eta \int_{-1}^1 \mathrm{d}\zeta \frac{(1-\eta)^2}{\eta^4} \zeta \\ &\times \exp\left(-\frac{\epsilon}{\xi\eta} - \xi\right)\left[\frac{2(1-\eta)^{1/2} - \zeta(2-\eta)}{2 - \eta - 2\zeta(1-\eta)^{1/2}}\right]^2. \end{aligned} \tag{5.13}$$

When we put the expression between square brackets equal to a new variable ζ' and eliminate ζ in favour of ζ' we obtain upon integration over ζ':

$$\Lambda_{\mathrm{B}}^{11} = -\lambda_0^{-1}\epsilon \int_0^\infty \mathrm{d}\xi \int_0^1 \frac{\mathrm{d}\eta}{\eta^2} \exp\left(-\frac{\epsilon}{\xi\eta} - \xi\right) F(\eta), \tag{5.14}$$

where $F(\eta)$ is defined as

$$F(\eta) = \tfrac{1}{2}(1 - \eta + \tfrac{3}{8}\eta^2) \log\left[\frac{2 - \eta - 2(1-\eta)^{1/2}}{2 - \eta + 2(1-\eta)^{1/2}}\right] + \tfrac{3}{4}(2-\eta)(1-\eta)^{1/2}. \tag{5.15}$$

An asymptotic expansion of (5.14) with respect to ϵ can be obtained by making a decomposition of the integration domain into parts in which different estimates for the integrand hold. This procedure is discussed in detail in appendix A. From (A.11) we infer

$$\Lambda_{\mathrm{B}}^{11} \simeq \lambda_0^{-1}(-\log\epsilon - 2\gamma + 2\log 2 - \tfrac{1}{2}). \tag{5.16}$$

For H_{B}^{00} an expression analogous to (5.8) follows from (4.17) with (4.10). After performing the contractions and the U-integration we obtain an expression which is proportional to (5.9) in the same way as we had in (5.2) for the B′ case. Using (5.16) we arrive at

$$H_{\tilde{\mathrm{B}}}^{00} \simeq \eta_0^{-1}(-\log\epsilon - 2\gamma + 2\log 2 - \tfrac{1}{2}). \tag{5.17}$$

Comparing the results (5.5), (5.6) for the B′ case with (5.16) and (5.17) for the $\tilde{\mathrm{B}}$ case we note that the asymptotic expansions are equal in their leading $\log\epsilon$ terms, but different in the contributions that are independent of ϵ. In both cases Λ^{11} and H^{00} satisfy the Eucken relation: $H^{00}/\Lambda^{11} = \frac{15}{4}k_\mathrm{B}/m$.

Let us consider now the BGL-like contributions. The ordinary BGL case has been treated before[5,6,14]). We briefly recall the essential steps in the calculation, because the integral expressions for the collision brackets in the modified B$\tilde{\mathrm{G}}$L case will be derived in an analogous way. From (4.15) with (4.4) we obtain by performing the differentiations with respect to $\boldsymbol{v}$ and $\boldsymbol{v}'$, carrying out the contraction and introducing centre-of-mass and relative velocities:

$$\Lambda_{\mathrm{BGL}}^{11} = \frac{e^4 m^2\beta^4}{300\pi^2 k_\mathrm{B}} \int \mathrm{d}\boldsymbol{U}\,\mathrm{d}\boldsymbol{u}\,\mathrm{d}\boldsymbol{q}\,\delta(\boldsymbol{q}\cdot\boldsymbol{u}) \frac{1}{q^4|\epsilon^{(0)}(\boldsymbol{q},\boldsymbol{q}\cdot\boldsymbol{U})|^2} \times \bar{f}_\mathrm{M}(\boldsymbol{U}+\tfrac{1}{2}\boldsymbol{u})\bar{f}_\mathrm{M}(\boldsymbol{U}-\tfrac{1}{2}\boldsymbol{u})\{[\boldsymbol{u}\cdot(\boldsymbol{U}-\boldsymbol{V})]^2 q^2 + u^2[(\boldsymbol{U}-\boldsymbol{V})\cdot\boldsymbol{q}]^2\}. \tag{5.18}$$

Owing to the presence of the δ-function it is convenient to make a decomposition of $\boldsymbol{u}$ and $\boldsymbol{U}-\boldsymbol{V}$ into components $\perp$ and $\|$ with respect to $\boldsymbol{q}$. The integrals over the $\perp$-components are of Gaussian type, while the integral over $\boldsymbol{u}_\|$ is easily performed with the help of the δ-function. At this stage one introduces the dimensionless variables $\zeta^2 = \frac{1}{2}\beta m(\boldsymbol{U}-\boldsymbol{V})_\|^2$ and $\eta = q/k_\mathrm{D}$. In terms of these variables the permeability $\epsilon^{(0)}$ is

$$\epsilon^{(0)} = 1 + \eta^{-2}[F_1(\zeta) + \mathrm{i}F_2(\zeta)], \tag{5.19}$$

where

$$F_1(\zeta) = 1 - 2\zeta\,\mathrm{e}^{-\zeta^2}\psi(\zeta), \quad F_2(\zeta) = \sqrt{\pi}\,\zeta\,\mathrm{e}^{-\zeta^2}, \tag{5.20}$$

with $\psi(\zeta) = \int_0^\zeta \mathrm{d}t\,\exp(t^2)$. Since the η integration is divergent for $\eta\to\infty$ a cut-off at an arbitrary upper limit $\Lambda(\gg 1)$ is necessary. (The same manipulation will be performed in the Landau contribution; consequently the composite collision bracket does not depend on Λ, as we shall see.) In this way one obtains

$$\Lambda_{\mathrm{BGL}}^{11} = \left(\frac{2}{\pi}\right)^{1/2}\lambda_0^{-1}\int_0^\Lambda \mathrm{d}\eta \int_0^\infty \mathrm{d}\zeta\,\frac{\eta^3(1+4\zeta^2)\,\mathrm{e}^{-2\zeta^2}}{[F_1(\zeta)+\eta^2]^2 + F_2(\zeta)^2}. \tag{5.21}$$

The η-integration is elementary. The leading contribution for $\Lambda\to\infty$ is found to be

$$\Lambda^{11}_{\mathrm{BGL}} \simeq \lambda_0^{-1}(\log \Lambda - I_\Lambda). \tag{5.22}$$

The constant λ_0 has been given in (5.7), while I_Λ is the finite integral

$$I_\Lambda = \left(\frac{2}{\pi}\right)^{1/2} \int_0^\infty \mathrm{d}\zeta\, \mathrm{e}^{-2\zeta^2}(1+4\zeta^2)G(\zeta), \tag{5.23}$$

where $G(\zeta)$ is the function

$$G(\zeta) = \tfrac{1}{4}\log[F_1(\zeta)^2 + F_2(\zeta)^2] + \frac{F_1(\zeta)}{2F_2(\zeta)}\left\{\frac{\pi}{2} - \operatorname{arctg}\left[\frac{F_1(\zeta)}{F_2(\zeta)}\right]\right\}. \tag{5.24}$$

The corresponding expression for H^{00}_{BGL} is obtained from (5.18) by replacing the factor in curly brackets by q^2u^2 (and inserting the correct multiplicative constant) or from (5.21) by omitting the factor $1+4\zeta^2$ in the integrand. Then we find on a par with (5.22)

$$H^{00}_{\mathrm{BGL}} \simeq \eta_0^{-1}(\log \Lambda - I_H), \tag{5.25}$$

with η_0 given by (5.7) and I_H by

$$I_H = 2\left(\frac{2}{\pi}\right)^{1/2} \int_0^\infty \mathrm{d}\zeta\, \mathrm{e}^{-2\zeta^2}G(\zeta). \tag{5.26}$$

By comparing (4.4) with (4.11) we infer that $\Lambda^{11}_{\mathrm{B\hat{G}L}}$ is obtained from (5.18) by replacing a factor e^2q^{-2} by $\hat{\varphi}_{\mathrm{B\hat{G}L}}(q)$. Correspondingly we must insert a factor $e^{-2}\eta^2k_{\mathrm{D}}^2\hat{\varphi}_{\mathrm{B\hat{G}L}}(\eta k_{\mathrm{D}})$ in (5.21). As a consequence the cut-off in (5.21) becomes superfluous. When we use the explicit representation (3.23) for $\hat{\varphi}_{\mathrm{B\hat{G}L}}$, with the dimensionless variable $\xi = r/r_{\mathrm{L}}$ instead of r, we find the following integral expression for $\Lambda^{11}_{\mathrm{B\hat{G}L}}$

$$\Lambda^{11}_{\mathrm{B\hat{G}L}} = \left(\frac{2}{\pi}\right)^{1/2}\lambda_0^{-1}\epsilon \int_0^\infty \mathrm{d}\xi \int_0^\infty \mathrm{d}\eta \int_0^\infty \mathrm{d}\zeta\, \xi(1-\mathrm{e}^{-1/\xi})\frac{\eta^4 j_1(\epsilon\eta\xi)(1+4\zeta^2)\,\mathrm{e}^{-2\zeta^2}}{[F_1(\zeta)+\eta^2]^2 + F_2(\zeta)^2}. \tag{5.27}$$

An asymptotic expansion for the (ξ, η)-integral will be derived in appendix B. Substituting (B.14) into (5.27) we obtain:

$$\Lambda^{11}_{\mathrm{B\hat{G}L}} \simeq \lambda_0^{-1}(-\log\epsilon - 2\gamma + 2 - I_\Lambda), \tag{5.28}$$

where I_Λ has been given in (5.23).

We use the same argument as in the ordinary BGL case to derive an asymptotic expression for H^{00}_{BGL} from (5.27), thus arriving at:

$$H^{00}_{\mathrm{B\tilde{G}L}} \simeq \eta_0^{-1}(-\log\epsilon - 2\gamma + 2 - I_H), \tag{5.29}$$

where I_H has been defined in (5.26).

Comparing the results (5.22), (5.25) for the BGL case with (5.28), (5.29) for the B$\tilde{\mathrm{G}}$L case we note that in the B$\tilde{\mathrm{G}}$L case the plasma parameter ϵ enters into the expressions for the brackets in a natural way, whereas in the BGL case the corresponding expressions depend on the cut-off parameter Λ.

Finally we consider the Landau-like contributions to the collision brackets. In the ordinary Landau case $\Lambda^{11}_{\mathrm{L'}}$ is obtained from $\Lambda^{11}_{\mathrm{BGL}}$, according to the remark following (4.4), by inserting a factor $\eta^4(1+\eta^2)^{-2}$ and putting $F_1 = 0 = F_2$ in the integrand of (5.21). This procedure yields the following expression:

$$\Lambda^{11}_{\mathrm{L'}} = \left(\frac{2}{\pi}\right)^{1/2} \lambda_0^{-1} \int_0^{\Lambda} \mathrm{d}\eta \int_0^{\infty} \mathrm{d}\zeta \, \frac{\eta^3(1+4\zeta^2)\,\mathrm{e}^{-2\zeta^2}}{(1+\eta^2)^2}. \tag{5.30}$$

The leading contributions for $\Lambda \to \infty$ are found to be

$$\Lambda^{11}_{\mathrm{L'}} \simeq \lambda_0^{-1}(\log\Lambda - \tfrac{1}{2}). \tag{5.31}$$

Similarly, we have, by omitting the term $4\zeta^2$ in (5.30) and changing the multiplicative constant:

$$H^{00}_{\mathrm{L'}} \simeq \eta_0^{-1}(\log\Lambda - \tfrac{1}{2}). \tag{5.32}$$

In the modified $\tilde{\mathrm{L}}$ case $\Lambda^{11}_{\tilde{\mathrm{L}}}$ is derived from the B$\tilde{\mathrm{G}}$L case by putting $\epsilon^{(0)}$ equal to 1 and replacing $\hat{\varphi}_{\mathrm{B\tilde{G}L}}$ by $\hat{\varphi}_{\tilde{\mathrm{L}}}$, which is given in (3.27). Correspondingly, we may put $F_1 = 0 = F_2$ and insert a factor $\exp(-\epsilon\xi)$ in (5.27). Hence we have:

$$\Lambda^{11}_{\tilde{\mathrm{L}}} = \left(\frac{2}{\pi}\right)^{1/2} \lambda_0^{-1}\epsilon \int_0^{\infty} \mathrm{d}\xi \int_0^{\infty} \mathrm{d}\eta \int_0^{\infty} \mathrm{d}\zeta \, \xi(1-\mathrm{e}^{-1/\xi})\,\mathrm{e}^{-\epsilon\xi} j_1(\epsilon\eta\xi)(1+4\zeta^2)\,\mathrm{e}^{-2\zeta^2}. \tag{5.33}$$

The ζ-integration is elementary while the (ξ,η)-integration is discussed in appendix B. The asymptotic expansion for $\Lambda^{11}_{\tilde{\mathrm{L}}}$ is found from (B.19) to be

$$\Lambda^{11}_{\tilde{\mathrm{L}}} \simeq \lambda_0^{-1}(-\log\epsilon - 2\gamma + 1). \tag{5.34}$$

Similarly we find:

$$H^{00}_{\tilde{\mathrm{L}}} \simeq \eta_0^{-1}(-\log\epsilon - 2\gamma + 1). \tag{5.35}$$

By comparison of the results (5.31), (5.32) and (5.34), (5.35) it is seen that, just as in the BGL-like contributions, there is a logarithmic dependence on the cut-off parameter in the L′ case, whereas in the $\tilde{\mathrm{L}}$ case the plasma parameter ϵ

enters into the logarithm. Both the L′ and $\tilde{\mathrm{L}}$ contributions satisfy the Eucken relation.

We can collect now the obtained results and calculate the transport coefficients in lowest order as they follow from the two alternative composite collision terms. From (4.2) with (5.5), (5.22) and (5.31) we have

$$\Lambda_{\mathrm{P}}^{11} \simeq \lambda_0^{-1}(-\log \epsilon - 2\gamma + 2\log 2 + \tfrac{1}{2} - I_\Lambda). \tag{5.36}$$

On the other hand, from (5.16), (5.28) and (5.34) one arrives at the very same expression for $\Lambda_{\tilde{\mathrm{P}}}^{11}$, so that we may write

$$\Lambda_{\mathrm{P}}^{11} \simeq \Lambda_{\tilde{\mathrm{P}}}^{11}, \tag{5.37}$$

for $\epsilon \to 0$. Similarly, we have from both (5.6), (5.25), (5.32) on the one hand and (5.17), (5.29), (5.35) on the other hand

$$H_{\mathrm{P}}^{00} \simeq \eta_0^{-1}(-\log \epsilon - 2\gamma + 2\log 2 + \tfrac{1}{2} - I_H) \simeq H_{\tilde{\mathrm{P}}}^{00}. \tag{5.38}$$

From (5.37), (5.38) with (4.14), (4.16) we conclude that in lowest Chapman–Cowling approximation the heat conductivities and the viscosities following from the alternative convergent kinetic equations (3.1a) and (3.1b) are in fact identical for small values of the plasma parameter ϵ:

$$\lambda_{\mathrm{P}} \simeq \lambda_{\tilde{\mathrm{P}}}, \quad \eta_{\mathrm{P}} \simeq \eta_{\tilde{\mathrm{P}}}. \tag{5.39}$$

Hence in this order of approximation the two kinetic equations are in fact equivalent in so far as the ensuing transport coefficients are concerned. In the next paper[11]) we shall generalize this statement by including higher orders of the Chapman–Cowling approximation scheme.

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Appendix A

Approximate evaluation of the modified Boltzmann ($\tilde{\mathrm{B}}$) *contribution to the inverse heat conductivity*

In section 5 we derived the expression (5.14) for the contribution $\Lambda_{\tilde{\mathrm{B}}}^{11}$ to the inverse heat conductivity. It contains the integral

$$J(\epsilon) = \int_0^\infty \mathrm{d}\xi \int_0^1 \frac{\mathrm{d}\eta}{\eta^2} \exp\left(-\frac{\epsilon}{\xi\eta} - \xi\right) F(\eta), \tag{A.1}$$

with the function

$$F(\eta) = \tfrac{1}{2}(1 - \eta + \tfrac{3}{8}\eta^2) \log\left[\frac{2 - \eta - 2(1-\eta)^{1/2}}{2 - \eta + 2(1-\eta)^{1/2}}\right] + \tfrac{3}{4}(2-\eta)(1-\eta)^{1/2}. \tag{A.2}$$

The function $F(\eta)$ is regular on $(0, 1]$. For $\eta \downarrow 0$ one may write:

$$F(\eta) \simeq \log \eta - 2 \log 2 + \tfrac{3}{2} + \mathcal{O}(\eta). \tag{A.3}$$

The dominant contribution to $J(\epsilon)$ for small ϵ follows by inserting (A.3) instead of (A.2) into (A.1). The error introduced in this way can be proven to be at most of order ϵ in comparison to the leading term. With the variable $\eta' = \epsilon/(\eta\xi)$ we obtain

$$J(\epsilon) \simeq \epsilon^{-1} \int_0^\infty \mathrm{d}\xi \int_{\epsilon/\xi}^\infty \mathrm{d}\eta' \xi \, \mathrm{e}^{-\xi-\eta'} (-\log \xi - \log \eta' + \log \epsilon - 2 \log 2 + \tfrac{3}{2}). \tag{A.4}$$

The integral over η' can trivially be carried out for all terms between the brackets except $\log \eta'$; if a partial integration is performed in the latter term we get

$$J(\epsilon) \simeq \epsilon^{-1} \int_0^\infty \mathrm{d}\xi \, \xi \, \mathrm{e}^{-\xi}\left[\mathrm{e}^{-\epsilon/\xi}(-2 \log 2 + \tfrac{3}{2}) - \int_{\epsilon/\xi}^\infty \frac{\mathrm{d}\eta'}{\eta'} \mathrm{e}^{-\eta'}\right]. \tag{A.5}$$

Using the identity $\xi \, \mathrm{e}^{-\xi} = -(\mathrm{d}/\mathrm{d}\xi)[(\xi + 1)\,\mathrm{e}^{-\xi}]$ we can eliminate the η'-integral by a partial integration with respect to ξ:

$$J(\epsilon) \simeq \epsilon^{-1} \int_0^\infty \mathrm{d}\xi \, \mathrm{e}^{-\xi-\epsilon/\xi}[\xi(-2 \log 2 + \tfrac{3}{2}) - 1 - \xi^{-1}]. \tag{A.6}$$

The asymptotic form of the integral for small ϵ is easily obtained now by approximating the exponential function. To that end we split the integration domain into two intervals[10]), viz. $I_1 = [0, \epsilon^{1/2}]$ and $I_2 = [\epsilon^{1/2}, \infty)$, with contributions J_1 and J_2, respectively. In the first interval we expand the function $\exp(-\xi)$ and introduce the integration variable $\xi' = \epsilon/\xi$; the contribution J_1 then becomes

$$J_1(\epsilon) \simeq -\epsilon^{-1} \int_{\epsilon^{1/2}}^\infty \frac{\mathrm{d}\xi'}{\xi'} \mathrm{e}^{-\xi'}. \tag{A.7}$$

Since one has[15]) for small $\delta > 0$

$$\int_\delta^\infty \frac{dt}{t} e^{-t} = \int_\delta^\infty \frac{dt}{t}\left(e^{-t} - \frac{1}{1+t}\right) + \int_\delta^\infty \frac{dt}{t(1+t)}$$
$$= -\log\delta - \gamma + \mathcal{O}(\delta), \tag{A.8}$$

with γ Euler's constant, we find for J_1

$$J_1(\epsilon) \simeq \epsilon^{-1}(\tfrac{1}{2}\log\epsilon + \gamma). \tag{A.9}$$

In the second interval I_2 the function $\exp(-\epsilon/\xi)$ is expanded; then one obtains with (A.8)

$$J_2(\epsilon) \simeq \epsilon^{-1}(\tfrac{1}{2}\log\epsilon + \gamma - 2\log 2 + \tfrac{1}{2}). \tag{A.10}$$

Adding (A.9) and (A.10) we arrive at the asymptotic expression of $J(\epsilon)$ for small ϵ:

$$J(\epsilon) \simeq \epsilon^{-1}(\log\epsilon + 2\gamma - 2\log 2 + \tfrac{1}{2}). \tag{A.11}$$

An examination of the correction terms in the above calculation yields a more precise result

$$J(\epsilon) = \epsilon^{-1}[\log\epsilon + 2\gamma - 2\log 2 + \tfrac{1}{2} + \mathcal{O}(\epsilon^{1/2-\delta})] \tag{A.12}$$

for arbitrarily small positive δ.

Appendix B

Approximate evaluation of the modified Balescu–Guernsey–Lenard ($\mathrm{B\tilde{G}L}$) *and the modified Landau* ($\tilde{\mathrm{L}}$) *contributions to the inverse heat conductivity*

The contribution (5.27) to the inverse heat conductivity contains the double integral

$$K(\zeta, \epsilon) = \int_0^\infty d\xi \int_0^\infty d\eta\, \xi(1 - e^{-1/\xi}) \frac{\eta^4 j_1(\epsilon\eta\xi)}{(F_1 + \eta^2)^2 + F_2^2}. \tag{B.1}$$

The denominator is positive for all η and ζ, since the functions $F_i(\zeta)$ have the properties $F_1(0) = 1$ and $F_2(\zeta) > 0$ for $\zeta > 0$.

The asymptotic form of $K(\zeta, \epsilon)$ for small ϵ can be obtained by using the following three approximate expressions for the constituents of the integrand:

$$1 - e^{-1/\xi} \simeq \xi^{-1}, \quad \xi \gg 1, \tag{B.2a}$$

$$\frac{\eta^4}{(F_1 + \eta^2)^2 + F_2^2} \simeq 1, \quad \eta \gg 1, \tag{B.2b}$$

$$j_l(z) \simeq \frac{z^l}{(2l+1)!!}, \quad z \ll 1. \tag{B.2c}$$

In the following, the integration domain will be decomposed such that in each subdomain at least one of these relations may be employed to simplify the integrand of (B.1).

Let us start by splitting the η-integration domain into the intervals $I_{\eta 1} = [\epsilon^{-1/4}, \infty)$ and $I_{\eta 2} = [0, \epsilon^{-1/4}]$. The contribution K_1 of the first interval to K follows by applying (B.2b) and performing the integration over η:

$$K_1(\zeta, \epsilon) \simeq \epsilon^{-1} \int_0^\infty d\xi (1 - e^{-1/\xi}) j_0(\epsilon^{3/4}\xi), \tag{B.3}$$

since $j_1(z) = -\,dj_0(z)/dz$.

To proceed further we split the ξ-integration domain in turn into the intervals $I_{\xi 1} = [\epsilon^{-1/4}, \infty)$ and $I_{\xi 2} = [0, \epsilon^{-1/4}]$, and evaluate the corresponding contributions K_{11} and K_{12}. For K_{11} we get with the help of (B.2a)

$$K_{11}(\zeta, \epsilon) \simeq \epsilon^{-1} \int_{\epsilon^{-1/4}}^\infty \frac{d\xi}{\xi} j_0(\epsilon^{3/4}\xi)$$

$$= \epsilon^{-1} \int_{\epsilon^{1/2}}^\infty \frac{d\xi'}{\xi'} j_0(\xi'), \tag{B.4}$$

where we introduced the variable $\xi' = \epsilon^{3/4}\xi$. We now employ the auxiliary relation[15]) valid for small positive δ:

$$\int_\delta^\infty \frac{dt}{t} j_0(t) = \int_\delta^\infty \frac{dt}{t}\left(\frac{\sin t}{t} - \frac{1}{1+t}\right) + \int_\delta^\infty \frac{dt}{t(1+t)}$$

$$\simeq -\log\delta - \gamma + 1 + \mathcal{O}(\delta), \tag{B.5}$$

with γ Euler's constant; as a consequence (B.4) becomes

$$K_{11}(\zeta, \epsilon) \simeq \epsilon^{-1}(-\tfrac{1}{2}\log\epsilon - \gamma + 1). \tag{B.6}$$

The second part K_{12} of K_1 (B.3) can be simplified by means of (B.2c):

$$K_{12}(\zeta, \epsilon) \simeq \epsilon^{-1} \int_0^{\epsilon^{-1/4}} d\xi (1 - e^{-1/\xi})$$

$$= \epsilon^{-1} \int_{\epsilon^{1/4}}^{\infty} \frac{d\xi'}{\xi'^2} (1 - e^{-\xi'}), \tag{B.7}$$

with $\xi' = 1/\xi$. Using the asymptotic relation for small δ:

$$\int_\delta^\infty \frac{dt}{t^2} (1 - e^{-t}) = -\frac{e^{-\delta} - 1}{\delta} + \int_\delta^\infty \frac{dt}{t} e^{-t}$$

$$\simeq -\log \delta - \gamma + 1 + \mathcal{O}(\delta), \tag{B.8}$$

which follows with the help of (A.8), we get

$$K_{12}(\zeta, \epsilon) \simeq \epsilon^{-1}(-\tfrac{1}{4} \log \epsilon - \gamma + 1). \tag{B.9}$$

To obtain the asymptotic form of the second part K_2 of (B.1) we again split the ξ-integration domain, this time into the intervals $I'_{\xi 1} = [\epsilon^{-1/2}, \infty)$ and $I'_{\xi 2} = [0, \epsilon^{-1/2}]$. The contribution K_{21} from the first interval becomes upon using the approximation (B.2a) and performing the ξ integral

$$K_{21}(\zeta, \epsilon) \simeq \epsilon^{-1} \int_0^{\epsilon^{-1/4}} d\eta \frac{\eta^3 j_0(\epsilon^{1/2}\eta)}{(F_1 + \eta^2)^2 + F_2^2}. \tag{B.10}$$

Employing moreover the approximation (B.2c) for j_0 we are left with an elementary integral, which yields

$$K_{21}(\zeta, \epsilon) \simeq \epsilon^{-1}[-\tfrac{1}{4} \log \epsilon - G(\zeta)], \tag{B.11}$$

with $G(\zeta)$ from (5.24). Finally, we consider the contribution K_{22} from the interval $I'_{\xi 2}$. Since we may use now the approximation (B.2c) at once, K_{22} factorizes into a product of two integrals each of which can be estimated easily:

$$K_{22}(\zeta, \epsilon) \simeq \tfrac{1}{3}\epsilon \int_0^{\epsilon^{-1/2}} d\xi\, \xi^2 (1 - e^{-1/\xi}) \int_0^{\epsilon^{-1/4}} d\eta \frac{\eta^5}{(F_1 + \eta^2)^2 + F_2^2}$$

$$= \epsilon \mathcal{O}(\epsilon^{-1}) \mathcal{O}(\epsilon^{-1/2}) = \epsilon^{-1} \mathcal{O}(\epsilon^{1/2}). \tag{B.12}$$

Adding (B.6), (B.9), (B.11) and (B.12) we arrive at the asymptotic form of the complete integral (B.1):

$$K(\zeta, \epsilon) \simeq \epsilon^{-1}[-\log \epsilon - 2\gamma + 2 - G(\zeta)]. \tag{B.13}$$

A detailed examination of the remainder terms in the various contributions to

K leads to the more precise statement:

$$K(\zeta, \epsilon) = \epsilon^{-1}[-\log \epsilon - 2\gamma + 2 - G(\zeta) + \mathcal{O}(\epsilon^{1/2-\delta})], \tag{B.14}$$

for arbitrarily small positive exponent δ.

In the main text we also need an asymptotic expression for the integral $K^0(\epsilon)$, which follows from (B.1) by putting F_1 and F_2 equal to 0 and inserting a factor $\exp(-\epsilon\xi)$ (see (5.33)):

$$K^0(\epsilon) = \int_0^\infty d\xi \int_0^\infty d\eta\, \xi(1 - e^{-1/\xi})\, e^{-\epsilon\xi} j_1(\epsilon\eta\xi). \tag{B.15}$$

By introducing the variable $\eta' = \epsilon\eta\xi$ and performing the η'-integral we get

$$K^0(\epsilon) = \epsilon^{-1} \int_0^\infty d\xi(1 - e^{-1/\xi})\, e^{-\epsilon\xi}. \tag{B.16}$$

As before we split the ξ-integration domain into the intervals $I'_{\xi 1}$ and $I'_{\xi 2}$. From the first interval we obtain with the use of (B.2a):

$$\begin{aligned} K_1^0(\epsilon) &\simeq \epsilon^{-1} \int_{\epsilon^{-1/2}}^\infty \frac{d\xi}{\xi}\, e^{-\epsilon\xi} = \epsilon^{-1} \int_{\epsilon^{1/2}}^\infty \frac{d\xi'}{\xi'}\, e^{-\xi'} \\ &\simeq \epsilon^{-1}(-\tfrac{1}{2}\log \epsilon - \gamma), \end{aligned} \tag{B.17}$$

where (A.8) has been used. In the second interval we may replace $\exp(-\epsilon\xi)$ by 1; with the help of (B.8) we then find

$$K_2^0(\epsilon) \simeq \epsilon^{-1}(-\tfrac{1}{2}\log \epsilon - \gamma + 1). \tag{B.18}$$

Addition of (B.17) and (B.18) yields

$$K^0(\epsilon) \simeq \epsilon^{-1}(-\log \epsilon - 2\gamma + 1), \tag{B.19}$$

or more precisely:

$$K^0(\epsilon) = \epsilon^{-1}[-\log \epsilon - 2\gamma + 1 + \mathcal{O}(\epsilon^{1/2-\delta})], \tag{B.20}$$

with $\delta > 0$, as follows by estimating the neglected terms in the asymptotic expansions.

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