de Groot, S. R.
Suttorp, L. G.
1968

Physica 39
41–60

THE RELATIVISTIC ENERGY-MOMENTUM TENSOR IN POLARIZED MEDIA

IV. THE MACROSCOPIC MATERIAL ENERGY-MOMENTUM TENSOR*)

by S. R. DE GROOT and L. G. SUTTORP

Instituut voor Theoretische Fysica, Universiteit van Amsterdam, Amsterdam, Nederland

Synopsis

The relativistic material energy-momentum tensor in polarized media, which contains bulk terms, fluctuation terms and correlation terms, is studied especially in the nonrelativistic limit. On this basis explicit forms of the energy-momentum and angular momentum laws are given. Contributions quadratic in the polarizations are proved to be contained in the correlation terms. For this reason a redefinition of the field and material part of the total energy-momentum tensor may be introduced.

§ 1. *Introduction.* In the preceding paper[1]) the total macroscopic energy-momentum tensor $T^{\alpha\beta}$ of a polarized medium in the presence of electromagnetic fields has been derived in terms of atomic parameters. It satisfies the conservation law of energy-momentum

$$\partial_\beta T^{\alpha\beta} = 0 \qquad (\alpha = 0, 1, 2, 3) \tag{1}$$

and it is symmetric

$$T^{\alpha\beta} = T^{\beta\alpha}. \tag{2}$$

The conservation law of angular momentum

$$\partial_\gamma(x^\alpha T^{\beta\gamma} - x^\beta T^{\alpha\gamma}) = 0 \tag{3}$$

follows from (1) and (2).

A part of $T^{\alpha\beta}$, which contains explicitly the macroscopic fields and polarizations, was called the macroscopic field energy-momentum tensor $T^{\alpha\beta}_{(f)}$; one has

$$T^{\alpha\beta} = T^{\alpha\beta}_{(f)} + T^{\alpha\beta}_{(m)}, \tag{4}$$

where the second contribution was called the macroscopic material energy-

*) Articles I–III appeared in Physica **37** (1967) 284, 297; **39** (1968) 28.

momentum tensor $T^{\alpha\beta}_{(m)}$. It consists of two contributions

$$T^{\alpha\beta}_{(m)} = T^{\alpha\beta}_{(m)1} + T^{\alpha\beta}_{(m)2}, \tag{5}$$

where the first arose from the averaging of the atomic material tensor $t^{\alpha\beta}_{(m)}$, and the second from (part of) the atomic field tensor $t^{\alpha\beta}_{(f)}$. They were found to be

$$\begin{aligned} T^{\alpha\beta}_{(m)1} = \langle t^{\alpha\beta}_{(m)} \rangle = \\ &= \varrho^{t'} U^\alpha U^\beta - \tfrac{1}{2}\Delta^\alpha_\varepsilon \Delta^\beta_\zeta \partial_\gamma(\Sigma^{\varepsilon\zeta} U^\gamma) + \tfrac{1}{2}c^{-2}(U^\alpha \Sigma^{\beta\gamma} DU_\gamma + U^\beta \Sigma^{\alpha\gamma} DU_\gamma) \\ &+ \tfrac{1}{2}\partial_\gamma(\Sigma^{\alpha\gamma} U^\beta + \Sigma^{\beta\gamma} U^\alpha) \\ &+ \int [\rho_1^{t'}(u_1^\alpha u_1^\beta - U^\alpha U^\beta) - \tfrac{1}{2}\Delta^\alpha_{1\varepsilon}\Delta^\beta_{1\zeta}\partial_\gamma(\sigma_1^{\varepsilon\zeta} u_1^\gamma) + \tfrac{1}{2}\Delta^\alpha_\varepsilon \Delta^\beta_\zeta \partial_\gamma(\sigma_1^{\varepsilon\zeta} U^\gamma) \\ &+ \tfrac{1}{2}c^{-2}\{u_1^\alpha D_1 u_{1\gamma} - U^\alpha DU_\gamma)\, \sigma_1^{\beta\gamma} + (u_1^\beta D_1 u_{1\gamma} - U^\beta DU_\gamma)\, \sigma_1^{\alpha\gamma}\} \\ &+ \tfrac{1}{2}\partial_\gamma\{\sigma_1^{\alpha\gamma}(u_1^\beta - U^\beta) + \sigma_1^{\beta\gamma}(u_1^\alpha - U^\alpha)\}]\, f_1^{\text{ret}}(1\,; \boldsymbol{R}, t)\, \mathrm{d}V_1, \end{aligned} \tag{6}$$

$$\begin{aligned} T^{\alpha\beta}_{(m)2} = \langle t^{\alpha\beta}_{(f)} \rangle - T^{\alpha\beta}_{(f)} = \\ &= c^{-2}F^{\alpha\gamma} \int (u_1^\beta u_1^\varepsilon - U^\beta U^\varepsilon)\, m_{1\gamma\varepsilon} f_1^{\text{ret}}(1\,; \boldsymbol{R}, t)\, \mathrm{d}V_1 \\ &- c^{-2}F_{\gamma\varepsilon} \int (u_1^\beta u_1^\varepsilon - U^\beta U^\varepsilon)\, m_1^{\alpha\gamma} f_1^{\text{ret}}(1\,; \boldsymbol{R}, t)\, \mathrm{d}V_1 \\ &- c^{-4}F_{\gamma\varepsilon} \int (u_1^\alpha u_1^\beta u_1^\gamma u_{1\zeta} - U^\alpha U^\beta U^\gamma U_\zeta)\, m_1^{\varepsilon\zeta} f_1^{\text{ret}}(1\,; \boldsymbol{R}, t)\, \mathrm{d}V_1 \\ &+ \int \{f_2^{\alpha\gamma} h_1{}^\beta{}_{\cdot\gamma} - \tfrac{1}{4} f_{1\gamma\varepsilon} f_2^{\gamma\varepsilon} g^{\alpha\beta} + c^{-2} u_1^\beta (f_2^{\alpha\gamma} m_{1\gamma\varepsilon} - m_1^{\alpha\gamma} f_{2\gamma\varepsilon})\, u_1^\varepsilon \\ &- c^{-4} u_1^\alpha u_1^\beta u_1^\gamma f_{2\gamma\varepsilon} m_1^{\varepsilon\zeta} u_{1\zeta}\}\, c_2^{\text{ret}}(1, 2;\, \boldsymbol{R}, t)\, \mathrm{d}V_1\, \mathrm{d}V_2, \end{aligned} \tag{7}$$

where $\rho_1^{t'}$, u_1^α and $D_1 u_1^\alpha$ are the mass density, four-velocity and four-acceleration of atom 1, $\Delta^\alpha_{1\beta} = \delta^\alpha_\beta + c^{-2}u_1^\alpha u_{1\beta}$, $\sigma_1^{\alpha\beta}$ is the atomic internal angular momentum density, $f_1^{\alpha\beta}$ and $h_1^{\alpha\beta}$ the atomic fields due to atom 1, and $m_1^{\alpha\beta}$ the atomic polarization tensor. Furthermore $\varrho^{t'}$, U^α and DU^α are the bulk mass-density, four-velocity and four-acceleration respectively, $\Delta^\alpha_\beta = \delta^\alpha_\beta + c^{-2}U^\alpha U_\beta$, $\Sigma^{\alpha\beta}$ the bulk internal angular momentum density and $F^{\alpha\beta}$ the macroscopic field. The retarded one-point distribution function $f_1^{\text{ret}}(1\,; \boldsymbol{R}, t)$ is defined in such a way that $f_1^{\text{ret}}(1\,; \boldsymbol{R}, t)\, \mathrm{d}V_1$ is the probability that a sphere shrinking with the speed of light towards the space-time point $\boldsymbol{R}, t$ encounters an atom with certain values of its position, velocity, acceleration, etc., and of its internal parameters within the element $\mathrm{d}V_1$. Finally $c_2^{\text{ret}}(1, 2;\, \boldsymbol{R}, t)$ is the retarded two-point correlation function.

Introduction $F^\alpha \equiv -\partial_\beta T^{\alpha\beta}_{(f)}$, we may write (1), using (4), as

$$\partial_\beta T^{\alpha\beta}_{(m)} = F^\alpha, \tag{8}$$

which shows that F^α is the ponderomotive force density (including the Lorentz force) exerted by the field on the medium. Similarly (3) becomes with (4)

$$\partial_\gamma(x^\alpha T^{\beta\gamma}_{(m)} - x^\beta T^{\alpha\gamma}_{(m)}) = x^\alpha F^\beta - x^\beta F^\alpha + T^{\alpha\beta}_{(f)} - T^{\beta\alpha}_{(f)}, \tag{9}$$

which is the balance of angular momentum. It may be written alternatively as

$$x^\alpha \partial_\gamma T^{\beta\gamma}_{(m)} - x^\beta \partial_\gamma T^{\alpha\gamma}_{(m)} + T^{\beta\alpha}_{(m)} - T^{\alpha\beta}_{(m)} = x^\alpha F^\beta - x^\beta F^\alpha + T^{\alpha\beta}_{(f)} - T^{\beta\alpha}_{(f)}. \quad (10)$$

The purpose of this paper is to investigate and discuss the content of the material energy-momentum tensor $T^{\alpha\beta}_{(m)}$ given above, in particular the expressions $\partial_\beta T^{\alpha\beta}_{(m)}$ and $T^{\alpha\beta}_{(m)} - T^{\beta\alpha}_{(m)}$ which occur in the laws (8)–(10).

§ 2. *The tensor* $T^{\alpha\beta}_{(m)1}$. The part of the material energy-momentum tensor, which arises from the atomic material energy-momentum tensor, was called $T^{\alpha\beta}_{(m)1}$ and given in (6). It consists of bulk terms and of fluctuations terms, due to the fact that the atomic velocities and accelerations may be different from their mean (bulk) values. In the laws (8)–(10) the expressions $\partial_\beta T^{\alpha\beta}_{(m)1}$ and $T^{\alpha\beta}_{(m)1} - T^{\beta\alpha}_{(m)1}$ occur. The first of these becomes, with (6),

$$\begin{aligned}\partial_\beta T^{\alpha\beta}_{(m)1} = {} & \partial_\beta\{\varrho^{t'} U^\alpha U^\beta + c^{-2}\Sigma^{\alpha\gamma}(DU_\gamma)\, U^\beta\} \\ & + \partial_\beta \int [\rho_1^{t'}(u_1^\alpha u_1^\beta - U^\alpha U^\beta) \\ & + c^{-2}\sigma_1^{\alpha\gamma}\{(D_1 u_{1\gamma})\, u_1^\beta - (DU_\gamma)\, U^\beta\}]\, f_1^{\text{ret}}\,(1\,;\,\boldsymbol{R},t)\,\mathrm{d}V_1. \end{aligned} \quad (11)$$

Here the bulk part consists of the four-divergence of the mass density times the dyadic product of the macroscopic four-velocity components of the medium, and a relativistic correction term involving the macroscopic internal angular momentum density $\Sigma^{\alpha\beta}$. Similarly the fluctuation part contains a term with the mass density and a relativistic correction. Since $\rho_1^{t'}$ and $\sigma_1^{\alpha\gamma}$ contain as a factor $\delta(\boldsymbol{R}_1 - \boldsymbol{R})$ with $\boldsymbol{R}_1$ the position of atom 1, one may use the ordinary distribution function $f_1(1;\, t)$ instead of the retarded distribution function[1]).

Considering the atomic velocities and accelerations as sums of a bulk term and a fluctuation

$$u_1^\alpha = U^\alpha + \hat{u}_1^\alpha, \quad (12)$$

$$D_1 u_1^\alpha = DU^\alpha + D_1\,\hat{}u_1^\alpha, \quad (13)$$

one may write (11) as

$$\begin{aligned}\partial_\beta T^{\alpha\beta}_{(m)1} = {} & \partial_\beta\{\varrho^{t'} U^\alpha U^\beta + c^{-2}\Sigma^{\alpha\gamma}(DU_\gamma)\, U^\beta\} \\ & + \partial_\beta \int [\rho_1^{t'}(U^\alpha \hat{u}_1^\beta + \hat{u}_1^\alpha U^\beta + \hat{u}_1^\alpha \hat{u}_1^\beta) \\ & + c^{-2}\sigma_1^{\alpha\gamma}\{(DU_\gamma)\, \hat{u}_1^\beta + (D_1\,\hat{}u_{1\gamma})\, U^\beta + (D_1\,\hat{}u_{1\gamma})\, \hat{u}_1^\beta\}]\, f_1(1\,;t)\,\mathrm{d}V_1. \end{aligned} \quad (14)$$

It is of interest to study also the nonrelativistic approximation of (11). In (11) the atomic mass density $\rho_1^{t'}$ consists of the rest mass density ρ_1' and the mass density $\delta\rho_1'$ corresponding to the internal Coulomb and kinetic energies of the atom:

$$\rho_1^{t'} = \rho_1' + \delta\rho_1'. \qquad (15)$$

The macroscopic rest mass density ϱ' and the macroscopic four-velocity U^α are defined by

$$\int \rho_1' u_1^\alpha f_1(1\,;t)\,\mathrm{d}V_1 = \varrho' U^\alpha, \quad \text{with} \quad U^\alpha U_\alpha = -c^2, \qquad (16)$$

where the velocities have components $u_1^\alpha = (\gamma_1 c, \gamma_1 \boldsymbol{v}_1)$ and $U^\alpha = (\gamma c, \gamma \boldsymbol{v})$. From the cases $\alpha = 0$ and $\alpha = i = 1, 2, 3$ of (16) one obtains

$$\int \rho_1' \gamma_1 (v_1^i - v^i)\, f_1(1\,;t)\,\mathrm{d}V_1 = 0. \qquad (17)$$

This formula contains the atomic rest mass density in the $\boldsymbol{R}$, t-reference frame

$$\rho_1 = \rho_1' \gamma_1, \qquad (18)$$

of which the average ϱ – the macroscopic rest mass energy in the $\boldsymbol{R}$, t-frame — is

$$\varrho \equiv \int \rho_1 f_1(1\,;t)\,\mathrm{d}V_1 = \varrho'\gamma, \qquad (19)$$

as follows from (16) for $\alpha = 0$. It satisfies a conservation law

$$\partial_0 \varrho c + \boldsymbol{\nabla}\cdot(\varrho \boldsymbol{v}) = 0, \qquad (20)$$

as follows from the atomic conservation law $\partial_\alpha(\rho_1' u_1^\alpha) = 0$. Similarly we define

$$\delta\rho_1 = \delta\rho_1' \gamma_1, \qquad (21)$$

$$\delta\varrho = \int \delta\rho_1 f_1^{\mathrm{ret}}(1\,;t)\,\mathrm{d}V_1. \qquad (22)$$

In the nonrelativistic approximation we write $\gamma_1 \approx 1 + \frac{1}{2}v_1^2/c^2$ in (11) with (15), (16), (18), (19), (21) and (22). This yields

$$\partial_\beta T^{0\beta}_{(m)1} = \partial_0\{(\varrho + \delta\varrho)\,c^2 + \int \tfrac{1}{2}\rho_1 v_1^2 f_1(1\,;t)\,\mathrm{d}V_1\}$$
$$+ \boldsymbol{\nabla}\cdot[\varrho \boldsymbol{v} c + \int \{\tfrac{1}{2}\rho_1 v_1^2(\boldsymbol{v}_1/c) + \delta\rho_1 \boldsymbol{v}_1 c\}\, f_1(1\,;t)\,\mathrm{d}V_1], \qquad (23)$$

$$\partial_\beta T^{i\beta}_{(m)1} = \partial_0(\varrho v^i c) + \boldsymbol{\nabla}\cdot\int \rho_1 \boldsymbol{v}_1 v_1^i f_1(1\,;t)\,\mathrm{d}V_1, \qquad (24)$$

where terms up to order c^{-1} and c^0 have been included in (23) and (24) respectively. With a splitting of the velocity into a bulk and fluctuation part

$$\boldsymbol{v}_1 = \boldsymbol{v} + \hat{\boldsymbol{v}}_1 \qquad (25)$$

and with (17)–(22) one obtains

$$\partial_\beta T^{0\beta}_{(m)1} = \partial_0(\tfrac{1}{2}\varrho v^2 + \delta\varrho c^2 + \int \tfrac{1}{2}\rho_1 \hat{v}_1^2 f_1(1\,;t)\,\mathrm{d}V_1)$$
$$+ c^{-1}\boldsymbol{\nabla}\cdot[(\tfrac{1}{2}\varrho v^2 + \delta\varrho c^2)\,\boldsymbol{v}$$
$$+ \int \{\tfrac{1}{2}\rho_1(\hat{v}_1^2 \boldsymbol{v} + 2\boldsymbol{v}\cdot\hat{\boldsymbol{v}}_1\hat{\boldsymbol{v}}_1 + \hat{v}_1^2\hat{\boldsymbol{v}}_1) + \delta\rho_1 \hat{\boldsymbol{v}}_1 c^2\}\, f_1(1\,;t)\,\mathrm{d}V_1], \qquad (26)$$

$$\partial_\beta T^{i\beta}_{(m)1} = \partial_0(\varrho v^i c) + \boldsymbol{\nabla}\cdot\{\varrho \boldsymbol{v} v^i + \int \rho_1 \hat{\boldsymbol{v}}_1 \hat{v}_1^i f_1(1\,;t)\,\mathrm{d}V_1\}. \qquad (27)$$

The right-hand side of (26) contains the time derivative of the bulk kinetic energy density, the internal atomic energy density, the thermal kinetic energy density and furthermore a space derivative of energy transport terms. The right-hand side of (27) contains the derivative of the bulk momentum density and a space derivative of the sum of the bulk momentum transport and the kinetic pressure tensor.

The other expression, which occurs in the laws (8)–(10), is $T^{\alpha\beta}_{(m)1} - T^{\beta\alpha}_{(m)1}$. Its value follows from (6):

$$T^{\beta\alpha}_{(m)1} - T^{\beta\alpha}_{(m)1} = -\Delta^{\alpha}_{\varepsilon}\Delta^{\beta}_{\zeta}\partial_{\gamma}(\Sigma^{\varepsilon\zeta}U^{\gamma}) - \int \{\Delta^{\alpha}_{1\varepsilon}\Delta^{\beta}_{1\zeta}\partial_{\gamma}(\sigma^{\varepsilon\zeta}_{1}u^{\gamma}_{1}) - \Delta^{\alpha}_{\varepsilon}\Delta^{\beta}_{\zeta}\partial_{\gamma}(\sigma^{\varepsilon\zeta}_{1}U^{\gamma})\} f^{\text{ret}}_{1}(1\,; \boldsymbol{R}, t)\, \mathrm{d}V_{1}. \qquad (28)$$

In the nonrelativistic approximation up to order c^0 the space-space components of this expression are

$$T^{ij}_{(m)1} - T^{ji}_{(m)1} = -c\partial_0\Sigma^k - \boldsymbol{\nabla}\cdot(\boldsymbol{v}\Sigma^k + \int \hat{\boldsymbol{v}}_1\sigma^k_1 f_1(1\,; t)\,\mathrm{d}V_1) \quad (i, j, k = 1, 2, 3 \text{ cycl.}), \qquad (29)$$

where $\Sigma^k = \Sigma^{ij}$ $(i, j, k = 1, 2, 3$ cycl.) and where (25) has been used.

§ 3. *The tensor* $T^{\alpha\beta}_{(m)2}$. The part $T^{\alpha\beta}_{(m)2}$ (7) of the material energy-momentum tensor arose from the atomic field energy-momentum tensor. It contains three velocity fluctuation terms, in which one might introduce the splitting (12), and a correlation term. In the latter the retarded solutions of the atomic field equations for the atomic fields $f^{\alpha\beta}_1$ and $h^{\alpha\beta}_1$ should be inserted, as well as the expressions for the polarization tensor $m^{\alpha\beta}_1$ in terms of atomic parameters. In this way the relativistic expression (7) can be completely specified.

In the laws (8)–(10) the tensor $T^{\alpha\beta}_{(m)2}$ occurs only in the expressions $\partial_\beta T^{\alpha\beta}_{(m)2}$ and $T^{\alpha\beta}_{(m)2} - T^{\beta\alpha}_{(m)2}$, which we shall now investigate. The first of these expressions may be rewritten if use is made of the lemma[1]), according to which averaging and space-time differentiations of a physical quantity a commute:

$$\partial_\beta\langle a\rangle = \langle\partial_\beta a\rangle, \qquad (30)$$

and of the field equations[2]) for the partial atomic fields $f^{\alpha\beta}_k$ and $h^{\alpha\beta}_k \equiv f^{\alpha\beta}_k - m^{\alpha\beta}_k$:

$$\partial_\beta h^{\alpha\beta}_k = c^{-1}j^{\alpha}_k, \qquad (31)$$

$$\partial^\alpha f^{\beta\gamma}_k + \partial^\beta f^{\alpha\gamma}_k + \partial^\gamma f^{\alpha\beta}_k = 0, \qquad (32)$$

where j^{α}_k is the atomic four-current, due to atom k. Then one obtains from (7):

$$\begin{aligned}\partial_\beta T^{\alpha\beta}_{(m)2} = \partial_\beta[&c^{-2}F^{\alpha\gamma}\int (u^{\beta}_1 u^{\varepsilon}_1 - U^{\beta}U^{\varepsilon})\, m_{1\gamma\varepsilon} f_1(1\,; t)\,\mathrm{d}V_1 \\ &- c^{-2}F_{\gamma\varepsilon}\int (u^{\beta}_1 u^{\varepsilon}_1 - U^{\beta}U^{\varepsilon})\, m^{\alpha\gamma}_1 f_1(1\,; t)\,\mathrm{d}V_1 \\ &- c^{-4}F_{\gamma\varepsilon}\int (u^{\alpha}_1 u^{\beta}_1 u^{\gamma}_1 u_{1\zeta} - U^{\alpha}U^{\beta}U^{\gamma}U_{\zeta})\, m^{\varepsilon\zeta}_1 f_1(1\,; t)\,\mathrm{d}V_1 \\ &+ \int \{c^{-2}u^{\beta}_1(f^{\alpha\gamma}_2 m_{1\gamma\varepsilon} - m^{\alpha\gamma}_1 f_{2\gamma\varepsilon})\, u^{\varepsilon}_1 - c^{-4}u^{\alpha}_1 u^{\beta}_1 u^{\gamma}_1 f_{2\gamma\varepsilon} m^{\varepsilon\zeta}_1 u_{1\zeta}\} \\ &c^{\text{ret}}_2(1, 2;\, \boldsymbol{R}, t)\,\mathrm{d}V_1\,\mathrm{d}V_2] \\ &- \int \{c^{-1}f^{\alpha\beta}_2 j_{1\beta} + \tfrac{1}{2}(\partial^\alpha f_{2\beta\gamma})\, m^{\beta\gamma}_1\}\, c^{\text{ret}}_2(1, 2;\, \boldsymbol{R}, t)\,\mathrm{d}V_1\,\mathrm{d}V_2. \qquad (33)\end{aligned}$$

Since the atomic polarization tensor $m_1^{\alpha\beta}$ contains a delta-function, we could use the ordinary distribution function $f_1(1;t)$.

The nonrelativistic approximation of (33), including terms up to order c^{-1} in $\partial_\beta T^{0\beta}_{(m)2}$ and terms up to order c^0 in $\partial_\beta T^{i\beta}_{(m)2}$ $(i = 1, 2, 3)$, will now be studied. Then expressions for the atomic current, polarization and field are needed up to order c^0 only. For atoms carrying charges e_k, electric dipoles $\boldsymbol{\mu}_k$ and magnetic dipoles*) $\boldsymbol{\nu}_k$, the atomic charge density $\rho_k^e = c^{-1}j_k^0$, current density $\boldsymbol{j}_k = (j_k^1, j_k^2, j_k^3)$, electric and magnetic polarization vectors $\boldsymbol{p}_k = -(m_k^{01}, m_k^{02}, m_k^{03})$ and $\boldsymbol{m}_k = (m_k^{23}, m_k^{31}, m_k^{12})$ are of the form

$$\rho_k^e = e_k\delta(\boldsymbol{R}_k - \boldsymbol{R}), \tag{34}$$

$$\boldsymbol{j}_k = e_k\boldsymbol{v}_k\delta(\boldsymbol{R}_k - \boldsymbol{R}), \tag{35}$$

$$\boldsymbol{p}_k = \boldsymbol{\mu}_k\delta(\boldsymbol{R}_k - \boldsymbol{R}), \tag{36}$$

$$\boldsymbol{m}_k = \boldsymbol{\nu}_k\delta(\boldsymbol{R}_k - \boldsymbol{R}). \tag{37}$$

These quantities are the sources of the atomic field equations (31), (32) of which the solutions are – up to order c^0 –

$$\boldsymbol{e}_k = (-e_k\boldsymbol{\nabla} + \boldsymbol{\mu}_k\cdot\boldsymbol{\nabla}\boldsymbol{\nabla})\frac{1}{4\pi\,|\boldsymbol{R} - \boldsymbol{R}_k|}, \tag{38}$$

$$\boldsymbol{b}_k = -\tfrac{1}{2}\boldsymbol{\nabla}\wedge\left\{(\boldsymbol{\nu}_k\wedge\boldsymbol{\nabla})\cdot\frac{\mathbf{T}(\boldsymbol{R} - \boldsymbol{R}_k)}{4\pi\,|\boldsymbol{R} - \boldsymbol{R}_k|}\right\}, \tag{39}$$

where $\mathbf{T}(\boldsymbol{s}) = \mathbf{U} + \boldsymbol{ss}/s^2$ with $\mathbf{U}$ the unit tensor.

We consider now the velocity fluctuation terms in $T^{\alpha\beta}_{(m)2}$ (7). We shall call the sum of these three terms $T^{\alpha\beta}_{(m)21}$. With (25), (36) and (37) and $\boldsymbol{E} = (F^{01}, F^{02}, F^{03})$ one finds for its divergence in the nonrelativistic approximation

$$\partial_\beta T^{0\beta}_{(m)21} = -\boldsymbol{\nabla}\cdot\int(\dot{\boldsymbol{v}}_1/c)\,\boldsymbol{E}\cdot\boldsymbol{\mu}_1 f_1(\boldsymbol{R}, \boldsymbol{v}_1, \boldsymbol{\mu}_1; t)\,\mathrm{d}\boldsymbol{v}_1\,\mathrm{d}\boldsymbol{\mu}_1, \tag{40}$$

$$\partial_\beta T^{i\beta}_{(m)} = 0 \qquad (i = 1, 2, 3), \tag{41}$$

where terms up to order c^{-1} and c^0 have been included in (40) and (41) respectively, and where f_1 is the ordinary (non-retarded) distribution function. The other term of (7) is the correlation term which we call $T^{\alpha\beta}_{(m)22}$. Its divergence is

$$\begin{aligned}\partial_\beta T^{0\beta}_{(m)22} = &-\partial_0\int\boldsymbol{p}_1\cdot\boldsymbol{e}_2 c_2(1, 2; t)\,\mathrm{d}V_1\,\mathrm{d}V_2\\ &-c^{-1}\boldsymbol{\nabla}\cdot\int\boldsymbol{v}_1\boldsymbol{p}_1\cdot\boldsymbol{e}_2 c_2(1, 2; t)\,\mathrm{d}V_1\,\mathrm{d}V_2\\ &-\int(c^{-1}\boldsymbol{j}_1\cdot\boldsymbol{e}_2 - \boldsymbol{p}_1\cdot\partial_0\boldsymbol{e}_2 - \boldsymbol{m}_1\cdot\partial_0\boldsymbol{b}_2)\,c_2(1, 2; t)\,\mathrm{d}V_1\,\mathrm{d}V_2,\end{aligned} \tag{42}$$

$$\partial_\beta T^{i\beta}_{(m)22} = -\int\{\rho_1^e e_2^i + (\boldsymbol{\nabla}^i\boldsymbol{e}_2)\cdot\boldsymbol{p}_1 + (\boldsymbol{\nabla}^i\boldsymbol{b}_2)\cdot\boldsymbol{m}_1\}\,c_2(1, 2; t)\,\mathrm{d}V_1\,\mathrm{d}V_2 \qquad (i = 1, 2, 3), \tag{43}$$

*) The electric and magnetic dipole moments are considered as parameters of order c^0, which characterize the atom, just as the atomic charge.

again up to order c^{-1} and c^0 respectively, and where $c_2(1,2;t)$ is the ordinary (non-retarded) correlation function. With (34)–(39) these expressions become

$$\partial_\beta T^{0\beta}_{(m)22} = \partial_0 \iint \left\{ \boldsymbol{\mu}_1 \cdot \boldsymbol{\nabla}(e_2 - \boldsymbol{\mu}_2 \cdot \boldsymbol{\nabla}) \frac{1}{4\pi |\boldsymbol{R} - \boldsymbol{R}_2|} \right\} \delta(\boldsymbol{R} - \boldsymbol{R}_1)\, c_2(1,2;t)\, \mathrm{d}V_1 \mathrm{d}V_2$$
$$+ c^{-1}\boldsymbol{\nabla} \cdot \int \boldsymbol{v}_1 \left\{ \boldsymbol{\mu}_1 \cdot \boldsymbol{\nabla}(e_2 - \boldsymbol{\mu}_2 \cdot \boldsymbol{\nabla}) \frac{1}{4\pi |\boldsymbol{R} - \boldsymbol{R}_2|} \right\} \delta(\boldsymbol{R} - \boldsymbol{R}_1)\, c_2(1,2;t)\, \mathrm{d}V_1 \mathrm{d}V_2$$
$$+ c^{-1} \int e_1 e_2 \left(\boldsymbol{v}_1 \cdot \boldsymbol{\nabla} \frac{1}{4\pi |\boldsymbol{R} - \boldsymbol{R}_2|} \right) \delta(\boldsymbol{R} - \boldsymbol{R}_1)\, c_2(1,2;t)\, \mathrm{d}V_1 \mathrm{d}V_2$$
$$- c^{-1} \iint \left\{ (e_1 \boldsymbol{v}_1 \cdot \boldsymbol{\nabla} \boldsymbol{\mu}_2 \cdot \boldsymbol{\nabla} - e_2 \boldsymbol{v}_2 \cdot \boldsymbol{\nabla} \boldsymbol{\mu}_1 \cdot \boldsymbol{\nabla}) \frac{1}{4\pi |\boldsymbol{R} - \boldsymbol{R}_2|} \right\} \delta(\boldsymbol{R} - \boldsymbol{R}_1)\, c_2(1,2;t)\, \mathrm{d}V_1 \mathrm{d}V_2$$
$$- c^{-1} \iint \left\{ \boldsymbol{\mu}_1 \cdot \boldsymbol{\nabla}(\boldsymbol{\mu}_2 \cdot \boldsymbol{\nabla} \boldsymbol{v}_2 \cdot \boldsymbol{\nabla} - \dot{\boldsymbol{\mu}}_2 \cdot \boldsymbol{\nabla}) \frac{1}{4\pi |\boldsymbol{R} - \boldsymbol{R}_2|} \right\} \delta(\boldsymbol{R} - \boldsymbol{R}_1)\, c_2(1,2;t)\, \mathrm{d}V_1 \mathrm{d}V_2$$
$$- \tfrac{1}{2} c^{-1} \iint \left\{ (\boldsymbol{\nu}_1 \wedge \boldsymbol{\nabla})(\dot{\boldsymbol{\nu}}_2 \wedge \boldsymbol{\nabla} - \boldsymbol{v}_2 \cdot \boldsymbol{\nabla} \boldsymbol{\nu}_2 \wedge \boldsymbol{\nabla}) : \frac{\mathsf{T}(\boldsymbol{R} - \boldsymbol{R}_2)}{4\pi |\boldsymbol{R} - \boldsymbol{R}_2|} \right\}$$
$$\delta(\boldsymbol{R} - \boldsymbol{R}_1)\, c_2(1,2;t)\, \mathrm{d}V_1 \mathrm{d}V_2, \tag{44}$$

$$\partial_\beta T^{i\beta}_{(m)22} = \int e_1 e_2 \left(\nabla^i \frac{1}{4\pi |\boldsymbol{R} - \boldsymbol{R}_2|} \right) \delta(\boldsymbol{R} - \boldsymbol{R}_1)\, c_2(1,2;t)\, \mathrm{d}V_1 \mathrm{d}V_2$$
$$- \int \left\{ \nabla^i (e_1 \boldsymbol{\mu}_2 \cdot \boldsymbol{\nabla} - e_2 \boldsymbol{\mu}_1 \cdot \boldsymbol{\nabla}) \frac{1}{4\pi |\boldsymbol{R} - \boldsymbol{R}_2|} \right\} \delta(\boldsymbol{R} - \boldsymbol{R}_1)\, c_2(1,2;t)\, \mathrm{d}V_1 \mathrm{d}V_2$$
$$- \int \left((\nabla^i \boldsymbol{\mu}_1 \cdot \boldsymbol{\nabla} \boldsymbol{\mu}_2 \cdot \boldsymbol{\nabla} \frac{1}{4\pi |\boldsymbol{R} - \boldsymbol{R}_2|} \right) \delta(\boldsymbol{R} - \boldsymbol{R}_1)\, c_2(1,2;t)\, \mathrm{d}V_1 \mathrm{d}V_2$$
$$+ \tfrac{1}{2} \int \left\{ \nabla^i (\boldsymbol{\nu}_1 \wedge \boldsymbol{\nabla})(\boldsymbol{\nu}_2 \wedge \boldsymbol{\nabla}) : \frac{\mathsf{T}(\boldsymbol{R} - \boldsymbol{R}_2)}{4\pi |\boldsymbol{R} - \boldsymbol{R}_2|} \right\} \delta(\boldsymbol{R} - \boldsymbol{R}_1)\, c_2(1,2;t)\, \mathrm{d}V_1 \mathrm{d}V_2. \tag{45}$$

Both these expressions contain in the integrands a delta function and a correlation distribution function. The integration over one of the particle coordinates can be carried out, *e.g.*:

$$\int \delta(\boldsymbol{R} - \boldsymbol{R}_1)\, c_2(1,2;t)\, \mathrm{d}\boldsymbol{R}_1 =$$
$$= c_2(\boldsymbol{R}_1 = \boldsymbol{R}, \boldsymbol{R}_2, \boldsymbol{v}_1, \boldsymbol{v}_2, e_1, e_2, \boldsymbol{\mu}_1, \boldsymbol{\mu}_2, \dot{\boldsymbol{\mu}}_1, \dot{\boldsymbol{\mu}}_2, \boldsymbol{\nu}_1, \boldsymbol{\nu}_2, \dot{\boldsymbol{\nu}}_1, \dot{\boldsymbol{\nu}}_2; t). \tag{46}$$

(Due to integrations correlation functions with a smaller number of variables will also occur in the following). The right-hand side will be written as $c_2(\boldsymbol{R}, \boldsymbol{R} - \boldsymbol{s}, 1, 2; t)$ where the relative coordinate $\boldsymbol{s} = \boldsymbol{R}_1 - \boldsymbol{R}_2$ has been introduced, and where the arguments 1 and 2 in c_2 now indicate the atomic parameters, apart from the positions. Let us consider systems where the correlation function diminishes rapidly if $\boldsymbol{s}$ becomes of the order of a distance over which the macroscopic properties change appreciably. This situation is frequently realized in fluid systems. It is formally expressed by neglecting terms of order s^2 and higher in the Taylor expansion of the correlation function:

$$c_2(\boldsymbol{R}, \boldsymbol{R}-\boldsymbol{s}, 1, 2; t) = c_2(\boldsymbol{R}+\tfrac{1}{2}\boldsymbol{s}, \boldsymbol{R}-\tfrac{1}{2}\boldsymbol{s}, 1, 2; t) - \tfrac{1}{2}\boldsymbol{s}\cdot\boldsymbol{\nabla} c_2(\boldsymbol{R}+\tfrac{1}{2}\boldsymbol{s}, \boldsymbol{R}-\tfrac{1}{2}\boldsymbol{s}, 1, 2; t), \tag{47}$$

where $\boldsymbol{\nabla} = \partial/\partial\boldsymbol{R}$. This formula constitutes the Irving-Kirkwood approximation[3]). With this relation and the conservation law for correlation functions

$$\begin{aligned}(\partial/\partial t)\, c_2(\boldsymbol{R}+\tfrac{1}{2}\boldsymbol{s}, \boldsymbol{R}-\tfrac{1}{2}\boldsymbol{s}, e_1, e_2, \boldsymbol{\mu}_1, \boldsymbol{\mu}_2, \boldsymbol{\nu}_1, \boldsymbol{\nu}_2; t) =\\ = -\int \{\tfrac{1}{2}(\boldsymbol{v}_1+\boldsymbol{v}_2)\cdot\boldsymbol{\nabla} + (\boldsymbol{v}_1-\boldsymbol{v}_2)\cdot\boldsymbol{\nabla}_s + \dot{\boldsymbol{\mu}}_1\cdot\boldsymbol{\nabla}_{\mu_1}\\ + \dot{\boldsymbol{\mu}}_2\cdot\boldsymbol{\nabla}_{\mu_2} + \dot{\boldsymbol{\nu}}_1\cdot\boldsymbol{\nabla}_{\nu_1} + \dot{\boldsymbol{\nu}}_2\cdot\boldsymbol{\nabla}_{\nu_2}\}\\ c_2(\boldsymbol{R}+\tfrac{1}{2}\boldsymbol{s}, \boldsymbol{R}-\tfrac{1}{2}\boldsymbol{s}, \boldsymbol{v}_1, \boldsymbol{v}_2, e_1, e_2, \boldsymbol{\mu}_1, \boldsymbol{\mu}_2, \dot{\boldsymbol{\mu}}_1, \dot{\boldsymbol{\mu}}_2, \boldsymbol{\nu}_1, \boldsymbol{\nu}_2, \dot{\boldsymbol{\nu}}_1, \dot{\boldsymbol{\nu}}_2; t)\\ \mathrm{d}\boldsymbol{v}_1\, \mathrm{d}\boldsymbol{v}_2\, \mathrm{d}\dot{\boldsymbol{\mu}}_1\, \mathrm{d}\dot{\boldsymbol{\mu}}_2\, \mathrm{d}\dot{\boldsymbol{\nu}}_1\, \mathrm{d}\dot{\boldsymbol{\nu}}_2,\end{aligned} \tag{48}$$

one obtains for (44)

$$\begin{aligned}\partial_\beta T^{0\beta}_{(m)22} = \partial_0 \Bigg[\sum_{e_1, e_2} \int \frac{e_1 e_2}{8\pi s}\, c_2(\boldsymbol{R}+\tfrac{1}{2}\boldsymbol{s}, \boldsymbol{R}-\tfrac{1}{2}\boldsymbol{s}, e_1, e_2; t)\, \mathrm{d}\boldsymbol{s}\\ - \sum_{e_1} \int \left(\boldsymbol{\mu}_2\cdot\boldsymbol{\nabla}_s \frac{e_1}{4\pi s}\right) c_2(\boldsymbol{R}+\tfrac{1}{2}\boldsymbol{s}, \boldsymbol{R}-\tfrac{1}{2}\boldsymbol{s}, e_1, \boldsymbol{\mu}_2; t)\, \mathrm{d}\boldsymbol{s}\, \mathrm{d}\boldsymbol{\mu}_2\\ - \int \left(\boldsymbol{\mu}_1\cdot\boldsymbol{\nabla}_s \boldsymbol{\mu}_2\cdot\boldsymbol{\nabla}_s \frac{1}{8\pi s}\right) c_2(\boldsymbol{R}+\tfrac{1}{2}\boldsymbol{s}, \boldsymbol{R}-\tfrac{1}{2}\boldsymbol{s}, \boldsymbol{\mu}_1, \boldsymbol{\mu}_2; t)\, \mathrm{d}\boldsymbol{s}\, \mathrm{d}\boldsymbol{\mu}_1\, \mathrm{d}\boldsymbol{\mu}_2\\ - \int \left\{(\boldsymbol{\nu}_1 \wedge \boldsymbol{\nabla}_s)(\boldsymbol{\nu}_2 \wedge \boldsymbol{\nabla}_s) : \frac{\mathsf{T}(\boldsymbol{s})}{16\pi s}\right\} c_2(\boldsymbol{R}+\tfrac{1}{2}\boldsymbol{s}, \boldsymbol{R}-\tfrac{1}{2}\boldsymbol{s}, \boldsymbol{\nu}_1, \boldsymbol{\nu}_2; t)\, \mathrm{d}\boldsymbol{s}\, \mathrm{d}\boldsymbol{\nu}_1\, \mathrm{d}\boldsymbol{\nu}_2 \Bigg]\\ + \frac{1}{c}\boldsymbol{\nabla}\cdot\left(\sum_{e_1, e_2} \int \frac{e_1 e_2 \mathsf{T}(\boldsymbol{s})}{16\pi s}\cdot(\boldsymbol{v}_1+\boldsymbol{v}_2)\, c_2(\boldsymbol{R}+\tfrac{1}{2}\boldsymbol{s}, \boldsymbol{R}-\tfrac{1}{2}\boldsymbol{s}, \boldsymbol{v}_1, \boldsymbol{v}_2, e_1, e_2; t)\, \mathrm{d}\boldsymbol{s}\, \mathrm{d}\boldsymbol{v}_1\, \mathrm{d}\boldsymbol{v}_2\right.\\ + \sum_{e_1} \int \left[\{\boldsymbol{s}\boldsymbol{\mu}_2\cdot\boldsymbol{\nabla}_s(\boldsymbol{v}_1+\boldsymbol{v}_2)\cdot\boldsymbol{\nabla}_s - (\boldsymbol{v}_1+\boldsymbol{v}_2)\, \boldsymbol{\mu}_2\cdot\boldsymbol{\nabla}_s - \boldsymbol{s}\dot{\boldsymbol{\mu}}_2\cdot\boldsymbol{\nabla}_s\} \frac{e_1}{8\pi s}\right]\\ c_2(\boldsymbol{R}+\tfrac{1}{2}\boldsymbol{s}, \boldsymbol{R}-\tfrac{1}{2}\boldsymbol{s}, \boldsymbol{v}_1, \boldsymbol{v}_2, e_1, \boldsymbol{\mu}_2, \dot{\boldsymbol{\mu}}_2; t)\, \mathrm{d}\boldsymbol{s}\, \mathrm{d}\boldsymbol{v}_1\, \mathrm{d}\boldsymbol{v}_2\, \mathrm{d}\dot{\boldsymbol{\mu}}_2\\ + \int \Big[\{\boldsymbol{s}\boldsymbol{\mu}_1\cdot\boldsymbol{\nabla}_s \boldsymbol{\mu}_2\cdot\boldsymbol{\nabla}_s(\boldsymbol{v}_1+\boldsymbol{v}_2)\cdot\boldsymbol{\nabla}_s\\ - (\boldsymbol{v}_1+\boldsymbol{v}_2)\, \boldsymbol{\mu}_1\cdot\boldsymbol{\nabla}_s \boldsymbol{\mu}_2\cdot\boldsymbol{\nabla}_s - 2\boldsymbol{s}\boldsymbol{\mu}_1\cdot\boldsymbol{\nabla}_s \dot{\boldsymbol{\mu}}_2\cdot\boldsymbol{\nabla}_s\} \frac{1}{16\pi s}\Big]\\ c_2(\boldsymbol{R}+\tfrac{1}{2}\boldsymbol{s}, \boldsymbol{R}-\tfrac{1}{2}\boldsymbol{s}, \boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{\mu}_1, \boldsymbol{\mu}_2, \dot{\boldsymbol{\mu}}_2; t)\, \mathrm{d}\boldsymbol{s}\, \mathrm{d}\boldsymbol{v}_1\, \mathrm{d}\boldsymbol{v}_2\, \mathrm{d}\boldsymbol{\mu}_1\, \mathrm{d}\boldsymbol{\mu}_2\, \mathrm{d}\dot{\boldsymbol{\mu}}_2\end{aligned}$$

$$+\int\Bigg[\{-s(\boldsymbol{\nu}_1\wedge\boldsymbol{\nabla}_s)(\boldsymbol{\nu}_2\wedge\boldsymbol{\nabla}_s)(\boldsymbol{v}_1+\boldsymbol{v}_2)\cdot\boldsymbol{\nabla}_s-(\boldsymbol{v}_1+\boldsymbol{v}_2)(\boldsymbol{\nu}_1\wedge\boldsymbol{\nabla}_s)(\boldsymbol{\nu}_2\wedge\boldsymbol{\nabla}_s)$$

$$+2s(\boldsymbol{\nu}_1\wedge\boldsymbol{\nabla}_s)(\dot{\boldsymbol{\nu}}_2\wedge\boldsymbol{\nabla}_s)\}:\frac{\mathrm{T}(s)}{32\pi s}\Bigg]$$

$$c_2(\boldsymbol{R}+\tfrac{1}{2}\boldsymbol{s},\boldsymbol{R}-\tfrac{1}{2}\boldsymbol{s},\boldsymbol{v}_1,\boldsymbol{v}_2,\boldsymbol{\nu}_1,\boldsymbol{\nu}_2,\dot{\boldsymbol{\nu}}_2;t)\,\mathrm{d}\boldsymbol{s}\,\mathrm{d}\boldsymbol{v}_1\,\mathrm{d}\boldsymbol{v}_2\,\mathrm{d}\boldsymbol{\nu}_1\,\mathrm{d}\boldsymbol{\nu}_2\,\mathrm{d}\dot{\boldsymbol{\nu}}_2\Bigg), \quad (49)$$

where the sum over e_1 (and e_2) indicates that in a mixture the stable groups (atoms, molecules, ions) of which the various components consists may carry different charges. The right-hand side contains the time derivative of the correlation part of the energy density and the divergence of the correlation part of the energy flow. The energy density consists of four terms arising from charge-charge, charge-electric dipole, electric dipole-dipole and magnetic dipole-dipole interactions. Contributions of the same origin appear in the energy flow.

Expression (45) becomes

$$\partial_\beta T^{i\beta}_{(m)22}=\boldsymbol{\nabla}\cdot\Bigg[-\sum_{e_1,e_2}\int\left(s\boldsymbol{\nabla}^i_s\frac{e_1e_2}{8\pi s}\right)c_2(\boldsymbol{R}+\tfrac{1}{2}\boldsymbol{s},\boldsymbol{R}-\tfrac{1}{2}\boldsymbol{s},e_1,e_2;t)\,\mathrm{d}\boldsymbol{s}$$

$$+\sum_{e_1}\int\left(s\boldsymbol{\nabla}^i_s\boldsymbol{\mu}_2\cdot\boldsymbol{\nabla}_s\frac{e_1}{4\pi s}\right)c_2(\boldsymbol{R}+\tfrac{1}{2}\boldsymbol{s},\boldsymbol{R}-\tfrac{1}{2}\boldsymbol{s},e_1,\boldsymbol{\mu}_2;t)\,\mathrm{d}\boldsymbol{s}\,\mathrm{d}\boldsymbol{\mu}_2$$

$$+\int\left(s\boldsymbol{\nabla}^i_s\boldsymbol{\mu}_1\cdot\boldsymbol{\nabla}_s\boldsymbol{\mu}_2\cdot\boldsymbol{\nabla}_s\frac{1}{8\pi s}\right)c_2(\boldsymbol{R}+\tfrac{1}{2}\boldsymbol{s},\boldsymbol{R}-\tfrac{1}{2}\boldsymbol{s},\boldsymbol{\mu}_1,\boldsymbol{\mu}_2;t)\,\mathrm{d}\boldsymbol{s}\,\mathrm{d}\boldsymbol{\mu}_1\,\mathrm{d}\boldsymbol{\mu}_2$$

$$-\int\left\{s\boldsymbol{\nabla}^i_s(\boldsymbol{\nu}_1\wedge\boldsymbol{\nabla}_s)(\boldsymbol{\nu}_2\wedge\boldsymbol{\nabla}_s):\frac{\mathrm{T}(s)}{16\pi s}\right\}c_2(\boldsymbol{R}+\tfrac{1}{2}\boldsymbol{s},\boldsymbol{R}-\tfrac{1}{2}\boldsymbol{s},\boldsymbol{\nu}_1,\boldsymbol{\nu}_2;t)\,\mathrm{d}\boldsymbol{s}\,\mathrm{d}\boldsymbol{\nu}_1\,\mathrm{d}\boldsymbol{\nu}_2\Bigg]. \quad (50)$$

The right-hand side is the divergence of the correlation part of the pressure tensor. It consists again of four terms: a charge-charge contribution, which in ionized systems usually overshadows the other terms, a cross term due to charge-electric dipole interaction, an electric dipole-dipole term, in which the London-Van der Waals potential pressure is contained, and finally a magnetic dipole-dipole term.

Let us consider now the difference $T^{\alpha\beta}_{(m)2}-T^{\beta\alpha}_{(m)2}$. According to (7) it can be written as

$$T^{\alpha\beta}_{(m)2}-T^{\beta\alpha}_{(m)2}=F^{\gamma\varepsilon}\int(\Delta^\alpha_{1\gamma}\Delta^{\beta\zeta}_1-\Delta^\alpha_\gamma\Delta^{\beta\zeta}$$
$$-\Delta^\beta_{1\gamma}\Delta^{\alpha\zeta}_1+\Delta^\beta_\gamma\Delta^{\alpha\zeta})\,m_{1\varepsilon\zeta}\,f^{\mathrm{ret}}_1(1;\boldsymbol{R},t)\,\mathrm{d}V_1$$
$$+\int\Delta^\alpha_{1\gamma}\Delta^{\beta\zeta}_1(f^{\gamma\varepsilon}_2m_{1\varepsilon\zeta}-m^{\gamma\varepsilon}_1f_{2\varepsilon\zeta})\,c^{\mathrm{ret}}_2(1,2;\boldsymbol{R},t)\,\mathrm{d}V_1\,\mathrm{d}V_2. \quad (51)$$

Here again the nonrelativistic approximation of the space-space components is interesting. Up to order c^0 one finds

$$T^{ij}_{(m)2}-T^{ji}_{(m)2}=\int(\boldsymbol{p}_1\wedge\boldsymbol{e}_2+\boldsymbol{m}_1\wedge\boldsymbol{b}_2)^k\,c_2(1,2;t)\,\mathrm{d}V_1\,\mathrm{d}V_2, \quad (52)$$

where i, j, $k = 1$, 2, 3 and cyclically and where the polarizations and fields are given by (36)–(39). Inserting these expressions gives the result:

$$T^{ij}_{(m)2} - T^{ji}_{(m)2} =$$

$$= \sum_{e_1} \int \left(\boldsymbol{\mu}_2 \wedge \boldsymbol{\nabla}_s \frac{e_1}{4\pi s} \right)^k c_2(\boldsymbol{R}+\boldsymbol{s}, \boldsymbol{R}, e_1, \boldsymbol{\mu}_2; t)\, \mathrm{d}\boldsymbol{s}\, \mathrm{d}\boldsymbol{\mu}_2$$

$$+ \int \left(\boldsymbol{\mu}_1 \cdot \boldsymbol{\nabla}_s \boldsymbol{\mu}_2 \wedge \boldsymbol{\nabla}_s \frac{1}{4\pi s} \right)^k c_2(\boldsymbol{R}+\boldsymbol{s}, \boldsymbol{R}, \boldsymbol{\mu}_1, \boldsymbol{\mu}_2; t)\, \mathrm{d}\boldsymbol{s}\, \mathrm{d}\boldsymbol{\mu}_1\, \mathrm{d}\boldsymbol{\mu}_2$$

$$- \int \left(\boldsymbol{\nu}_2 \wedge \left[\boldsymbol{\nabla}_s \wedge \left\{ (\boldsymbol{\nu}_1 \wedge \boldsymbol{\nabla}_s) \cdot \frac{\mathsf{T}(s)}{8\pi s} \right\} \right] \right)^k c_2(\boldsymbol{R}+\boldsymbol{s}, \boldsymbol{R}, \boldsymbol{\nu}_1, \boldsymbol{\nu}_2; t)\, \mathrm{d}\boldsymbol{s}\, \mathrm{d}\boldsymbol{\nu}_1\, \mathrm{d}\boldsymbol{\nu}_2. \qquad (53)$$

The three terms are due to charge-electric dipole, electric dipole-dipole and magnetic dipole-dipole interaction respectively.

In this section we derived the nonrelativistic expressions (40), (41), (49), (50) and (53). From these formulae it can be inferred that actually it suffices that the relative atomic motion is nonrelativistic *within the correlation domain*, *i.e.*, for values of $|\boldsymbol{s}|$ such that $c_2(\boldsymbol{R} + \frac{1}{2}\boldsymbol{s}, \boldsymbol{R} - \frac{1}{2}\boldsymbol{s}, \ldots; t)$ and $c_2(\boldsymbol{R} + \boldsymbol{s}, \boldsymbol{R}, \ldots; t)$ are appreciably different from zero.

§ 4. *The energy-momentum and angular momentum laws.* In the first member of the energy-momentum conservation law (8) with (5) we can now substitute (14) and the divergence of (7). The second member was found[1]) to have the form (III. 59–61)

$$F^\alpha = c^{-1} F^{\alpha\beta} J_\beta + \tfrac{1}{2}(\partial^\alpha F^{\beta\gamma})\, M_{\beta\gamma}$$

$$-c^{-2} \varrho' D\{v'(F^{\alpha\gamma} M_{\gamma\varepsilon} - M^{\alpha\gamma} F_{\gamma\varepsilon})\, U^\varepsilon\} + c^{-4} \varrho' D(v' U^\alpha U^\beta F_{\beta\gamma} M^{\gamma\varepsilon} U_\varepsilon), \qquad (54)$$

where $F^{\alpha\beta}$ is the field tensor, J^α the four-current density, $M^{\alpha\beta}$ the polarization tensor, $v' = (\varrho')^{-1}$ and $D = U^\beta \partial_\beta$.

The nonrelativistic energy law is given by (c times)the component $\alpha = 0$ of (8) with (5), (26), (40), (49) and the nonrelativistic approximation of (54):

$$(\partial/\partial t)(\tfrac{1}{2}\varrho v^2 + \delta\varrho c^2 + \textstyle\int \tfrac{1}{2} m_1 \hat{v}_1^2 f_1(\boldsymbol{R}, \boldsymbol{v}_1; t)\, \mathrm{d}\boldsymbol{v}_1 + C_{\mathrm{I}})$$

$$+ \boldsymbol{\nabla} \cdot [(\tfrac{1}{2}\varrho v^2 + \delta\varrho c^2)\, \boldsymbol{v} + \textstyle\int \{\tfrac{1}{2} m_1 (\hat{v}_1^2 \boldsymbol{v} + 2\boldsymbol{v} \cdot \hat{\boldsymbol{v}}_1 \hat{\boldsymbol{v}}_1 + \hat{v}_1^2 \hat{\boldsymbol{v}}_1) + \delta m_1 c^2 \hat{\boldsymbol{v}}_1\}$$

$$f_1(\boldsymbol{R}, \boldsymbol{v}_1; t)\, \mathrm{d}\boldsymbol{v}_1 - \textstyle\iint \hat{\boldsymbol{v}}_1 \boldsymbol{E} \cdot \boldsymbol{\mu}_1 f_1(\boldsymbol{R}, \boldsymbol{v}_1, \boldsymbol{\mu}_1; t)\, \mathrm{d}\boldsymbol{v}_1\, \mathrm{d}\boldsymbol{\mu}_1 + C_{\mathrm{II}}] =$$

$$= \boldsymbol{J} \cdot \boldsymbol{E} + \varrho \boldsymbol{E} \cdot \frac{\mathrm{d}(v\boldsymbol{P})}{\mathrm{d}t} - \frac{\mathrm{d}\boldsymbol{B}}{\mathrm{d}t} \cdot \boldsymbol{M} + \boldsymbol{v} \cdot (\boldsymbol{\nabla} \boldsymbol{E}) \cdot \boldsymbol{P} + \boldsymbol{v} \cdot (\boldsymbol{\nabla} \boldsymbol{B}) \cdot \boldsymbol{M}, \qquad (55)$$

where $\mathrm{d}/\mathrm{d}t$ is the substantial time derivative $(\partial/\partial t) + \boldsymbol{v} \cdot \boldsymbol{\nabla}$ and where the quantities C_{I} and C_{II} are abbreviations of the correlation integrals:

$$C_{\mathrm{I}} = \sum_{e_1, e_2} \int \frac{e_1 e_2}{8\pi s} c_2(\boldsymbol{R} + \tfrac{1}{2}\boldsymbol{s}, \boldsymbol{R} - \tfrac{1}{2}\boldsymbol{s}, e_1, e_2; t)\, \mathrm{d}\boldsymbol{s}$$

$$- \sum_{e_1} \int \left(\boldsymbol{\mu}_2 \cdot \boldsymbol{\nabla}_s \frac{e_1}{4\pi s}\right) c_2(\boldsymbol{R} + \tfrac{1}{2}\boldsymbol{s}, \boldsymbol{R} - \tfrac{1}{2}\boldsymbol{s}, e_1, \boldsymbol{\mu}_2; t)\, \mathrm{d}\boldsymbol{s}\, \mathrm{d}\boldsymbol{\mu}_2$$

$$- \int \left(\boldsymbol{\mu}_1 \cdot \boldsymbol{\nabla}_s \boldsymbol{\mu}_2 \cdot \boldsymbol{\nabla}_s \frac{1}{8\pi s}\right) c_2(\boldsymbol{R} + \tfrac{1}{2}\boldsymbol{s}, \boldsymbol{R} - \tfrac{1}{2}\boldsymbol{s}, \boldsymbol{\mu}_1, \boldsymbol{\mu}_2; t)\, \mathrm{d}\boldsymbol{s}\, \mathrm{d}\boldsymbol{\mu}_1\, \mathrm{d}\boldsymbol{\mu}_2$$

$$- \int \left\{(\boldsymbol{\nu}_1 \wedge \boldsymbol{\nabla}_s)(\boldsymbol{\nu}_2 \wedge \boldsymbol{\nabla}_s) : \frac{\mathbf{T}(\boldsymbol{s})}{16\pi s}\right\} c_2(\boldsymbol{R} + \tfrac{1}{2}\boldsymbol{s}, \boldsymbol{R} - \tfrac{1}{2}\boldsymbol{s}, \boldsymbol{\nu}_1, \boldsymbol{\nu}_2; t)\, \mathrm{d}\boldsymbol{s}\, \mathrm{d}\boldsymbol{\nu}_1\, \mathrm{d}\boldsymbol{\nu}_2, \qquad (56)$$

$$C_{\mathrm{II}} = \sum_{e_1, e_2} \int \frac{e_1 e_2 \mathbf{T}(\boldsymbol{s})}{16\pi s} \cdot (\boldsymbol{v}_1 + \boldsymbol{v}_2)\, c_2(\boldsymbol{R} + \tfrac{1}{2}\boldsymbol{s}, \boldsymbol{R} - \tfrac{1}{2}\boldsymbol{s}, \boldsymbol{v}_1, \boldsymbol{v}_2, e_1, e_2; t)\, \mathrm{d}\boldsymbol{s}\, \mathrm{d}\boldsymbol{v}_1\, \mathrm{d}\boldsymbol{v}_2$$

$$+ \sum_{e_1} \int \left[\{\boldsymbol{s} \cdot \boldsymbol{\mu}_2 \cdot \boldsymbol{\nabla}_s (\boldsymbol{v}_1 + \boldsymbol{v}_2) \cdot \boldsymbol{\nabla}_s - (\boldsymbol{v}_1 + \boldsymbol{v}_2)\, \boldsymbol{\mu}_2 \cdot \boldsymbol{\nabla}_s - \boldsymbol{s} \dot{\boldsymbol{\mu}}_2 \cdot \boldsymbol{\nabla}_s\} \frac{e_1}{8\pi s}\right]$$

$$c_2(\boldsymbol{R} + \tfrac{1}{2}\boldsymbol{s}, \boldsymbol{R} - \tfrac{1}{2}\boldsymbol{s}, \boldsymbol{v}_1, \boldsymbol{v}_2, e_1, \boldsymbol{\mu}_2, \dot{\boldsymbol{\mu}}_2; t)\, \mathrm{d}\boldsymbol{s}\, \mathrm{d}\boldsymbol{v}_1\, \mathrm{d}\boldsymbol{v}_2\, \mathrm{d}\boldsymbol{\mu}_2\, \mathrm{d}\dot{\boldsymbol{\mu}}_2$$

$$+ \int \Big[\{\boldsymbol{s} \boldsymbol{\mu}_1 \cdot \boldsymbol{\nabla}_s \boldsymbol{\mu}_2 \cdot \boldsymbol{\nabla}_s (\boldsymbol{v}_1 + \boldsymbol{v}_2) \cdot \boldsymbol{\nabla}_s$$

$$- (\boldsymbol{v}_1 + \boldsymbol{v}_2)\, \boldsymbol{\mu}_1 \cdot \boldsymbol{\nabla}_s \boldsymbol{\mu}_2 \cdot \boldsymbol{\nabla}_s - 2\boldsymbol{s} \boldsymbol{\mu}_1 \cdot \boldsymbol{\nabla}_s \dot{\boldsymbol{\mu}}_2 \cdot \boldsymbol{\nabla}_s\} \frac{1}{16\pi s}\Big]$$

$$c_2(\boldsymbol{R} + \tfrac{1}{2}\boldsymbol{s}, \boldsymbol{R} - \tfrac{1}{2}\boldsymbol{s}, \boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{\mu}_1, \boldsymbol{\mu}_2, \dot{\boldsymbol{\mu}}_2; t)\, \mathrm{d}\boldsymbol{s}\, \mathrm{d}\boldsymbol{v}_1\, \mathrm{d}\boldsymbol{v}_2\, \mathrm{d}\boldsymbol{\mu}_1\, \mathrm{d}\boldsymbol{\mu}_2\, \mathrm{d}\dot{\boldsymbol{\mu}}_2$$

$$+ \int \Big[\{-\boldsymbol{s}(\boldsymbol{\nu}_1 \wedge \boldsymbol{\nabla}_s)(\boldsymbol{\nu}_2 \wedge \boldsymbol{\nabla}_s)(\boldsymbol{v}_1 + \boldsymbol{v}_2) \cdot \boldsymbol{\nabla}_s$$

$$- (\boldsymbol{v}_1 + \boldsymbol{v}_2)(\boldsymbol{\nu}_1 \wedge \boldsymbol{\nabla}_s)(\boldsymbol{\nu}_2 \wedge \boldsymbol{\nabla}_s) + 2\boldsymbol{s}(\boldsymbol{\nu}_1 \wedge \boldsymbol{\nabla}_s)(\dot{\boldsymbol{\nu}}_2 \wedge \boldsymbol{\nabla}_s)\} : \frac{\mathbf{T}(\boldsymbol{s})}{32\pi s}\Big]$$

$$c_2(\boldsymbol{R} + \tfrac{1}{2}\boldsymbol{s}, \boldsymbol{R} - \tfrac{1}{2}\boldsymbol{s}, \boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{\nu}_1, \boldsymbol{\nu}_2, \dot{\boldsymbol{\nu}}_2; t)\, \mathrm{d}\boldsymbol{s}\, \mathrm{d}\boldsymbol{v}_1\, \mathrm{d}\boldsymbol{v}_2\, \mathrm{d}\boldsymbol{\nu}_1\, \mathrm{d}\boldsymbol{\nu}_2\, \mathrm{d}\dot{\boldsymbol{\nu}}_2. \qquad (57)$$

In (55) use has been made of the definitions $\rho_1 = m_1 \delta(\boldsymbol{R}_1 - \boldsymbol{R})$, with m_1 the rest mass of atom 1, and $\delta\rho_1 = \delta m_1\, \delta(\boldsymbol{R}_1 - \boldsymbol{R})$. (In the nonrelativistic case $\varrho = \varrho'$ according to (19) and also $v \equiv \varrho^{-1} = (\varrho')^{-1} \equiv v'$). The first member contains the time derivative of the energy density, which consists of the bulk kinetic energy density, the bulk internal atomic energy, the thermal agitation energy and four potential energy terms, including contributions due to the atomic charges and dipole moments (*v.* (56)). Furthermore in the left-hand side the divergence of the energy flow appears. The right-hand side contains the work performed (per unit of time and volume) by the Lorentz and ponderomotive forces.

The nonrelativistic momentum law follows from (8) for $\alpha = 1, 2, 3$ with (27), (41), (50) and the nonrelativistic approximation of (54). This law becomes, if only terms of order c^0 are included,

$$(\partial/\partial t)(\varrho \boldsymbol{v}) + \boldsymbol{\nabla} \cdot (\varrho \boldsymbol{v}\boldsymbol{v} + \textstyle\int m_1 \hat{\boldsymbol{v}}_1 \hat{\boldsymbol{v}}_1 f_1(\boldsymbol{R}, \boldsymbol{v}_1; t)\, \mathrm{d}\boldsymbol{v}_1 + \mathbf{C}_{\mathrm{III}}) =$$

$$= \varrho^e \boldsymbol{E} + (\boldsymbol{\nabla}\boldsymbol{E}) \cdot \boldsymbol{P} + (\boldsymbol{\nabla}\boldsymbol{B}) \cdot \boldsymbol{M}, \qquad (58)$$

where the abbreviation $\mathbf{C}_{\mathrm{III}}$ is introduced

$$\mathbf{C}_{\mathrm{III}} = -\sum_{e_1, e_2} \int \left(s\boldsymbol{\nabla}_s \frac{e_1 e_2}{8\pi s} \right) c_2(\boldsymbol{R} + \tfrac{1}{2}\boldsymbol{s}, \boldsymbol{R} - \tfrac{1}{2}\boldsymbol{s}, e_1, e_2; t) \, \mathrm{d}\boldsymbol{s}$$

$$+ \sum_{e_1} \int \left(s\boldsymbol{\nabla}_s \boldsymbol{\mu}_2 \cdot \boldsymbol{\nabla}_s \frac{e_1}{4\pi s} \right) c_2(\boldsymbol{R} + \tfrac{1}{2}\boldsymbol{s}, \boldsymbol{R} - \tfrac{1}{2}\boldsymbol{s}, e_1, \boldsymbol{\mu}_2; t) \, \mathrm{d}\boldsymbol{s} \, \mathrm{d}\boldsymbol{\mu}_2$$

$$+ \int \left(s\boldsymbol{\nabla}_s \boldsymbol{\mu}_1 \cdot \boldsymbol{\nabla}_s \boldsymbol{\mu}_2 \cdot \boldsymbol{\nabla}_s \frac{1}{8\pi s} \right) c_2(\boldsymbol{R} + \tfrac{1}{2}\boldsymbol{s}, \boldsymbol{R} - \tfrac{1}{2}\boldsymbol{s}, \boldsymbol{\mu}_1, \boldsymbol{\mu}_2; t) \, \mathrm{d}\boldsymbol{s} \, \mathrm{d}\boldsymbol{\mu}_1 \, \mathrm{d}\boldsymbol{\mu}_2$$

$$- \int \left\{ s\boldsymbol{\nabla}_s (\boldsymbol{\nu}_1 \wedge \boldsymbol{\nabla}_s)(\boldsymbol{\nu}_2 \wedge \boldsymbol{\nabla}_s) : \frac{\mathsf{T}(\boldsymbol{s})}{16\pi s} \right\} c_2(\boldsymbol{R} + \tfrac{1}{2}\boldsymbol{s}, \boldsymbol{R} - \tfrac{1}{2}\boldsymbol{s}, \boldsymbol{\nu}_1, \boldsymbol{\nu}_2; t) \, \mathrm{d}\boldsymbol{s} \, \mathrm{d}\boldsymbol{\nu}_1 \, \mathrm{d}\boldsymbol{\nu}_2. \quad (59)$$

The first two terms of (58) form together the product of mass density ϱ and the acceleration $\mathrm{d}\boldsymbol{v}/\mathrm{d}t$, where $\mathrm{d}/\mathrm{d}t$ is the substantial time derivative. The other terms at the left-hand side are equal to the divergence of a pressure tensor, which consists of the kinetic and four potential terms (see (59)).The right-hand side contains the Lorentz and ponderomotive force (ϱ^e is the macroscopic charge density). The term $(\boldsymbol{\nabla}\boldsymbol{E})\cdot\boldsymbol{P}$ is called the Kelvin force, found already in a statistical theory of the static, electric dipole case[4]).

The angular momentum law (10) with (5) is found explicitly if one substitutes the expressions (14), (7), (28), (51), (54) and also

$$T^{\alpha\beta}_{(f)} - T^{\beta\alpha}_{(f)} = \Delta^{\alpha}_{\gamma}\Delta^{\beta}_{\varepsilon}(F^{\gamma\zeta}M_{\zeta}^{\cdot\varepsilon} - M^{\gamma\zeta}F_{\zeta}^{\cdot\varepsilon}), \quad (60)$$

which follows from the form of $T^{\alpha\beta}_{(f)}$, given by (III.42). The nonrelativistic approximation of the space-space part of (10) ($\alpha = i$; $\beta = j$; $i, j = 1, 2, 3$) contains (27), (29), (41), (50), (53) and the nonrelativistic approximations of (54) and (60).

It may be of interest to consider separately the balance equation of intrinsic angular momentum, which follows from (8) and (10)

$$T^{\beta\alpha}_{(m)} - T^{\alpha\beta}_{(m)} = T^{\alpha\beta}_{(f)} - T^{\beta\alpha}_{(f)}, \quad (61)$$

where (5), (28), (51) and (60) must be inserted. The nonrelativistic approximation to this equation is obtained with the help of (29), (53) and the nonrelativistic limit of (60). Its space-space components ($\alpha = i$, $\beta = j$; $i, j, k = 1, 2, 3$ cycl.) become

$$c\partial_0 \Sigma^k + \boldsymbol{\nabla}\cdot\left(\boldsymbol{v}\Sigma^k + \int \hat{\boldsymbol{v}}_1 \sigma_1^k f_1(1; t) \, \mathrm{d}\boldsymbol{v}_1\right)$$

$$- \sum_{e_1} \int \left(\boldsymbol{\mu}_2 \wedge \boldsymbol{\nabla}_s \frac{e_1}{4\pi s} \right)^k c_2(\boldsymbol{R} + \boldsymbol{s}, \boldsymbol{R}, e_1, \boldsymbol{\mu}_2; t) \, \mathrm{d}\boldsymbol{s} \, \mathrm{d}\boldsymbol{\mu}_2$$

$$- \int \left(\boldsymbol{\mu}_1 \cdot \boldsymbol{\nabla}_s \boldsymbol{\mu}_2 \wedge \boldsymbol{\nabla}_s \frac{1}{4\pi s} \right)^k c_2(\boldsymbol{R} + \boldsymbol{s}, \boldsymbol{R}, \boldsymbol{\mu}_1, \boldsymbol{\mu}_2; t) \, \mathrm{d}\boldsymbol{s} \, \mathrm{d}\boldsymbol{\mu}_1 \, \mathrm{d}\boldsymbol{\mu}_2$$

$$+ \int \left(\boldsymbol{v}_2 \wedge \left[\boldsymbol{\nabla}_s \wedge \left\{ (\boldsymbol{v}_1 \wedge \boldsymbol{\nabla}_s) \cdot \frac{\mathsf{T}(s)}{8\pi s} \right\} \right] \right)^k c_2(\boldsymbol{R} + \boldsymbol{s}, \boldsymbol{R}, \boldsymbol{v}_1, \boldsymbol{v}_2; t) \, \mathrm{d}\boldsymbol{s} \, \mathrm{d}\boldsymbol{v}_1 \, \mathrm{d}\boldsymbol{v}_2$$

$$= (\boldsymbol{P} \wedge \boldsymbol{E} + \boldsymbol{M} \wedge \boldsymbol{B})^k \qquad (k = 1, 2, 3). \tag{62}$$

The first two terms form $\varrho \, \mathrm{d}(v\Sigma)/\mathrm{d}t$ where $\mathrm{d}/\mathrm{d}t$ is the substantial time derivative. The third term is a fluctuation term corresponding to the second term. The fourth and fifth terms are correlation terms of the same structure as the first term at the right-hand side; the sixth term of the left-hand side is related in the same way to the last term at the right-hand side.

§ 5. *Quadratic polarization contributions to the correlation integrals.* In the nonrelativistic energy-momentum laws time derivatives and divergences of correlation integrals occur, which are given by (49) and (50). These equalities will now be written as

$$\partial_\beta T^{0\beta}_{(m)22} = \partial_0 C_{\mathrm{I}} + c^{-1} \nabla_i C^i_{\mathrm{II}}, \tag{63}$$

$$\partial_\beta T^{i\beta}_{(m)22} = \nabla_j \mathsf{C}^{ji}_{\mathrm{III}}. \tag{64}$$

where the quantities C_{I}, C^i_{II}, $\mathsf{C}^{ij}_{\mathrm{III}}$ $(i, j = 1, 2, 3)$ are the abbreviations (56), (57) and (59). No time derivative occurs at the right-hand side of (64) as a consequence of the nonrelativistic approximation, according to which the momentum conservation law is considered up to order c^0.

Let us now define a tensor

$$\bar{T}^{\alpha\beta}_{(m)22} = \begin{pmatrix} \bar{T}^{00}_{(m)22} & \bar{T}^{0i}_{(m)22} \\ \bar{T}^{i0}_{(m)22} & \bar{T}^{ij}_{(m)22} \end{pmatrix}, \tag{65}$$

which in the local rest frame reduces to the array

$$\begin{pmatrix} C_{\mathrm{I}} & c^{-1} C^i_{\mathrm{II}} \\ c^{-1} C^i_{\mathrm{II}} & \mathsf{C}^{ji}_{\mathrm{III}} \end{pmatrix}. \tag{66}$$

The tensor defined as

$$\bar{T}^{\alpha\beta}_{(m)} = T^{\alpha\beta}_{(m)1} + T^{\alpha\beta}_{(m)21} + \bar{T}^{\alpha\beta}_{(m)22} \tag{67}$$

can be looked upon as the material tensor for a system of which the relative atomic motion is nonrelativistic within the correlation domain, since then the energy-momentum conservation laws have the form

$$\partial_\beta (\bar{T}^{\alpha\beta}_{(m)} + T^{\alpha\beta}_{(f)}) = 0, \tag{68}$$

as follows from the treatment of the preceding two sections. From now on we limit ourselves to the study of systems of the kind described above.

The correlation integrals C_{I}, C^i_{II} and $\mathsf{C}^{ij}_{\mathrm{III}}$ contain the correlation functions, which are defined in terms of the one- and two-point distribution functions:

$$c_2(\boldsymbol{R} + \tfrac{1}{2}\boldsymbol{s}, \boldsymbol{R} - \tfrac{1}{2}\boldsymbol{s}, 1, 2; t) = f_2(\boldsymbol{R} + \tfrac{1}{2}\boldsymbol{s}, \boldsymbol{R} - \tfrac{1}{2}\boldsymbol{s}, 1, 2; t) - f_1(\boldsymbol{R} + \tfrac{1}{2}\boldsymbol{s}, 1; t) \, f_1(\boldsymbol{R} - \tfrac{1}{2}\boldsymbol{s}, 2; t). \tag{69}$$

If $\boldsymbol{s} = 0$ the function f_2 vanishes as a consequence of its definition[1]), which means that then c_2 is given in terms of the product of two one-point distribution functions only:

$$c_2(\boldsymbol{R}, \boldsymbol{R}, 1, 2; t) = -f_1(\boldsymbol{R}, 1; t)\, f_1(\boldsymbol{R}, 2; t). \tag{70}$$

Hence the correlation integrals contain in fact a noncorrelated part. For this reason we shall write the correlation integrals as sums of a principal value, obtained by excluding from the integration an infinitesimally small sphere around $\boldsymbol{s} = 0$, and a contribution from the small sphere. Let us treat as an example the following integral, which occurs in (56):

$$I \equiv \int \left(\boldsymbol{\mu}_1 \cdot \boldsymbol{\nabla}_s\, \boldsymbol{\mu}_2 \cdot \boldsymbol{\nabla}_s \frac{1}{8\pi s} \right) c_2(\boldsymbol{R} + \tfrac{1}{2}\boldsymbol{s}, \boldsymbol{R} - \tfrac{1}{2}\boldsymbol{s}, 1, 2; t)\, \mathrm{d}\boldsymbol{s}\, \mathrm{d}\boldsymbol{\mu}_1\, \mathrm{d}\boldsymbol{\mu}_2. \tag{71}$$

With the identity (see appendix)

$$\boldsymbol{\nabla}_s \boldsymbol{\nabla}_s \frac{1}{4\pi s} = \mathscr{P} \boldsymbol{\nabla}_s \boldsymbol{\nabla}_s \frac{1}{4\pi s} - \tfrac{1}{3} \mathsf{U} \delta(\boldsymbol{s}), \tag{72}$$

where U is the unit-tensor and $\delta(\boldsymbol{s})$ the three-dimensional delta function, and the property (70), one gets for the integral (71):

$$I = \mathscr{P} \int \left(\boldsymbol{\mu}_1 \cdot \boldsymbol{\nabla}_s \boldsymbol{\mu}_2 \cdot \boldsymbol{\nabla}_s \frac{1}{8\pi s} \right) c_2(\boldsymbol{R} + \tfrac{1}{2}\boldsymbol{s}, \boldsymbol{R} - \tfrac{1}{2}\boldsymbol{s}, 1, 2; t)\, \mathrm{d}\boldsymbol{s}\; \mathrm{d}\boldsymbol{\mu}_1\, \mathrm{d}\boldsymbol{\mu}_2 + \tfrac{1}{6}\boldsymbol{P}^2, \tag{73}$$

with the (nonrelativistic) electric polarization vector

$$\boldsymbol{P} = \int \boldsymbol{\mu}_1 f_1(\boldsymbol{R}, 1; t)\, \mathrm{d}\boldsymbol{\mu}_1. \tag{74}$$

In this way all correlation integrals occurring in $\overline{T}^{\alpha\beta}_{(m)22}$ can be treated (see appendix). The result is

$$C_{\mathrm{I}} = \mathscr{P} C_{\mathrm{I}} - \tfrac{1}{6}\boldsymbol{P}^2 - \tfrac{1}{3}\boldsymbol{M}^2, \tag{75}$$

$$\begin{aligned} C^i_{\mathrm{II}} = \mathscr{P} C^i_{\mathrm{II}} - \{ &\tfrac{1}{30}\boldsymbol{P} \cdot \textstyle\int (3\hat{\boldsymbol{v}}_1 \boldsymbol{\mu}_1 + 8 \boldsymbol{\mu}_1 \hat{\boldsymbol{v}}_1 + 3 \hat{\boldsymbol{v}}_1 \cdot \boldsymbol{\mu}_1 \mathsf{U})\, f_1(\boldsymbol{R}, 1; t)\, \mathrm{d}\boldsymbol{v}_1\, \mathrm{d}\boldsymbol{\mu}_1 \\ &+ \tfrac{1}{30}\boldsymbol{M} \cdot \textstyle\int (3\hat{\boldsymbol{v}}_1 \boldsymbol{\nu}_1 - 2\boldsymbol{\nu}_1 \hat{\boldsymbol{v}}_1 + 3\hat{\boldsymbol{v}}_1 \cdot \boldsymbol{\nu}_1 \mathsf{U})\, f_1(\boldsymbol{R}, 1; t)\, \mathrm{d}\boldsymbol{v}_1\, \mathrm{d}\boldsymbol{\nu}_1 \\ &+ \tfrac{1}{5}\boldsymbol{P}\boldsymbol{P} \cdot \boldsymbol{v} + \tfrac{4}{15}\boldsymbol{P}^2 \boldsymbol{v} + \tfrac{1}{5}\boldsymbol{M}\boldsymbol{M} \cdot \boldsymbol{v} - \tfrac{1}{15}\boldsymbol{M}^2 \boldsymbol{v}\}^i, \end{aligned} \tag{76}$$

$$\mathsf{C}^{ij}_{\mathrm{III}} = \mathscr{P} \mathsf{C}^{ij}_{\mathrm{III}} - (\tfrac{1}{5}\boldsymbol{P}\boldsymbol{P} + \tfrac{1}{10}\boldsymbol{P}^2 \mathsf{U} + \tfrac{1}{5}\boldsymbol{M}\boldsymbol{M} - \tfrac{2}{5}\boldsymbol{M}^2 \mathsf{U})^{ij}. \tag{77}$$

These expressions show that terms quadratic in the macroscopic polarizations $\boldsymbol{P}$ and $\boldsymbol{M}$ are contained in the correlation part of the material energy-momentum tensor (67). In the local rest frame of the system ($\boldsymbol{v} = 0$) these quadratic terms form an array

$$\begin{pmatrix} -\tfrac{1}{6}\boldsymbol{P}^2 - \tfrac{1}{3}\boldsymbol{M}^2 & 0 \\ 0 & -\tfrac{1}{5}\boldsymbol{P}\boldsymbol{P} - \tfrac{1}{10}\boldsymbol{P}^2\mathsf{U} - \tfrac{1}{5}\boldsymbol{M}\boldsymbol{M} + \tfrac{2}{5}\boldsymbol{M}^2\mathsf{U} \end{pmatrix}. \tag{78}$$

We now define $T_*^{\alpha\beta}$ as the tensor, which in the rest frame reduces to (78). (One may check that then in the nonrelativistic approximation the velocity terms of (76) are obtained). This tensor, which in covariant form reads

$$T_*^{\alpha\beta} = \tfrac{1}{5}\Delta^\alpha_\gamma \Delta^\beta_\varepsilon (M^{\gamma}_{\cdot\zeta} M^{\varepsilon\zeta} + \tfrac{1}{2} g^{\gamma\varepsilon} M_{\zeta\eta} M^{\zeta\eta} + \tfrac{1}{2} c^{-2} g^{\gamma\varepsilon} M^{\zeta\eta} U_\eta M_{\zeta\vartheta} U^\vartheta) \\ - \tfrac{1}{6} c^{-2} U^\alpha U^\beta (M_{\gamma\varepsilon} M^{\gamma\varepsilon} + 3c^{-2} M^{\gamma\varepsilon} U_\varepsilon M_{\gamma\zeta} U^\zeta), \qquad (79)$$

will be used in the next section.

§ 6. *A redefinition of the field and material parts of the energy-momentum tensor.* As shown in the preceding section the material energy-momentum tensor contains a part $T_*^{\alpha\beta}$, which depends quadratically on the macroscopic polarizations $\boldsymbol{P}$ and $\boldsymbol{M}$. The field tensor $T^{\alpha\beta}_{(f)}$ consisted of terms quadratic in the fields ($\boldsymbol{E}$ and $\boldsymbol{B}$) and of terms which are bilinear in the fields and polarizations. In view of this situation one might introduce a new splitting of the total energy-momentum tensor $\overline{T}^{\alpha\beta} \equiv \overline{T}^{\alpha\beta}_{(m)} + T^{\alpha\beta}_{(f)}$, as it occurs in (68),

$$\overline{T}^{\alpha\beta} = T^{\alpha\beta}_{(f*)} + T^{\alpha\beta}_{(m*)}, \qquad (80)$$

with a new field tensor

$$T^{\alpha\beta}_{(f*)} = T^{\alpha\beta}_{(f)} + T_*^{\alpha\beta}, \qquad (81)$$

which depends quadratically on the fields and polarizations, and correspondingly a new material tensor

$$T^{\alpha\beta}_{(m*)} = \overline{T}^{\alpha\beta}_{(m)} - T_*^{\alpha\beta}. \qquad (82)$$

The new field energy-momentum tensor, which according to (81) is the sum of (III.42) and (79), reads

$$T^{\alpha\beta}_{(f*)} = F^{\alpha\gamma} H^{\beta}_{\cdot\gamma} - \tfrac{1}{4} F_{\gamma\varepsilon} F^{\gamma\varepsilon} g^{\alpha\beta} \\ + c^{-2} U^\beta (F^{\alpha\gamma} M_{\gamma\varepsilon} - M^{\alpha\gamma} F_{\gamma\varepsilon})\, U^\varepsilon - c^{-4} U^\alpha U^\beta U^\gamma F_{\gamma\varepsilon} M^{\varepsilon\zeta} U_\zeta \\ + \tfrac{1}{5}\Delta^\alpha_\gamma \Delta^\beta_\varepsilon (M^{\gamma}_{\cdot\zeta} M^{\varepsilon\zeta} + \tfrac{1}{2} g^{\gamma\varepsilon} M_{\zeta\eta} M^{\zeta\eta} + \tfrac{1}{2} c^{-2} g^{\gamma\varepsilon} M^{\zeta\eta} U_\eta M_{\zeta\vartheta} U^\vartheta) \\ - \tfrac{1}{6} c^{-2} U^\alpha U^\beta (M_{\gamma\varepsilon} M^{\gamma\varepsilon} + 3c^{-2} M^{\gamma\varepsilon} U_\varepsilon M_{\gamma\zeta} U^\zeta). \qquad (83)$$

In the local momentary rest frame this field tensor reads – in three-dimensional notation –

$$T^{\alpha\beta}_{(f*)} = \begin{bmatrix} \tfrac{1}{2}\boldsymbol{E}^2 + \tfrac{1}{2}\boldsymbol{B}^2 - \tfrac{1}{6}\boldsymbol{P}^2 - \tfrac{1}{3}\boldsymbol{M}^2 & \boldsymbol{E}\wedge\boldsymbol{H} \\ \boldsymbol{E}\wedge\boldsymbol{H} & \begin{matrix} -\boldsymbol{ED} - \boldsymbol{HB} - \tfrac{1}{5}\boldsymbol{PP} - \tfrac{1}{5}\boldsymbol{MM} \\ + (\tfrac{1}{2}\boldsymbol{E}^2 + \tfrac{1}{2}\boldsymbol{H}^2 - \tfrac{1}{10}\boldsymbol{P}^2 - \tfrac{1}{10}\boldsymbol{M}^2)\,\mathrm{U} \end{matrix} \end{bmatrix}, \qquad (84)$$

where $\boldsymbol{D} \equiv \boldsymbol{E} + \boldsymbol{P}$ en $\boldsymbol{H} \equiv \boldsymbol{B} - \boldsymbol{M}$. The field tensor $T^{\alpha\beta}_{(f*)}$ corresponds to a material tensor (82), in which nonrelativistically only bulk materials terms, fluctuation terms and principal values of correlation integrals occur. In fact the energy law – up to order c^0 – (55) becomes with the help of the equalities (75)–(76)

$$
\begin{aligned}
&(\partial/\partial t)(\tfrac{1}{2}\varrho v^2 + \delta\varrho c^2 + \textstyle\int \tfrac{1}{2}m_1\hat{v}_1^2 f_1(\boldsymbol{R}, \boldsymbol{v}_1; t)\,\mathrm{d}\boldsymbol{v}_1 + \mathscr{P}C_{\mathrm{I}})\\
&\quad + \boldsymbol{\nabla}\cdot[(\tfrac{1}{2}\varrho v^2 + \delta\varrho c^2)\,\boldsymbol{v} + \textstyle\int \{\tfrac{1}{2}m_1(\hat{v}_1^2\boldsymbol{v} + 2\boldsymbol{v}\cdot\hat{\boldsymbol{v}}_1\hat{\boldsymbol{v}}_1 + \hat{v}_1^2\hat{\boldsymbol{v}}_1)\\
&\quad + \delta m_1 c^2\hat{\boldsymbol{v}}_1\}\, f_1(\boldsymbol{R}, \boldsymbol{v}_1; t)\,\mathrm{d}\boldsymbol{v}_1 - \textstyle\int \hat{\boldsymbol{v}}_1\boldsymbol{E}\cdot\boldsymbol{\mu}_1 f_1(\boldsymbol{R}, \boldsymbol{v}_1, \boldsymbol{\mu}_1; t)\,\mathrm{d}\boldsymbol{v}_1\,\mathrm{d}\boldsymbol{\mu}_1\\
&\quad - \tfrac{1}{30}\boldsymbol{P}\cdot\textstyle\int (3\hat{\boldsymbol{v}}_1\boldsymbol{\mu}_1 + 8\boldsymbol{\mu}_1\hat{\boldsymbol{v}}_1 + 3\hat{\boldsymbol{v}}_1\cdot\boldsymbol{\mu}_1\mathsf{U})\, f_1(\boldsymbol{R}, \boldsymbol{v}_1, \boldsymbol{\mu}_1; t)\,\mathrm{d}\boldsymbol{v}_1\,\mathrm{d}\boldsymbol{\mu}_1\\
&\quad - \tfrac{1}{30}\boldsymbol{M}\cdot\textstyle\int (3\hat{\boldsymbol{v}}_1\boldsymbol{\nu}_1 - 2\boldsymbol{\nu}_1\hat{\boldsymbol{v}}_1 + 3\hat{\boldsymbol{v}}_1\cdot\boldsymbol{\nu}_1\mathsf{U})\, f_1(\boldsymbol{R}, \boldsymbol{v}_1, \boldsymbol{\nu}_1; t)\,\mathrm{d}\boldsymbol{v}_1\,\mathrm{d}\boldsymbol{\nu}_1\\
&\quad + \mathscr{P}\boldsymbol{C}_{\mathrm{II}}] =\\
&\quad = \boldsymbol{J}\cdot\boldsymbol{E} + \varrho\boldsymbol{E}\cdot\frac{\mathrm{d}(v\boldsymbol{P})}{\mathrm{d}t} - \frac{\mathrm{d}\boldsymbol{B}}{\mathrm{d}t}\cdot\boldsymbol{M} + \tfrac{1}{6}\frac{\partial P^2}{\partial t} + \tfrac{1}{3}\frac{\partial M^2}{\partial t}\\
&\quad + \boldsymbol{v}\cdot(\boldsymbol{\nabla}\boldsymbol{E})\cdot\boldsymbol{P} + \boldsymbol{v}\cdot(\boldsymbol{\nabla}\boldsymbol{B})\cdot\boldsymbol{M} + \boldsymbol{\nabla}\cdot(\tfrac{1}{5}\boldsymbol{P}\boldsymbol{P}\cdot\boldsymbol{v} + \tfrac{4}{15}P^2\boldsymbol{v} + \tfrac{1}{5}\boldsymbol{M}\boldsymbol{M}\cdot\boldsymbol{v} - \tfrac{1}{15}M^2\boldsymbol{v}),
\end{aligned}
\tag{85}
$$

where the principal value of the correlation integrals C_{I}, and $\boldsymbol{C}_{\mathrm{II}}$ are given by

$$
\begin{aligned}
\mathscr{P}C_{\mathrm{I}} = &\sum_{e_1,e_2}\int \frac{e_1e_2}{8\pi s}\, c_2(\boldsymbol{R} + \tfrac{1}{2}\boldsymbol{s}, \boldsymbol{R} - \tfrac{1}{2}\boldsymbol{s}, e_1, e_2; t)\,\mathrm{d}\boldsymbol{s}\\
&-\sum_{e_1}\int\left(\boldsymbol{\mu}_2\cdot\boldsymbol{\nabla}_s\frac{e_1}{4\pi s}\right) c_2(\boldsymbol{R} + \tfrac{1}{2}\boldsymbol{s}, \boldsymbol{R} - \tfrac{1}{2}\boldsymbol{s}, e_1, \boldsymbol{\mu}_2; t)\,\mathrm{d}\boldsymbol{s}\,\mathrm{d}\boldsymbol{\mu}_2\\
&-\mathscr{P}\int\left(\boldsymbol{\mu}_1\cdot\boldsymbol{\nabla}_s\,\boldsymbol{\mu}_2\cdot\boldsymbol{\nabla}_s\frac{1}{8\pi s}\right) c_2(\boldsymbol{R} + \tfrac{1}{2}\boldsymbol{s}, \boldsymbol{R} - \tfrac{1}{2}\boldsymbol{s}, \boldsymbol{\mu}_1, \boldsymbol{\mu}_2; t)\,\mathrm{d}\boldsymbol{s}\,\mathrm{d}\boldsymbol{\mu}_1\,\mathrm{d}\boldsymbol{\mu}_2\\
&+\mathscr{P}\int\left(\boldsymbol{\nu}_1\cdot\boldsymbol{\nabla}_s\,\boldsymbol{\nu}_2\cdot\boldsymbol{\nabla}_s\frac{1}{8\pi s}\right) c_2(\boldsymbol{R} + \tfrac{1}{2}\boldsymbol{s}, \boldsymbol{R} - \tfrac{1}{2}\boldsymbol{s}, \boldsymbol{\nu}_1, \boldsymbol{\nu}_2; t)\,\mathrm{d}\boldsymbol{s}\,\mathrm{d}\boldsymbol{\nu}_1\,\mathrm{d}\boldsymbol{\nu}_2,
\end{aligned}
\tag{86}
$$

$$
\begin{aligned}
\mathscr{P}\boldsymbol{C}_{\mathrm{II}} = &\sum_{e_1,e_2}\int \frac{e_1e_1\mathsf{T}(\boldsymbol{s})}{16\pi s}\cdot(\boldsymbol{v}_1 + \boldsymbol{v}_2)\, c_2(\boldsymbol{R} + \tfrac{1}{2}\boldsymbol{s}, \boldsymbol{R} - \tfrac{1}{2}\boldsymbol{s}, \boldsymbol{v}_1, \boldsymbol{v}_2, e_1, e_2; t)\,\mathrm{d}\boldsymbol{s}\,\mathrm{d}\boldsymbol{v}_1\,\mathrm{d}\boldsymbol{v}_2\\
&+\sum_{e_1}\int\left[\{\boldsymbol{s}\boldsymbol{\mu}_2\cdot\boldsymbol{\nabla}_s(\boldsymbol{v}_1 + \boldsymbol{v}_2)\cdot\boldsymbol{\nabla}_s - (\boldsymbol{v}_1 + \boldsymbol{v}_2)\,\boldsymbol{\mu}_2\cdot\boldsymbol{\nabla}_s - \boldsymbol{s}\dot{\boldsymbol{\mu}}_2\cdot\boldsymbol{\nabla}_s\}\frac{e_1}{8\pi s}\right]\\
&c_2(\boldsymbol{R} + \tfrac{1}{2}\boldsymbol{s}, \boldsymbol{R} - \tfrac{1}{2}\boldsymbol{s}, \boldsymbol{v}_1, \boldsymbol{v}_2, e_1, \boldsymbol{\mu}_2, \dot{\boldsymbol{\mu}}_2; t)\,\mathrm{d}\boldsymbol{s}\,\mathrm{d}\boldsymbol{v}_1\,\mathrm{d}\boldsymbol{v}_2\,\mathrm{d}\boldsymbol{\mu}_2\,\mathrm{d}\dot{\boldsymbol{\mu}}_2\\
&+\mathscr{P}\int\Big[\{\boldsymbol{s}\boldsymbol{\mu}_1\cdot\boldsymbol{\nabla}_s\boldsymbol{\mu}_2\cdot\boldsymbol{\nabla}_s(\boldsymbol{v}_1 + \boldsymbol{v}_2)\cdot\boldsymbol{\nabla}_s - (\boldsymbol{v}_1 + \boldsymbol{v}_2)\,\boldsymbol{\mu}_1\cdot\boldsymbol{\nabla}_s\boldsymbol{\mu}_2\cdot\boldsymbol{\nabla}_s\\
&- 2\boldsymbol{s}\boldsymbol{\mu}_1\cdot\boldsymbol{\nabla}_s\dot{\boldsymbol{\mu}}_2\cdot\boldsymbol{\nabla}_s\}\frac{1}{16\pi s}\Big]\\
&c_2(\boldsymbol{R} + \tfrac{1}{2}\boldsymbol{s}, \boldsymbol{R} - \tfrac{1}{2}\boldsymbol{s}, \boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{\mu}_1, \boldsymbol{\mu}_2, \dot{\boldsymbol{\mu}}_2; t)\,\mathrm{d}\boldsymbol{s}\,\mathrm{d}\boldsymbol{v}_1\,\mathrm{d}\boldsymbol{v}_2\,\mathrm{d}\boldsymbol{\mu}_1\,\mathrm{d}\boldsymbol{\mu}_2\,\mathrm{d}\dot{\boldsymbol{\mu}}_2\\
&+\mathscr{P}\int\{\boldsymbol{s}\boldsymbol{\nu}_1\cdot\boldsymbol{\nabla}_s\boldsymbol{\nu}_2\cdot\boldsymbol{\nabla}_s(\boldsymbol{v}_1 + \boldsymbol{v}_2)\cdot\boldsymbol{\nabla}_s +
\end{aligned}
$$

$$+ (v_1 + v_2)\, \nu_1 \cdot \nabla_s \nu_2 \cdot \nabla_s - 2s \nu_1 \cdot \nabla_s \dot{\nu}_2 \cdot \nabla_s\} \frac{1}{16\pi s}\Big]$$

$$c_2(R + \tfrac{1}{2}s, R - \tfrac{1}{2}s, v_1, v_2, \nu_1, \nu_2, \dot{\nu}_2; t)\, \mathrm{d}s\, \mathrm{d}v_1\, \mathrm{d}v_2\, \mathrm{d}\nu_1\, \mathrm{d}\nu_2\, \mathrm{d}\dot{\nu}_2. \qquad (87)$$

Similarly the momentum law (58) becomes up to order c^0 with the help of (77):

$$(\partial/\partial t)(\varrho v) + \nabla \cdot (\varrho vv + \textstyle\int m_1 \hat{v}_1 \hat{v}_1 f_1(R, v_1; t)\, \mathrm{d}v_1 + \mathscr{P}\mathrm{C}_{\mathrm{III}}) =$$
$$= \varrho^e E + (\nabla E) \cdot P + (\nabla H) \cdot M + \tfrac{1}{5}\{(\nabla \cdot P)\, P + P \cdot \nabla P +$$
$$+ (\nabla P) \cdot P + (\nabla \cdot M)\, M + M \cdot \nabla M + (\nabla M) \cdot M\}, \qquad (88)$$

where the principal value of the correlation integral $\mathrm{C}_{\mathrm{III}}$ is:

$$\mathscr{P}\mathrm{C}_{\mathrm{III}} = -\sum_{e_1, e_2} \int s \nabla_s \frac{e_1 e_2}{8\pi s}\, c_2(R + \tfrac{1}{2}s, R - \tfrac{1}{2}s, e_1, e_2; t)\, \mathrm{d}s$$
$$+ \sum_{e_1} \int \left(s \nabla_s\, \mu_2 \cdot \nabla_s \frac{e_1}{4\pi s} \right) c_2(R + \tfrac{1}{2}s, R - \tfrac{1}{2}s, e_1, \mu_2; t)\, \mathrm{d}s\, \mathrm{d}\mu_2$$
$$+ \mathscr{P} \int \left(s \nabla_s\, \mu_1 \cdot \nabla_s\, \mu_2 \cdot \nabla_s \frac{1}{8\pi s} \right) c_2(R + \tfrac{1}{2}s, R - \tfrac{1}{2}s, \mu_1, \mu_2; t)\, \mathrm{d}s\, \mathrm{d}\mu_1\, \mathrm{d}\mu_2$$
$$+ \mathscr{P} \int \left(s \nabla_s\, \nu_1 \cdot \nabla_s\, \nu_2 \cdot \nabla_s \frac{1}{8\pi s} \right) c_2(R + \tfrac{1}{2}s, R - \tfrac{1}{2}s, \nu_1, \nu_2; t)\, \mathrm{d}s\, \mathrm{d}\nu_1\, \mathrm{d}\nu_2. \qquad (89)$$

The correlation integrals containing the magnetic dipoles in (86), (87) and (89) could be rewritten in a form analogous to the correlation integrals with electric dipoles by application of the identity (A 13) of the appendix.

With the "Lorentz fields" defined by

$$E^L = E + \tfrac{1}{3}P, \quad B^L = B - \tfrac{2}{3}M, \qquad (90)$$

and the use of (20) the right-hand sides of (85) and (88) get the forms

$$J \cdot E + \varrho E^L \cdot \frac{\mathrm{d}(vP)}{\mathrm{d}t} - \frac{\mathrm{d}B^L}{\mathrm{d}t} \cdot M + v \cdot (\nabla E^L) \cdot P$$
$$+ v \cdot (\nabla B^L) \cdot M + \nabla \cdot (\tfrac{1}{5} P^{\circ} P \cdot v + \tfrac{1}{5} M^{\circ} M \cdot v), \qquad (91)$$

$$\varrho^e E + (\nabla E^L) \cdot P + (\nabla B^L) \cdot M + \nabla \cdot (\tfrac{1}{5} P^{\circ} P + \tfrac{1}{5} M^{\circ} M), \qquad (92)$$

where $\mathrm{d}/\mathrm{d}t$ is the substantial derivative and where the symbol $\circ$ indicates the traceless part of a tensor: $P^{\circ}P \equiv PP - \tfrac{1}{3}P^2 \mathrm{U}$. The law (88) with (89) contains at the left-hand side the divergence of a pressure tensor, which consists of a kinetic part and a potential part, which is the *principal value* of correlation integrals. The right-hand side of (88) or (92) shows now an expression for the force density, which includes terms quadratic in the polarizations.

The right-hand sides of (85) and (88) are the nonrelativistic approximations of the components $\alpha = 0$ and $\alpha = 1, 2, 3$ of the four-vector

$$F^*_\alpha = -\partial_\beta T^{\alpha\beta}_{(f*)}, \tag{93}$$

where the field tensor is given by (83). In the local rest frame and for a system of constant and uniform velocity they read explicitly

$$F^0_* = c^{-1}\boldsymbol{J}\cdot\boldsymbol{E} + \boldsymbol{E}^L\cdot\partial_0\boldsymbol{P} - (\partial_0\boldsymbol{B}^L)\cdot\boldsymbol{M}, \tag{94}$$

$$\begin{aligned} \boldsymbol{F}_* = {} & \varrho^e\boldsymbol{E} + c^{-1}\boldsymbol{J}\wedge\boldsymbol{B} + (\boldsymbol{\nabla}\boldsymbol{E}^L)\cdot\boldsymbol{P} \\ & + (\boldsymbol{\nabla}\boldsymbol{B}^L)\cdot\boldsymbol{M} + \boldsymbol{\nabla}\cdot(\tfrac{1}{5}\boldsymbol{P}^{\circ}\boldsymbol{P} + \tfrac{1}{5}\boldsymbol{M}^{\circ}\boldsymbol{M}) \\ & + \partial_0(\boldsymbol{P}\wedge\boldsymbol{B}) - \partial_0(\boldsymbol{M}\wedge\boldsymbol{E}). \end{aligned} \tag{95}$$

The terms of order c^0 in (94) times c and in (95) were already obtained in the right-hand sides of (85) and (88).

In this section we have shown that it is possible to introduce a different splitting of the total energy-momentum tensor in a field part and a material part. Such a redefinition has been performed here by taking principal values of the correlation integrals. It is obvious that this procedure is by no means unique since the shape of the small region which is excluded from the integration over $\boldsymbol{s}$ may be chosen arbitrarily. Therefore the terms in the field energy-momentum tensor, which are quadratic in the polarizations are not fixed. In the next papers it will be shown however that thermodynamical considerations permit us to select a field energy-momentum tensor which corresponds to a material tensor containing a scalar pressure tensor in the local momentary rest frame.

APPENDIX

While studying the properties of the correlation integrals (§ 5, formulae (71)–(77)) the following equalities have been employed

$$\nabla_i\nabla_j\frac{1}{4\pi s} = \mathscr{P}\nabla_i\nabla_j\frac{1}{4\pi s} - \tfrac{1}{3}\delta_{ij}\delta(\boldsymbol{s}), \tag{A 1}$$

$$\varepsilon_{ikl}\varepsilon_{jmn}\nabla_k\nabla_m\frac{\mathrm{T}_{ln}(\boldsymbol{s})}{4\pi s} = \mathscr{P}\left\{\varepsilon_{ikl}\varepsilon_{jmn}\nabla_k\nabla_m\frac{\mathrm{T}_{ln}(\boldsymbol{s})}{4\pi s}\right\} - \tfrac{4}{3}\delta_{ij}\delta(\boldsymbol{s}), \tag{A 2}$$

$$s_i\nabla_j\nabla_k\nabla_l\frac{1}{4\pi s} = \mathscr{P}s_i\nabla_j\nabla_k\nabla_l\frac{1}{4\pi s} + \frac{1}{5}(\delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})\,\delta(\boldsymbol{s}), \tag{A 3}$$

$$\begin{aligned} s_i\nabla_j\,\varepsilon_{kmn}\,\varepsilon_{lpq}\nabla_m\nabla_p\frac{\mathrm{T}_{nq}(\boldsymbol{s})}{4\pi s} = {} & \mathscr{P}s_i\nabla_j\,\varepsilon_{kmn}\,\varepsilon_{lpq}\nabla_m\nabla_p\frac{\mathrm{T}_{nq}(\boldsymbol{s})}{4\pi s} \\ & + (\tfrac{8}{5}\delta_{ij}\delta_{kl} - \tfrac{2}{5}\delta_{ik}\delta_{jl} - \tfrac{2}{5}\delta_{il}\delta_{jk})\,\delta(\boldsymbol{s}), \end{aligned} \tag{A 4}$$

where i, j, k, l, m, n, p and q may have the values 1, 2 and 3,

$$\mathrm{T}_{ij}(\boldsymbol{s}) \equiv \delta_{ij} + s_i s_j/s^2, \tag{A 5}$$

and where δ_{ij} and ε_{ijk} are the Kronecker and Levi-Civita symbols. The symbol $\mathscr{P}$ denotes the principal value of the volume integrals in the sense that an infinitesimally small sphere around the origin is excluded from the integration. The proofs of (A 1) and (A 3) are sketched here as examples.

The left-hand side of (A 1) multiplied by a function $f(\boldsymbol{s})$ and integrated over the $\boldsymbol{s}$-space becomes equal to the sum of a principal value and a term which is obtained after a Taylor-expansion of $f(\boldsymbol{s})$ around the origin:

$$\int\left(\nabla_i\nabla_j\frac{1}{4\pi s}\right)f(\boldsymbol{s})\,\mathrm{d}\boldsymbol{s} = \mathscr{P}\int\left(\nabla_i\nabla_j\frac{1}{4\pi s}\right)f(\boldsymbol{s})\,\mathrm{d}\boldsymbol{s} + f(0)\lim_{\varepsilon\to 0}\int_{U(\varepsilon)}\left(\nabla_i\nabla_j\frac{1}{4\pi s}\right)\mathrm{d}\boldsymbol{s}, \tag{A 6}$$

where $U(\varepsilon)$ is a sphere of radius ε around the origin. The integral in the last term is equal to a surface integral

$$\int_{S(\varepsilon)} n_i\left(\nabla_j\frac{1}{4\pi s}\right)\mathrm{d}S = \int_{\Omega} n_i\left(\nabla_j\frac{1}{4\pi s}\right)s^2\,\mathrm{d}\Omega, \tag{A 7}$$

where $n_i (i = 1, 2, 3)$ is the unit vector in the direction of $\boldsymbol{s}$ and $\mathrm{d}\Omega$ an element of solid angle. This expression becomes

$$-\int n_i n_j\,\mathrm{d}\Omega/4\pi = -\tfrac{1}{3}\delta_{ij}, \tag{A 8}$$

which proves (A 1).

The left-hand side of (A 3) may be treated similarly

$$\int\left(s_i\nabla_j\nabla_k\nabla_l\frac{1}{4\pi s}\right)f(\boldsymbol{s})\,\mathrm{d}\boldsymbol{s} = \mathscr{P}\int\left(s_i\nabla_j\nabla_k\nabla_l\frac{1}{4\pi s}\right)f(\boldsymbol{s})\,\mathrm{d}\boldsymbol{s} + f(0)\lim_{\varepsilon\to 0}\int_{U(\varepsilon)}\left(s_i\nabla_j\nabla_k\nabla_l\frac{1}{4\pi s}\right)\mathrm{d}\boldsymbol{s}. \tag{A 9}$$

The last integral becomes equal to a surface integral after a partial integration:

$$\int\left\{(n_j s_i\nabla_k - \delta_{ij}n_k)\nabla_l\frac{1}{4\pi s}\right\}s^2\,\mathrm{d}\Omega. \tag{A 10}$$

Performing the differentiations one gets

$$\int(\delta_{ij}n_k n_l - \delta_{kl}n_i n_j + 3n_i n_j n_k n_l)\,\mathrm{d}\Omega/4\pi. \tag{A 11}$$

This yields

$$\tfrac{1}{5}(\delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}), \tag{A 12}$$

which completes the proof of (A 3).

It may be remarked that from differentiation for $\boldsymbol{s} \neq 0$ one can verify the identity

$$\mathscr{P}\varepsilon_{ikl}\varepsilon_{jmn}\nabla_k\nabla_m \frac{\mathrm{T}_{ln}(\boldsymbol{s})}{4\pi s} = -2\mathscr{P}\nabla_i\nabla_j \frac{1}{4\pi s}. \tag{A 13}$$

This means that the principal values in (A 2) and (A 4) are essentially the same as those in (A 1) and (A 3) respectively.

This investigation is part of the research programme of the "Stichting voor Fundamenteel Onderzoek der Materie (F.O.M.)", which is financially supported by the "Organisatie voor Zuiver Wetenschappelijk Onderzoek (Z.W.O.)".

Received 4-9-67

REFERENCES

1) de Groot, S. R. and Suttorp, L. G., Physica **39** (1968) 28 (article III of this series).
2) de Groot, S. R. and Suttorp, L. G., Physica **31** (1965) 1713.
3) Irving, J. H. and Kirkwood, J. G., J. chem. Phys. **18** (1950) 817.
4) Mazur, P. and de Groot, S. R., Physica **22** (1956) 657.