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THE RELATIVISTIC ENERGY-MOMENTUM TENSOR IN POLARIZED MEDIA

II. THE ANGULAR MOMENTUM LAWS AND THE SYMMETRY OF THE ENERGY-MOMENTUM TENSOR

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Synopsis

The angular momentum balance of matter in an electromagnetic field on the atomic level is derived from microscopic theory. As a consequence an energy-momentum tensor can be constructed, which is completely symmetrical. Both its material and field parts are given explicitly in terms of the atomic parameters.

§ 1. *Introduction.* From microscopic theory the conservation laws of energy and momentum for a system in an electromagnetic field have been obtained[1]). In the same way we shall derive balance equations and conservation laws for angular momentum on the "atomic" level. This means that we suppose that the constituent charged point particles of the system are grouped into stable structures such as atoms, ions, molecules etc., which carry electric and magnetic moments (for brevity we shall refer to these stable groups as atoms).

§ 2. *The atomic angular momentum balance.* The equations of motion of the point particle ki (constituent i of atom k) with mass m_{ki}, charge e_{ki}, time-space coordinates $R^{\alpha}_{ki} = (c\, t_{ki}, \boldsymbol{R}_{ki})$ and proper time τ_{ki} reads:

$$c m_{ki} \frac{\mathrm{d}}{\mathrm{d}s}\left\{\frac{\mathrm{d}R^{\alpha}_{ki}}{\mathrm{d}s}\left(\frac{\mathrm{d}\tau_{ki}}{\mathrm{d}s}\right)^{-1}\right\} = e_{ki}\, f^{\alpha\gamma}_{(t)}(R_{ki})\,\frac{\mathrm{d}R_{ki\gamma}}{\mathrm{d}s} \quad (\alpha = 0, 1, 2, 3), \tag{1}$$

where the parameter s is the proper time of the privileged point R^{α}_{k}, which characterizes the motion of the atom as a whole. Inner coordinates r^{α}_{ki} are defined by means of

$$R^{\alpha}_{ki} = R^{\alpha}_{k} + r^{\alpha}_{ki}. \tag{2}$$

The following conditions are imposed on r^{α}_{ki}: the orthogonality relation

$$r_{ki\alpha}\,\frac{\mathrm{d}R^{\alpha}_{k}}{\mathrm{d}s} = 0, \tag{3}$$

and the centre of mass relation

$$\sum_i m_{ki}\, r^{\alpha}_{ki} = 0, \tag{4}$$

which is valid up to second order[1]). After multiplication of (1) by r^{β}_{ki} and expansion up to second order (using (2)) one gets, adding an integration over a four-dimensional delta-function:

$$\sum_i c \int m_{ki} \frac{\mathrm{d}}{\mathrm{d}s}\left\{\frac{\mathrm{d}R^{\alpha}_k}{\mathrm{d}s} + \frac{\mathrm{d}r^{\alpha}_{ki}}{\mathrm{d}s} + \frac{1}{c^2}\frac{\mathrm{d}R^{\alpha}_k}{\mathrm{d}s}\frac{\mathrm{d}r_{ki\gamma}}{\mathrm{d}s}\frac{\mathrm{d}R^{\gamma}_k}{\mathrm{d}s}\right\} r^{\beta}_{ki}\, \delta(R_k - R)\,\mathrm{d}s =$$
$$= \sum_i \int e_{ki}\, f^{\alpha\gamma}_{(t)}(R_{ki})\left(\frac{\mathrm{d}R_{k\gamma}}{\mathrm{d}s} + \frac{\mathrm{d}r_{ki\gamma}}{\mathrm{d}s}\right) r^{\beta}_{ki}\, \delta(R_k - R)\,\mathrm{d}s. \tag{5}$$

Using (4), the first term at the left-hand side disappears. Taking (twice) the antisymmetric part of eq. (5) one obtains with the use of (3), (I.13) *), the definition of the atomic angular momentum density

$$\sigma^{\alpha\beta}_k = \sum_i c \int m_{ki}\, \Delta^{\alpha}_{k\gamma}\, \Delta^{\beta}_{k\varepsilon}\left(r^{\gamma}_{ki}\frac{\mathrm{d}r^{\varepsilon}_{ki}}{\mathrm{d}s} - r^{\varepsilon}_{ki}\frac{\mathrm{d}r^{\gamma}_{ki}}{\mathrm{d}s}\right)\delta(R_k - R)\,\mathrm{d}s \tag{6}$$

(a purely space-space-like tensor in the rest frame), the tensor

$$\Delta^{\beta}_{k\alpha} = \delta^{\beta}_{\alpha} + \frac{1}{c^2}\frac{\mathrm{d}R_{k\alpha}}{\mathrm{d}s}\frac{\mathrm{d}R^{\beta}_k}{\mathrm{d}s}, \tag{7}$$

the four-velocity u^{α}_k of the atom k (I.38) and the four-acceleration $D_k u^{\alpha}_k$ (I.43):

$$\partial_\gamma(u^{\gamma}_k\sigma^{\alpha\beta}_k) + \frac{1}{2c^2}u^{\alpha}_k\sigma^{\beta\gamma}_k D_k u_{k\gamma} - \frac{1}{2c^2}u^{\beta}_k\sigma^{\alpha\gamma}_k D_k u_{k\gamma} =$$
$$= \frac{1}{2c^2}\sum_i \int e_{ki}\frac{\mathrm{d}R^{\alpha}_k}{\mathrm{d}s} f_{(t)\gamma\varepsilon}(R_{ki})\frac{\mathrm{d}R^{\varepsilon}_k}{\mathrm{d}s}\frac{\mathrm{d}}{\mathrm{d}s}(r^{\gamma}_{ki}r^{\beta}_{ki})\,\delta(R_k - R)\,\mathrm{d}s$$
$$- \sum_i \int e_{ki}\, f^{\alpha\gamma}_{(t)}(R_{ki})\frac{\mathrm{d}R_{k\gamma}}{\mathrm{d}s} r^{\beta}_{ki}\,\delta(R_k - R)\,\mathrm{d}s$$
$$- \sum_i \int e_{ki}\, f^{\alpha\gamma}_{(t)}(R_{ki})\frac{\mathrm{d}r_{ki\gamma}}{\mathrm{d}s} r^{\beta}_{ki}\,\delta(R_k - R)\,\mathrm{d}s - (\alpha, \beta), \tag{8}$$

where (α, β) stands for the preceding terms of the right-hand side with α and β interchanged. This equation is the balance of the atomic angular momentum density $\sigma^{\alpha\beta}_k$.

§ 3. *The intra-atomic electromagnetic field.* The field $f^{\alpha\beta}_{(t)}(R_{ki})$ acting on particle ki may be split into a part $f^{\alpha\beta}_{(in)}(R_{ki})$ due to the action of the other

*) (I.13) means formula (13) of the first paper [1]).

constituent particles of atom k and a part $f^{\alpha\beta}(R_{ki})$ due to the other atoms $l \neq k$:

$$f^{\alpha\beta}_{(t)}(R_{ki}) = f^{\alpha\beta}_{(in)}(R_{ki}) + f^{\alpha\beta}(R_{ki}). \tag{9}$$

Let us first consider the right-hand side of (8), but for the intra-atomic field $f^{\alpha\beta}_{(in)}$ only. The space-space part ($\alpha, \beta = 1, 2, 3$) reads in the rest frame (denoted by (0)) in which $\mathrm{d}R^{\alpha}_{k}/\mathrm{d}s = (c, 0, 0, 0)$:

$$-\sum_i \int e_{ki}\, f^{\alpha\gamma(0)}_{(in)}(R^{(0)}_{ki}) \frac{\mathrm{d}R^{(0)}_{k\gamma}}{\mathrm{d}s}\, r^{\beta(0)}_{ki}\, \delta(R^{(0)}_{k} - R^{(0)})\, \mathrm{d}s$$
$$-\sum_i \int e_{ki}\, \Delta^{\alpha(0)}_{k\varepsilon}\, f^{\varepsilon\gamma(0)}_{(in)}(R_{ki})\, \frac{\mathrm{d}r^{(0)}_{ki\gamma}}{\mathrm{d}s}\, r^{\beta(0)}_{ki}\, \delta(R^{(0)}_{k} - R^{(0)})\, \mathrm{d}s - (\alpha, \beta). \tag{10}$$

In this expression we substitute the formulae (I.17) and (I.18) for the intra-atomic fields up to order c^{-2}. Then one obtains transforming back to the reference frame:

$$-\sum_i \delta\rho^{I'}_{ki}(r^{\alpha}_{ki}\, D_k u^{\beta}_{k} - r^{\beta}_{ki}\, D_k u^{\alpha}_{k}) - \Delta^{\alpha}_{k\varepsilon}\, \Delta^{\beta}_{k\zeta}\, \partial_{\gamma}(u^{\gamma}_{k}\lambda^{\varepsilon\zeta}_{k}), \tag{11}$$

where the Coulomb mass density is given by

$$\delta\rho^{I'}_{ki} = c \int \delta m^{I}_{ki}\, \delta(R_k - R)\, \mathrm{d}s, \tag{12a}$$

$$\delta m^{I}_{ki} = c^{-2} \sum_{j(\neq i)} \frac{e_{ki}\, e_{kj}}{8\pi\, |\boldsymbol{r}^{(0)}_{ki} - \boldsymbol{r}^{(0)}_{kj}|}, \tag{12b}$$

and the angular momentum density $\lambda^{\alpha\beta}_{k}$ of the intra-atomic field is defined by its components in the momentary rest frame:

$$\lambda^{\alpha\beta(0)}_{k} = -\frac{1}{2c} \int \sum_{i,j(i\neq j)} \frac{e_{ki}\, e_{kj}}{4\pi\, |\boldsymbol{r}^{(0)}_{ki} - \boldsymbol{r}^{(0)}_{kj}|} \{\dot{\boldsymbol{r}}^{(0)}_{kj} \cdot \mathrm{T}(\boldsymbol{r}^{(0)}_{ki}, \boldsymbol{r}^{(0)}_{kj})\}^{\alpha}$$
$$\boldsymbol{r}^{\beta(0)}_{ki}\, \delta(R^{(0)}_{k} - R^{(0)})\, \mathrm{d}s - (\alpha, \beta), \quad (\alpha, \beta = 1, 2, 3), \tag{13a}$$

$$\lambda^{\alpha 0(0)}_{k} = -\lambda^{0\alpha(0)}_{k} = 0 \ (\alpha = 1, 2, 3); \quad \lambda^{00(0)}_{k} = 0, \tag{13b}$$

with

$$\mathrm{T}(\boldsymbol{r}^{(0)}_{ki}, \boldsymbol{r}^{(0)}_{kj}) \equiv \mathrm{U} + \frac{(\boldsymbol{r}^{(0)}_{ki} - \boldsymbol{r}^{(0)}_{kj})(\boldsymbol{r}^{(0)}_{ki} - \boldsymbol{r}^{(0)}_{kj})}{|\boldsymbol{r}^{(0)}_{ki} - \boldsymbol{r}^{(0)}_{kj}|^2}. \tag{13c}$$

(A simple calculation of the total moment of the Lorentz forces, with fields up to order c^{-2}, acting on a group of charged point particles leads to the time derivative of a tensor of exactly the same form as $\lambda^{\alpha\beta}_{k}$.)

In contrast with the result (11) for the space-space part in the rest frame the space-time part ($\alpha = 0$, $\beta = 1, 2, 3$) of the right-hand side of

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*) The dot indicates a (three-dimensional) contraction of $\dot{\boldsymbol{r}}_{kj}{}^{(0)}$ with the first index of the tensor (13c); its second index is α.

(8) for $f^{\alpha\beta}_{(in)}$ contains in the rest frame not only terms of order c^{-2} but also terms of order 1. These terms are somewhat difficult to interpret physically but they will not be needed for the following discussions. In fact from now on we shall consider equation (8) with both sides multiplied by $\Delta^{\varepsilon}_{k\alpha}\,\Delta^{\zeta}_{k\beta}$ (with $\Delta^{\beta}_{k\alpha} = \delta^{\alpha}_{\beta} + c^{-2}\,u_{k\alpha}u^{\beta}_{k}$); then in the rest frame it only contains space-space components (ε, $\zeta = 1, 2, 3$):

$$\begin{aligned}\Delta^{\varepsilon}_{k\alpha}\,\Delta^{\zeta}_{k\beta}\,\partial_{\gamma}\{u^{\gamma}_{k}(\sigma^{\alpha\beta}_{k} + \lambda^{\alpha\beta}_{k})\} = {}& \sum_{i}\delta\rho^{I'}_{ki}(D_{k}u^{\varepsilon}_{k})\,r^{\zeta}_{ki}\\ &- \sum_{i}\int e_{ki}\,f^{\varepsilon\gamma}(R_{ki})\,\frac{\mathrm{d}R_{k\gamma}}{\mathrm{d}s}\,r^{\zeta}_{ki}\,\delta(R_{k}-R)\,\mathrm{d}s\\ &- \sum_{i}\int e_{ki}\,\Delta^{\varepsilon}_{k\alpha}\,f^{\alpha\gamma}(R_{ki})\,\frac{\mathrm{d}r_{ki\gamma}}{\mathrm{d}s}\,r^{\zeta}_{ki}\,\delta(R_{k}-R)\,\mathrm{d}s - (\varepsilon,\zeta).\end{aligned} \tag{14}$$

This is a form of the balance equation of angular momentum, in which the interatomic fields remain to be studied.

§ 4. *The interatomic field.* In contrast to the intra-atomic field the interatomic field $f^{\alpha\beta}(R_{ki})$ may be expanded in powers of r_{ki}. Up to second order in r_{ki} the equation (14) gets the form:

$$\begin{aligned}\Delta^{\alpha}_{k\varepsilon}\,\Delta^{\beta}_{k\zeta}\,\partial_{\gamma}(u^{\gamma}_{k}\,\sigma^{+\varepsilon\zeta}) = {}& \sum_{i}\delta\rho^{I'}_{ki}(D_{k}u^{\alpha}_{k})\,r^{\beta}_{ki}\\ &- \sum_{i}\int e_{ki}\,f^{\alpha\gamma}(R)\frac{\mathrm{d}R_{k\gamma}}{\mathrm{d}s}\,r^{\beta}_{ki}\,\delta(R_{k}-R)\,\mathrm{d}s\\ &- \sum_{i}\int e_{ki}\,r^{\varepsilon}_{ki}\,\partial_{\varepsilon}f^{\alpha\gamma}(R)\frac{\mathrm{d}R_{k\gamma}}{\mathrm{d}s}\,r^{\beta}_{ki}\,\delta(R_{k}-R)\,\mathrm{d}s\\ &- \sum_{i}\int e_{ki}\,\Delta^{\alpha}_{k\varepsilon}\,f^{\varepsilon\gamma}(R)\,\frac{\mathrm{d}r_{ki\gamma}}{\mathrm{d}s}\,r^{\beta}_{ki}\,\delta(R_{k}-R)\,\mathrm{d}s - (\alpha,\beta),\end{aligned} \tag{15}$$

where the angular momentum density $\sigma^{+\alpha\beta}_{k}$ is defined a

$$\sigma^{+\alpha\beta}_{k} = \sigma^{\alpha\beta}_{k} + \lambda^{\alpha\beta}_{k}. \tag{16}$$

Retaining only terms with electric and magnetic dipoles, but discarding electric quadrupoles, (15) becomes with the use of (3):

$$\begin{aligned}\Delta^{\alpha}_{k\varepsilon}\,\Delta^{\beta}_{k\zeta}\,\partial_{\gamma}(u^{\gamma}_{k}\sigma^{+\varepsilon\zeta}_{k}) = {}& \sum_{i}\delta\rho^{I'}_{ki}(D_{k}u^{\alpha}_{k})\,r^{\beta}_{ki}\\ &+ c^{-2}\,f^{\alpha\gamma}\,u_{k\gamma}\,m^{(1)\beta\varepsilon}_{k}\,u_{k\varepsilon}\\ &- \Delta^{\alpha}_{k\varepsilon}\,f^{\varepsilon\gamma}\,m^{(2)\beta}_{k\ \cdot\gamma} - (\alpha,\beta),\end{aligned} \tag{17}$$

where the polarization tensors are in first and second order

$$m^{(1)\alpha\beta}_{k} = \sum_{i}\int e_{ki}\left(r^{\alpha}_{ki}\,\frac{\mathrm{d}R^{\beta}_{k}}{\mathrm{d}s} - r^{\beta}_{ki}\frac{\mathrm{d}R^{\alpha}_{k}}{\mathrm{d}s}\right)\delta(R_{k}-R)\,\mathrm{d}s, \tag{18}$$

$$m_k^{(2)\alpha\beta} = \tfrac{1}{2} \sum_i \int e_{ki}\, \Delta^\alpha_{k\gamma}\, \Delta^\beta_{k\varepsilon} \left(r^\gamma_{ki} \frac{\mathrm{d}r^\varepsilon_{ki}}{\mathrm{d}s} - r^\varepsilon_{ki} \frac{\mathrm{d}r^\gamma_{ki}}{\mathrm{d}s} \right) \delta(R_k - R)\, \mathrm{d}s. \tag{19}$$

Using (I.28) and the properties (I.36) and (I.37) the equation (17) can be written (with a summation over k) in the form of a balance equation for the atomic angular momentum density:

$$\sum_k \Delta^\alpha_{k\varepsilon}\, \Delta^\beta_{k\zeta}\, \partial_\gamma (u^\gamma_k\, \sigma^{+\varepsilon\zeta}_k) - \sum_{ki} \delta\rho^{I'}_{ki} (D_k u^\alpha_k\, r^\beta_{ki} - D_k u^\beta_k\, r^\alpha_{ki}) =$$
$$= \sum_{k,l(k \neq l)} \Delta^\alpha_{k\varepsilon}\, \Delta^\beta_{k\zeta} (m^{\varepsilon\gamma}_k\, f^{\zeta}_{l\cdot\gamma} - f^{\varepsilon\gamma}_l\, m^{\zeta}_{k\cdot\gamma}), \tag{20}$$

with the total polarization tensor

$$m^{\alpha\beta}_k = m^{(1)\alpha\beta}_k + m^{(2)\alpha\beta}_k. \tag{21}$$

With the use of the conservation law (I.44) of proper mass (I.23) the angular momentum balance (20) gets the form

$$\sum_k \rho'_k\, \Delta^\alpha_{k\varepsilon}\, \Delta^\beta_{k\zeta}\, D_k (v'_k\, \sigma^{+\varepsilon\zeta}_k)$$
$$- \sum_{ki} \delta\rho^{I'}_{ki} \{ (D_k u^\alpha_k)\, r^\beta_{ki} - (D_k u^\beta_k)\, r^\alpha_{ki} \} =$$
$$= \sum_{k,l(k \neq l)} \Delta^\alpha_{k\varepsilon}\, \Delta^\beta_{k\zeta} (m^{\varepsilon\gamma}_k\, f^{\zeta}_{l\cdot\gamma} - f^{\varepsilon\gamma}_l\, m^{\zeta}_{k\cdot\gamma}), \tag{22}$$

where $D_k = u^\alpha_k\, \partial_\alpha$ and $v'_k = (\rho'_k)^{-1}$.

In the momentary rest frame of atom k the contribution of k to equation (22) reads (with the notation $\boldsymbol{\sigma}^{+(0)m}_k \equiv \sigma^{+(0)ij}_k$, $i, j, m = 1, 2, 3$ cycl.):

$$\rho'_k\, D_k (v'_k \boldsymbol{\sigma}^{+(0)}_k) - \sum_i \delta\rho^{I'}_{ki}\, D_k\, \boldsymbol{v}^{(0)}_k \wedge \boldsymbol{r}^{(0)}_{ki} =$$
$$= \sum_{l(\neq k)} (\boldsymbol{p}^{(0)}_k \wedge \boldsymbol{e}^{(0)}_l + \boldsymbol{m}^{(0)}_k \wedge \boldsymbol{b}^{(0)}_l). \tag{23}$$

If the atom k is isotropic as far as polarization and magnetization are concerned, the term at the right-hand side vanishes.

§ 5. *The symmetry of the energy-momentum tensor.* The conservation laws of energy-momentum at the atomic level are

$$\partial_\beta (t^{\alpha\beta}_{(m)} + t^{\alpha\beta}_{(f)}) = 0. \tag{24}$$

Here the atomic material energy-momentum tensor is given by[1])

$$t^{\alpha\beta}_{(m)} = \sum_k (\rho'_k + \delta\rho^{I'}_k + \delta\rho^{II'}_k)\, u^\alpha_k u^\beta_k + \sum_k c^{-2}\, \sigma^{+\alpha\gamma}_k (D_k u_{k\gamma})\, u^\beta_k$$
$$+ \sum_k \partial_\gamma \{ u^\gamma_k (\sum_i \delta\rho^{I'}_{ki}\, r^\alpha_{ki}) \}\, u^\beta_k, \tag{25}$$

where for convenience the angular momentum density σ^+_k, instead of σ_k, is written in the second term. The difference $c^{-2}\, \lambda^{\alpha\gamma}_k (D_k u_{k\gamma})\, u^\beta_k$ is negligible since it contains the intra-atomic field and is of order c^{-4}. The atomic energy-momentum tensor of the field is[1])

$$t^{\alpha\beta}_{(f)} = \sum_{k,l(k\neq l)} \{f^{\alpha\gamma}_l h^{\beta}_{k\cdot\gamma} - \tfrac{1}{4} f_{l\gamma\varepsilon} f^{\gamma\varepsilon}_k g^{\alpha\beta}$$
$$+ c^{-2}\, u^{\beta}_k (f^{\alpha\gamma}_l m_{k\gamma\varepsilon} - m^{\alpha\gamma}_k f_{l\gamma\varepsilon})\, u^{\varepsilon}_k$$
$$- c^{-4}\, u^{\alpha}_k u^{\beta}_k u^{\gamma}_k f_{l\gamma\varepsilon} m^{\varepsilon\zeta}_k u_{k\zeta}\}, \tag{26}$$

where $h^{\alpha\beta}_k = f^{\alpha\beta}_k - m^{\alpha\beta}_k$.

Twice the antisymmetric part of this tensor $t^{\alpha\beta}_{(f)}$ is:

$$t^{\alpha\beta}_{(f)} - t^{\beta\alpha}_{(f)} = \sum_{k,l(k\neq l)} \Delta^{\alpha}_{k\varepsilon}\, \Delta^{\beta}_{k\zeta} (m^{\varepsilon\gamma}_k f^{\zeta}_{l\cdot\gamma} - f^{\varepsilon\gamma}_l m^{\zeta}_{k\cdot\gamma}), \tag{27}$$

which turns out to be equal to the right-hand side of (20). We wish to relate the left-hand side of (20) to the *material* energy-momentum tensor. In this respect it should first be remarked that the energy-momentum tensor is determined up to a divergence-free part only, as is clear from (24). It will be shown that the addition of the divergence-free expression

$$-\tfrac{1}{2} \sum_k \partial_\gamma (\sigma^{+\alpha\beta}_k u^{\gamma}_k - \sigma^{+\alpha\gamma}_k u^{\beta}_k - \sigma^{+\beta\gamma}_k u^{\alpha}_k) \tag{28}$$

to the material energy-momentum tensor (25) is of advantage. A similar procedure has been followed by Belinfante[2]) and Rosenfeld[3]) in the field theory of particles with spin. The sum of (25) and (28) gives a *new* material energy-momentum tensor (which will again be denoted by the symbol $t^{\alpha\beta}_{(m)}$):

$$t^{\alpha\beta}_{(m)} = \sum_k (\rho'_k + \delta\rho^{I'}_k + \delta\rho^{II'}_k)\, u^{\alpha}_k u^{\beta}_k$$
$$- \tfrac{1}{2} \sum_k \Delta^{\alpha}_{k\varepsilon}\, \Delta^{\beta}_{k\zeta}\, \partial_\gamma (\sigma^{+\varepsilon\zeta}_k u^{\gamma}_k)$$
$$+ \tfrac{1}{2} c^{-2} \sum_k (u^{\alpha}_k\, \sigma^{+\beta\gamma}_k\, D_k u_{k\gamma} + u^{\beta}_k\, \sigma^{+\alpha\gamma}_k\, D_k u_{k\gamma}) + \tfrac{1}{2} \sum_k \partial_\gamma (\sigma^{+\alpha\gamma}_k\, u^{\beta}_k + \sigma^{+\beta\gamma}_k\, u^{\alpha}_k)$$
$$+ \sum_k \partial_\gamma \{u^{\gamma}_k (\sum_i \delta\rho^{I'}_{ki}\, r^{\alpha}_{ki})\}\, u^{\beta}_k, \tag{29}$$

where (6), (13), the relation

$$u_{k\alpha}\, \sigma^{+\alpha\beta}_k = 0 \tag{30}$$

(which follows from (6) and (13)), and (7) have been used. In this way (twice) the antisymmetric part of the tensor $t^{\alpha\beta}_{(m)}$ (29) is nearly equal to the left-hand side of (20): in fact

$$t^{\beta\alpha}_{(m)} - t^{\alpha\beta}_{(m)} = \sum_k \Delta^{\alpha}_{k\varepsilon}\, \Delta^{\beta}_{k\zeta}\, \partial_\gamma (\sigma^{+\varepsilon\zeta}_k\, u^{\gamma}_k) - \sum_k \partial_\gamma \{u^{\gamma}_k (\sum_i \delta\rho^{I'}_{ki}\, r^{\alpha}_{ki})\}\, u^{\beta}_k$$
$$+ \sum_k \partial_\gamma \{u^{\gamma}_k (\sum_i \delta\rho^{I'}_{ki}\, r^{\beta}_{ki})\}\, u^{\alpha}_k. \tag{31}$$

The difference between the right-hand side of this equation and the left-hand side of (20) consists only in small terms of intra-atomic origin. If one desires so one can get rid of these terms by an appropriate localization of the atomic energy

$$(m_k + \delta m_k^I)\, c^2 = m_k c^2 + \sum_{i,j(i\neq j)} \frac{e_{ki}\, e_{kj}}{8\pi\, |\boldsymbol{r}_{ki}^{(0)} - \boldsymbol{r}_{kj}^{(0)}|}, \tag{32}$$

which occurs in the first term of $t_{(m)}^{\alpha\beta}$ (29). In fact, defining a point

$$\tilde{R}_k^\alpha = R_k^\alpha + \frac{1}{m_k + \delta m_k^I} \sum_i \delta m_{ki}^I\, r_{ki}^\alpha, \tag{33}$$

where δm_{ki}^I is defined by (12b), one derives by means of a Taylor expansion of $\delta(R_k - R)$ around $\tilde{R}_k$:

$$\sum_k c \int (m_k + \delta m_k^I) \frac{\mathrm{d}R_k^\alpha}{\mathrm{d}s} \frac{\mathrm{d}R_k^\beta}{\mathrm{d}s}\, \delta(R_k - R)\, \mathrm{d}s =$$

$$= \sum_k c \int (m_k + \delta m_k^I) \frac{\mathrm{d}R_k^\alpha}{\mathrm{d}s} \frac{\mathrm{d}R_k^\beta}{\mathrm{d}s}\, \delta(\tilde{R}_k - R)\, \mathrm{d}s$$

$$+ \sum_{ki} c \int \frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{\mathrm{d}R_k^\alpha}{\mathrm{d}s}\, \delta m_{ki}^I\, r_{ki}^\beta\right) \delta(R_k - R)\, \mathrm{d}s + A^{\alpha\beta} + B^{\alpha\beta}, \tag{34}$$

where $A^{\alpha\beta}$ is a series of terms symmetric in α, β and $B^{\alpha\beta}$ is divergence-free. One could thus define a new material tensor $t_{(m)}^{\alpha\beta} - B^{\alpha\beta}$ of which (twice) the antisymmetric part is given (cf. (31)) by:

$$\sum_k \Delta_{k\varepsilon}^\alpha\, \Delta_{k\zeta}^\beta\, \partial_\gamma(\sigma_k^{+\varepsilon\zeta}\, u_k^\gamma) + \sum_{ki} \delta\rho_{ki}^{I'}(r_{ki}^\alpha\, D_k u_k^\beta - r_{ki}^\beta\, D_k u_k^\alpha), \tag{35}$$

which is exactly equal to the left-hand side of (20). Thus, according to (27) and (35), the balance of angular momentum (20) is precisely the expression of the fact that the total energy-momentum tensor $t_{(m)}^{\alpha\beta} + t_{(f)}^{\alpha\beta}$ is *symmetric*.

As shown here the conclusion about the symmetry of the energy-momentum tensor is only reached if the atomic energy $m_k + \delta m_k^I$ is properly localized. However in the rest frame of atom k the (purely spacelike) difference $\tilde{R}_k^\alpha - R_k^\alpha$ is small (of the order 10^{-5}) compared with atomic dimensions as (33) shows. Such a refinement is hardly useful in a theory in which nuclei and electrons are considered as point particles. Therefore from now on the material energy-momentum tensor will be written as:

$$t_{(m)}^{\alpha\beta} = \sum_k \rho_k^{t'}\, u_k^\alpha u_k^\beta - \tfrac{1}{2} \sum_k \Delta_{k\varepsilon}^\alpha\, \Delta_{k\zeta}^\beta\, \partial_\gamma(\sigma_k^{+\varepsilon\zeta}\, u_k^\gamma)$$

$$+ \tfrac{1}{2} c^{-2} \sum_k (u_k^\alpha\, \sigma_k^{+\beta\gamma}\, D_k u_{k\gamma} + u_k^\beta\, \sigma_k^{+\alpha\gamma}\, D_k u_{k\gamma}) + \tfrac{1}{2} \sum_k \partial_\gamma(\sigma_k^{+\alpha\gamma} u_k^\beta + \sigma_k^{+\beta\gamma}\, u_k^\alpha), \tag{36}$$

where $\rho_k^{t'}$ is the sum of ρ_k', $\delta\rho_k^{I'}$ and $\delta\rho_k^{II'}$. In the approximation described above the sum of this tensor and the field energy-momentum tensor (26) is symmetric and conserved.

§ 6. *The conservation laws of angular momentum.* The total energy-momentum tensor of matter and field is

$$t^{\alpha\beta} = t^{\alpha\beta}_{(m)} + t^{\alpha\beta}_{(f)}, \tag{37}$$

with the tensors at the right-hand side given by (36) and (26). As discussed in the preceding section it is symmetric

$$t^{\alpha\beta} = t^{\beta\alpha}, \tag{38}$$

and conserved

$$\partial_\beta t^{\alpha\beta} = 0. \tag{39}$$

From (38) and (39) one finds the conservation law of angular momentum:

$$\partial_\gamma(x^\alpha t^{\beta\gamma} - x^\beta t^{\alpha\gamma}) = 0. \tag{40}$$

From this equation it follows that

$$I^{\alpha\beta} = \int (x^\alpha t^{\beta 0} - x^\beta t^{\alpha 0})\, \mathrm{d}V \tag{41}$$

is a conserved quantity:

$$\mathrm{d}I^{\alpha\beta}/\mathrm{d}t = 0. \tag{42}$$

Moreover, $I^{\alpha\beta}$ is a tensor according to Klein's theorem.

The contents of these conservation laws are well-known: for the space-space components ($\alpha, \beta = 1, 2, 3$) one has with $\boldsymbol{p}^i = c^{-1}t^{i0}$:

$$\frac{\mathrm{d}I^{ij}}{c\,\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\int (\boldsymbol{R} \wedge \boldsymbol{p})^k\, \mathrm{d}V = 0 \quad (i, j, k = 1, 2, 3 \text{ cycl.}), \tag{43}$$

which expresses the conservation of angular momentum properly speaking; for the space-time components one has

$$\frac{\mathrm{d}}{\mathrm{d}t}\int x^i t^{00}\, \mathrm{d}V = \frac{\mathrm{d}}{\mathrm{d}t}\int x^0 t^{i0}\, \mathrm{d}V = \\ = c\int t^{i0}\, \mathrm{d}V + x^0 \frac{\mathrm{d}}{\mathrm{d}t}\int t^{i0}\, \mathrm{d}V \quad (i = 1, 2, 3). \tag{44}$$

The last term vanishes since total momentum is conserved. Thus we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\int x^i t^{00}\, \mathrm{d}V = c^2 \boldsymbol{P}^i, \tag{45}$$

where $\boldsymbol{P}^i = c^{-1}\int t^{i0}\, \mathrm{d}V$ is the (conserved) total momentum. Dividing both members of this equation by the (conserved) total energy $E = \int t^{00}\, \mathrm{d}V$

of the system one obtains

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\int x^i t^{00}\,\mathrm{d}V}{\int t^{00}\,\mathrm{d}V}\right) = \frac{c^2 \boldsymbol{P}^i}{E}. \tag{46}$$

This equation expresses the fact that the centre of energy of the system moves with a uniform velocity $c^2\boldsymbol{P}/E$.

In the laws (43) and (46) the tensor components t^{00} and t^{i0} occur. Explicit expressions in three-dimensional notation for these quantities are given in the next section.

§ 7. *The atomic energy-momentum tensor in three-dimensional notation.* The components of the material energy-momentum tensor (36) may be written in vector notation if one uses the expressions for the components of u_k^α, $D_k u_k^\alpha$ and $\Delta_k^{\alpha\beta}$:

$$u_k^0 = c\gamma_k, \quad u_k^i = c\gamma_k \beta_k^i, \tag{47}$$

$$D_k u_k^0 = c^2 \gamma_k\, \partial_0 \gamma_k, \quad D_k u_k^i = c^2 \gamma_k\, \partial_0(\gamma_k \beta_k^i), \tag{48}$$

$$\Delta_k^{00} = -1 + \gamma_k^2, \quad \Delta_k^{0i} = \Delta_k^{i0} = \gamma_k^2 \beta_k^i, \quad \Delta_k^{ij} = g^{ij} + \gamma_k^2 \beta_k^i \beta_k^j, \tag{49}$$

where $i, j = 1, 2, 3$. Here $c\boldsymbol{\beta}_k$ is the velocity of atom k, $\gamma_k = (1 - \beta_k^2)^{-\frac{1}{2}}$ and $g^{ij} = 1$ if $i = j$, $g^{ij} = 0$ if $i \neq j$. Since the antisymmetric tensor $\sigma_k^{+\alpha\beta}$ is purely space-space-like in the (dashed) rest frame, the Lorentz transformation from the rest frame to the reference frame yields:

$$\sigma_k^{+ij} \equiv \boldsymbol{\sigma}_k^{+m} = (\gamma_k \boldsymbol{\Omega}_k \cdot \boldsymbol{\sigma}_k^{+\prime})^m \quad (i, j, m = 1, 2, 3 \text{ cycl.}), \tag{50}$$

$$\sigma_k^{+i0} = (\gamma_k \boldsymbol{\beta}_k \wedge \boldsymbol{\sigma}_k^{+\prime})^i \qquad (i = 1, 2, 3), \tag{51}$$

where we have used the three-tensor

$$\boldsymbol{\Omega}_k^{ij} = g^{ij} + \frac{\gamma_k^{-1} - 1}{\beta_k^2}\, \beta_k^i \beta_k^j, \tag{52}$$

which has the properties

$$(\boldsymbol{\Omega}_k^2)^{ij} = g^{ij} - \beta_k^i \beta_k^j, \tag{53a}$$

$$(\boldsymbol{\Omega}_k^{-2})^{ij} = g^{ij} + \gamma_k^2 \beta_k^i \beta_k^j. \tag{53b}$$

With the help of the abbreviation

$$\underline{\boldsymbol{\sigma}}_k = \boldsymbol{\Omega}_k \cdot \boldsymbol{\sigma}_k^{+\prime}, \tag{54}$$

one gets then for the components of (36):

$$t_{(m)}^{00} = \sum_k \left[\rho_k^{t\prime} c^2 \gamma_k^2 - c\gamma_k^4 (\boldsymbol{\beta}_k \wedge \underline{\boldsymbol{\sigma}}_k) \cdot \partial_0 \boldsymbol{\beta}_k + c\gamma_k^2 \boldsymbol{\beta}_k \cdot (\boldsymbol{\nabla} \wedge \underline{\boldsymbol{\sigma}}_k)\right], \tag{55}$$

$$\begin{aligned} t_{(m)}^{0i} = \sum_k \Big[&\rho_k^{t\prime} c^2 \gamma_k^2 \boldsymbol{\beta}_k + c\, \boldsymbol{\Omega}_k^{-2} \cdot \partial_0(\gamma_k^2 \boldsymbol{\beta}_k \wedge \underline{\boldsymbol{\sigma}}_k) + \tfrac{1}{2} c\gamma_k^2 \boldsymbol{\nabla} \wedge \underline{\boldsymbol{\sigma}}_k \\ &+ \tfrac{1}{2} c\gamma_k^2 (\boldsymbol{\beta}_k \cdot \boldsymbol{\nabla})(\boldsymbol{\beta}_k \wedge \underline{\boldsymbol{\sigma}}_k) - \tfrac{1}{2} c\gamma_k^2 \boldsymbol{\beta}_k \boldsymbol{\nabla} \cdot (\boldsymbol{\beta}_k \wedge \underline{\boldsymbol{\sigma}}_k)\Big]^i \quad (i = 1, 2, 3), \end{aligned} \tag{56}$$

$$t^{i0}_{(m)} = \sum_k [\rho_k^{t'} c^2\gamma_k^2\beta_k + c\gamma_k^4\partial_0\beta_k \wedge \underline{\sigma}_k + \tfrac{1}{2}c\gamma_k^2\nabla \wedge \underline{\sigma}_k - \tfrac{1}{2}c\gamma_k^2(\beta_k\cdot\nabla)$$

$$(\beta_k \wedge \underline{\sigma}_k) - \tfrac{1}{2}c\gamma_k^2\beta_k\nabla\cdot(\beta_k \wedge \underline{\sigma}_k)]^i \quad (i=1,2,3), \tag{57}$$

$$t^{ij}_{(m)} = \sum_k [\rho_k^{t'} c^2\gamma_k^2\beta_k^i\beta_k^j - \tfrac{1}{2}c\,\partial_0(\gamma_k^2\underline{\sigma}_k^m)$$

$$+ c\gamma_k^4(\partial_0\beta_k \wedge \underline{\sigma}_k)^i\beta_k^j + \tfrac{1}{2}c\,\partial_0\{\gamma_k^2(\beta_k \wedge \underline{\sigma}_k)^i\beta_k^j + \gamma_k^2\beta_k^i(\beta_k \wedge \underline{\sigma}_k)^j\}$$

$$- \tfrac{1}{2}c\,\gamma_k^2(\beta_k\cdot\nabla)\,\underline{\sigma}_k^m + \tfrac{1}{2}c\,\gamma_k^2\{(\nabla \wedge \underline{\sigma}_k)^i\beta_k^j + \beta_k^i(\nabla \wedge \underline{\sigma}_k)^j\}]$$

$$(i, j, m = 1, 2, 3 \text{ cycl.}). \tag{58}$$

Similarly, if one uses (47) and the vector notations for the fields and polarizations:

$$f_k^{0i} = e_k^i, \qquad f_k^{ij} = b_k^m \qquad (i, j, m = 1, 2, 3 \text{ cycl.}), \tag{59}$$

$$h_k^{0i} = d_k^i, \qquad h_k^{ij} = h_k^m \qquad (i, j, m = 1, 2, 3 \text{ cycl.}), \tag{60}$$

$$p_k^{0i} = -p_k^i, \quad p_k^{ij} = m_k^m \qquad (i, j, m = 1, 2, 3 \text{ cycl.}), \tag{61}$$

the components of the field energy-momentum tensor (26) can be calculated:

$$t^{00}_{(f)} = \sum_{k,l(k\neq l)} [\tfrac{1}{2}\boldsymbol{e}_k\cdot\boldsymbol{e}_l + \tfrac{1}{2}\boldsymbol{b}_k\cdot\boldsymbol{b}_l + \boldsymbol{p}_k\cdot\boldsymbol{e}_l - \gamma_k^2\beta_k\cdot(\boldsymbol{p}_k \wedge \boldsymbol{b}_l - \boldsymbol{m}_k \wedge \boldsymbol{e}_l)$$

$$- \gamma_k^4(\boldsymbol{p}_k - \beta_k \wedge \boldsymbol{m}_k)\cdot\boldsymbol{\Omega}_k^2\cdot(\boldsymbol{e}_l + \beta_k \wedge \boldsymbol{b}_l)], \tag{62}$$

$$t^{0i}_{(f)} = \sum_{k,l(k\neq l)} [\boldsymbol{e}_l \wedge \boldsymbol{h}_k - \gamma_k^2\beta_k\cdot(\boldsymbol{p}_k \wedge \boldsymbol{b}_l - \boldsymbol{m}_k \wedge \boldsymbol{e}_l)\,\beta_k$$

$$- \gamma_k^4(\boldsymbol{p}_k - \beta_k \wedge \boldsymbol{m}_k)\cdot\boldsymbol{\Omega}_k^2\cdot(\boldsymbol{e}_l + \beta_k \wedge \boldsymbol{b}_l)\,\beta_k]^i \quad (i = 1, 2, 3), \tag{63}$$

$$t^{i0}_{f} = \sum_{k,l(k\neq l)} [\boldsymbol{e}_l \wedge \boldsymbol{h}_k - \gamma_k^2\beta_k^2(\boldsymbol{p}_k \wedge \boldsymbol{b}_l - \boldsymbol{m}_k \wedge \boldsymbol{e}_l)$$

$$+ \gamma_k^2\beta_k \wedge (\boldsymbol{p}_k \wedge \boldsymbol{e}_l + \boldsymbol{m}_k \wedge \boldsymbol{b}_l)$$

$$- \gamma_k^4(\boldsymbol{p}_k - \beta_k \wedge \boldsymbol{m}_k)\cdot\boldsymbol{\Omega}_k^2\cdot(\boldsymbol{e}_l + \beta_k \wedge \boldsymbol{b}_l)\,\beta_k]^i \quad (i = 1, 2, 3), \tag{64}$$

$$t^{ij}_{(f)} = \sum_{k,l(k\neq l)} [-e_l^i d_k^j - h_k^i b_l^j + (\tfrac{1}{2}\boldsymbol{e}_k\cdot\boldsymbol{e}_l + \tfrac{1}{2}\boldsymbol{b}_k\cdot\boldsymbol{b}_l - \boldsymbol{m}_k\cdot\boldsymbol{b}_l)\,g^{ij}$$

$$+ \gamma_k^2\{\beta_k \wedge (\boldsymbol{p}_k \wedge \boldsymbol{e}_l + \boldsymbol{m}_k \wedge \boldsymbol{b}_l) - \boldsymbol{p}_k \wedge \boldsymbol{b}_l + \boldsymbol{m}_k \wedge \boldsymbol{e}_l\}^i\beta_k^j$$

$$- \gamma_k^4(\boldsymbol{p}_k - \beta_k \wedge \boldsymbol{m}_k)\cdot\boldsymbol{\Omega}_k^2\cdot(\boldsymbol{e}_l + \beta_k \wedge \boldsymbol{b}_l)\,\beta_k^i\beta_k^j] \quad (i, j = 1, 2, 3). \tag{65}$$

These formulae will enable us to give explicit expressions for the quantities which occur in the angular momentum laws.

§ 8. *Explicit expressions for the quantities occurring in the angular momentum laws.* In § 6 two angular momentum laws (43) and (46) were derived. They contain four global quantities, which we shall now write in explicit form with the help of the results of § 7. Using (57) and (64) for

$\boldsymbol{p}^i = c^{-1}\, t^{i0}$ one gets:

$$\int \boldsymbol{R} \wedge \boldsymbol{p}\, \mathrm{d}V = \sum_k \int \{\boldsymbol{R} \wedge (\rho_k^{t'} \gamma_k^2\, c\boldsymbol{\beta}_k + \gamma_k^4\, \partial_0\boldsymbol{\beta}_k \wedge \underline{\boldsymbol{\sigma}}_k) + \gamma_k^2 \underline{\boldsymbol{\sigma}}_k\}\, \mathrm{d}V$$
$$+ c^{-1} \sum_{k,l(k\neq l)} \int \boldsymbol{R} \wedge \{\boldsymbol{e}_l \wedge \boldsymbol{h}_k - \gamma_k^2\beta_k^2(\boldsymbol{p}_k \wedge \boldsymbol{b}_l - \boldsymbol{m}_k \wedge \boldsymbol{e}_l)$$
$$+ \gamma_k^2\boldsymbol{\beta}_k \wedge (\boldsymbol{p}_k \wedge \boldsymbol{e}_l + \boldsymbol{m}_k \wedge \boldsymbol{b}_l)$$
$$- \gamma_k^4(\boldsymbol{p}_k - \boldsymbol{\beta}_k \wedge \boldsymbol{m}_k)\cdot\boldsymbol{\Omega}_k^2\cdot(\boldsymbol{e}_l + \boldsymbol{\beta}_k \wedge \boldsymbol{b}_l)\boldsymbol{\beta}_k\}\, \mathrm{d}V, \tag{66}$$

where a partial integration has been performed. The first integrand contains the orbital angular momentum density $\boldsymbol{R}_k \wedge \rho_k^{t'} \gamma_k^2\, c\boldsymbol{\beta}_k$, the atomic angular momentum density $\gamma_k^2\underline{\boldsymbol{\sigma}}_k$ and a relativistic term in which $\underline{\boldsymbol{\sigma}}_k$ multiplied by the atomic acceleration occurs. Furthermore the second integrand contains the moment of the field momentum density with the leading term $\sum_{k,l(k\neq l)} c^{-1}\boldsymbol{R} \wedge (\boldsymbol{e}_l \wedge \boldsymbol{h}_k)$; the remaining terms vanish in the rest frame.

The equation (46) contains three global quantities, viz. $\int x^i\, t^{00}\, \mathrm{d}V$, $\boldsymbol{P}^i = c^{-1} \int t^{i0}\, \mathrm{d}V$ and $E = \int t^{00}\, \mathrm{d}V$. With the help of (55) and (62) we can write for the first of these, after partial integrations:

$$\int x^i\, t^{00}\, \mathrm{d}V = \sum_k \int [\, \boldsymbol{R}^i\{\rho_k^{t'}\, c^2\gamma_k^2 - c\gamma_k^4(\boldsymbol{\beta}_k \wedge \underline{\boldsymbol{\sigma}}_k)\cdot \partial_0\boldsymbol{\beta}_k\} + c\gamma_k^2(\boldsymbol{\beta}_k \wedge \underline{\boldsymbol{\sigma}}_k)^i]\, \mathrm{d}V$$
$$+ \sum_{k,l(k\neq l)} \int \boldsymbol{R}^i[\tfrac{1}{2}\boldsymbol{e}_k\cdot\boldsymbol{e}_l + \tfrac{1}{2}\boldsymbol{b}_k\cdot\boldsymbol{b}_l + \boldsymbol{p}_k\cdot\boldsymbol{e}_l - \gamma_k^2\boldsymbol{\beta}_k\cdot(\boldsymbol{p}_k \wedge \boldsymbol{b}_l - \boldsymbol{m}_k \wedge \boldsymbol{e}_l)$$
$$- \gamma_k^4(\boldsymbol{p}_k - \boldsymbol{\beta}_k \wedge \boldsymbol{m}_k)\cdot\boldsymbol{\Omega}_k^2\cdot(\boldsymbol{e}_l + \boldsymbol{\beta}_k \wedge \boldsymbol{b}_l)]\, \mathrm{d}V. \tag{67}$$

The total momentum becomes with the help of (57) and (64)

$$\boldsymbol{P} = c^{-1} \sum_k \int \{\rho_k^{t'}\, c^2\gamma_k^2\boldsymbol{\beta}_k + c\gamma_k^4(\partial_0\boldsymbol{\beta}_k \wedge \underline{\boldsymbol{\sigma}}_k)\}\mathrm{d}V$$
$$+ c^{-1} \sum_{k,l(k\neq l)} \int [\boldsymbol{e}_l \wedge \boldsymbol{h}_k - \gamma_k^2\beta_k^2(\boldsymbol{p}_k \wedge \boldsymbol{b}_l - \boldsymbol{m}_k \wedge \boldsymbol{e}_l)$$
$$+ \gamma_k^2\boldsymbol{\beta}_k \wedge (\boldsymbol{p}_k \wedge \boldsymbol{e}_l + \boldsymbol{m}_k \wedge \boldsymbol{b}_l)$$
$$- \gamma_k^4(\boldsymbol{p}_k - \boldsymbol{\beta}_k \wedge \boldsymbol{m}_k)\cdot\boldsymbol{\Omega}_k^2\cdot(\boldsymbol{e}_l + \boldsymbol{\beta}_k \wedge \boldsymbol{b}_l)\,\boldsymbol{\beta}_k]\, \mathrm{d}V. \tag{68}$$

The first integrand contains the material momentum density and a relativistic correction. The second integrand contains the field momentum density i.e. $c^{-1} \sum_{k,l(k\neq l)} \boldsymbol{e}_l \wedge \boldsymbol{h}_k$ as a leading term and a number of terms which vanish in the rest frame.

The total energy of the system gets the form:

$$E = \sum_k \int \{\rho_k^{t'}\, c^2\gamma_k^2 + c\gamma_k^4\boldsymbol{\beta}_k\cdot(\partial_0\boldsymbol{\beta}_k \wedge \underline{\boldsymbol{\sigma}}_k)\}\, \mathrm{d}V$$
$$+ \sum_{k,l(k\neq l)} \int [\tfrac{1}{2}\boldsymbol{e}_k\cdot\boldsymbol{e}_l + \tfrac{1}{2}\boldsymbol{b}_k\cdot\boldsymbol{b}_l + \boldsymbol{p}_k\cdot\boldsymbol{e}_l - \gamma_k^2\boldsymbol{\beta}_k\cdot(\boldsymbol{p}_k \wedge \boldsymbol{b}_l - \boldsymbol{m}_k \wedge \boldsymbol{e}_l)$$
$$- \gamma_k^4(\boldsymbol{p}_k - \boldsymbol{\beta}_k \wedge \boldsymbol{m}_k)\cdot\boldsymbol{\Omega}_k^2\cdot(\boldsymbol{e}_l + \boldsymbol{\beta}_k \wedge \boldsymbol{b}_l)]\, \mathrm{d}V. \tag{69}$$

The material part contains the mass energy and a relativistic correction.

The field part contains the leading terms $\frac{1}{2}\sum_{k,l(k\neq l)} (\boldsymbol{e}_k \cdot \boldsymbol{e}_l + \boldsymbol{b}_k \cdot \boldsymbol{b}_l)$ and a number of terms which disappear in the rest frame.

In this way the conservation laws of angular momentum (43) and (46) are completely specified.

§ 9. *Concluding remarks.* In a following paper we shall treat the macroscopic conservation laws of energy and momentum obtained by averaging the corresponding atomic laws. The macroscopic energy-momentum tensor which occurs in these laws is symmetric as a consequence of the symmetry of the corresponding atomic tensor. (In this connection it may be remarked that the *macroscopic* field and material energy-momentum tensors separately are not simply the average of the corresponding *atomic* energy-momentum tensors.) From the symmetry and conservation of the macroscopic energy-momentum tensor will follow the macroscopic conservation laws of angular momentum.

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